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A necessary condition for similarity of indefinite sturm-liouville operators

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ABSTRACT

We consider a singular Sturm-Liouville differential expression with an indefinite weight function and we present a necessary condition for similarity of indefinite Sturm-Liouville operators to self-adjoint operators. Using this result, we construct two examples and prove that none of them is similar to a self-adjoint operator.

Introduction

In this paper, we investigate the singular Sturm-Liouville expression

$$a(y) = 1/r(x)\ ((-py')' + qy\ ), \qquad (1.1)$$

where the weight function r changes its sign. Differential operators with indefinite weights have intensely been investigated in the recent years (see [1, 2, 3, 12, 5, 15, 18, 19]). We assume that (1.1) is in the limit point case at both $-\infty$ and $+\infty$ and that the functions p, q, r are real, $r \neq 0$ a.e.. Then the maximal operator A associated to (1.1) is self-adjoint in the Krein space $L^2_r(\mathbb{R})$, where the indefinite inner product is defined by

$$[f,g] := \int_{\mathbb{R}} f(x)\overline{g(x)}r(x)\,dx, \quad f,g \in L^2_r(\mathbb{R}). \qquad (1.2)$$

The operator $J: f(x) \mapsto \mathrm{sgn}(r(x))f(x)$ is a fundamental symmetry in the Krein space $L^2_r(\mathbb{R})$. Let us define the operator $L = JA$. Then L is a self-adjoint Sturm-Liouville operator in the Hilbert space $L^2_{|r|}(\mathbb{R})$. Two closed operators T_1 and T_2 in a Hilbert space H are called similar if there exist a bounded operator S with the bounded inverse S^{-1} in H such that $S[\mathrm{dom}(T]_1) =$ T_2 and $T_2 = ST_1S^{-1}$.

In general, the operator (1.1) considered on $L^2_r(\mathbb{R})$ has continuous spectrum. In the case, one considers the property of similarity either to a normal or to a self-adjoint operator. Using the Krein-Langer technique of definitizable operators in Krein spaces, Ćurgus and Langer [6] have obtained the first result in the direction. In particular, their result yields that the J-self adjoint operator with $r(x) = \mathrm{sgn}x$ is similar to a selfadjoint if L is a uniformly positive operator. Next Ćurgus and Najman [7] showed that the operator $\frac{(\mathrm{sgn}x)d^2}{dx^2}$ is similar to a self-adjoint one. In the paper [8], similarity of $\mathrm{sgn}x\left(-\frac{d^2}{dx^2} + c\delta\right)$ type operators to normal and self-adjoint operators were described. In [4, 15, 13] several necessary similarity conditions in terms of Weyl functions were obtained. Based on the concept of boundary triplet and the resolvent similarity criterion, references [16] and [20] investigate the main spectral properties of quasi-self-adjoint extensions of corresponding operator.

Here we are interested in more general indefinite differential expression of the form (1.1) and the main goal is the similarity of the operator A to a self-adjoint operator. In Section 2, we summarize necessary definitions and statements from the spectral theory of

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Sturm-Liouville operator A . In Section 3, we give boundary triplets for Sturm-Liouville operator. Finally, in Section 4, we show that a necessary condition for the operator A to be similar to a self-adjoint operator in Hilbert space.

Throughout the article we use the following notations: Let T be a linear operator in a Hilbert space H . In what follows dom(T), ker(T), ran(T) are the domain, kernel, range of T , respectively. We denote by ρ (T), $\sigma(T$) and $\sigma_p(T)$ the resolvent set, the spectrum and the point spectrum. $R_T(\lambda) \coloneqq (T-\lambda I)^{-1}$, $\lambda \in \rho$ (T) is the resolvent of T . We set $\mathbf{C}_\pm$:={ $\lambda \in \mathbf{C}$: Imλ >0}.

Preliminaries

Indefinite Sturm-Liouville Operators in $L_r^2(\mathbb{R})$

Consider the differential expression

$$a(y) = 1/r((-py^{\prime})^{\prime} + qy), \tag{2.1}$$

where p^{-1}, q , $r \in L_{loc}^1(\mathbb{R})$ are assumed to be real valued functions such that p >0 and $r \neq 0$ for a.e., $x \in \mathbf{R}$. Here we assume that the following condition holds:

There exist $a, b \in \mathbf{R}$, $a < b$, such that the restrictions $r_+ := r \upharpoonright (b,+\infty)$ and $r_- = r \upharpoonright (-\infty, a)$ satisfy $r_+(x) > 0$ for a.e., $x \in (b,+\infty)$ and $r_-(x) < 0$ for a.e., $x \in (-\infty, a)$.

By $L_r^2(\mathbb{R})$ we denote the Krein space of all equivalence of measurable functions f defined on $\mathbf{R}$ for which $\int_{-\infty}^{+\infty} |f(x)|^2 |r(x)|\, dx < +\infty$. The indefinite and definite inner products on $L_r^2(\mathbf{R})$ are

$$[f,g] \coloneqq \int_{-\infty}^{+\infty} f\bar{g} r\, dx \quad \text{and} \quad (f,g) \coloneqq \int_{-\infty}^{+\infty} f\bar{g}|r|\, dx . \tag{2.2}$$

Evidently, the operator J

$$(Jf)(x) = (\operatorname{sgn} r(x)) f(x), \ x \in \mathbf{R} \tag{2.3}$$

is the fundamental symmetry connecting the inner products in (2.2). By the space $L_{|r|}^2(\mathbb{R})$ we denote the Hilbert space $L_{|r|}^2 (\mathbb{R}, (\cdot,\cdot))$.

Let us assume that the Sturm-Liouville differential expression

$$l(y) := 1/(|r|)((-py^{\prime})^{\prime} + qy) \tag{2.4}$$

is in the limit point case at both singular endpoints $-\infty$ and $+\infty$. Then it is well known that the operator $L(y) = l(y)$ defined on the usual maximal domain

$$D_{\max} = \{y \in L_{|r|}^2(\mathbb{R}) : y, py' \in AC_{loc}(\mathbb{R}), l(y) \in L_r^2(\mathbb{R})\} \tag{2.5}$$

is self-adjoint in the Hilbert space $L_r^2(\mathbf{R})$. In the following we set

$$A(y) := JLy = 1/r((-py^{\prime})^{\prime} + qy),\ \mathbf{dom}(A) = \mathbf{dom}(JL) = D_{\max}.$$

The operator A is self-adjoint in the Krein space $L_r^2(\mathbb{R})$. Under some additional assumptions a first result on the spectral structure of L was proved in [12] with the help of a general perturbation result from [7]. Under semi-bounded from below of the operator L , reference [18] gave the conclusion that the local definitizability in some open neighborhood of ∞ of the operator $A = JB$.

Similarity to [12], we shall interpret the operator A as a finite rank perturbation in resolvent sense of the direct sum of three differential operators A_-, A_{ab} and A_+ defined in the sequel. We identify function $f \in L_r^2(\mathbf{R})$ with $f = f_- + f_{ab} + f_+$, where $f_- \in L_{r_-}^2((-\infty,a))$, $f_{ab} \in L_{r_{ab}}^2((a,b))$ and $f_+ \in L_{r_+}^2((b,+\infty))$ respectively. Similarly we denote the restrictions of p , q and

r onto the intervals $(-\infty, a)$ and $(b, +\infty)$ by p_-, p_+, q_-, q_+ and r_-, r_+, respectively. Moreover we denote the restriction of r, p and q onto the interval (a, b) by r_{ab}, p_{ab} and q_{ab}. Besides the differential expression l in (2.4) we set

$$l_- f_- := 1/r_- \left((p_- f_-')' - q_- f_- \right),$$

$$l_+ f_+ := 1/r_+ \left((-p_+ f_+')' + q_+ f_+ \right)$$

and

$$l_{ab} f_{ab} := 1/r_{ab} \left((-p_{ab} f_{ab}')' + q_{ab} f_{ab} \right)$$

respectively, and operators associated to them. Note that l_- and l_+ are in the limit point case at $-\infty$ and $+\infty$ and regular at the endpoints a and b, respectively, whereas l_{ab} is regular at both endpoints a and b. By $D_{\max}^-$ ($D_{\max}^+$ and $D_{\max}^{ab}$) we denote the set in (2.5) if r, $\mathbb{R}$ and l are replaced by r_-, $(-\infty, a)$ and l_- (resp. r_+, $(b, +\infty)$, l_+ and r_{ab}, (a, b), l_{ab}). Therefore the operators

$$A_{\min -} f_- := 1/r_- \left((p_- f_-')' - q_- f_- \right),$$

$$A_{\min +} f_+ := 1/r_+ \left((-p_+ f_+')' + q_+ f_+ \right)$$

and

$$S_{ab} f_{ab} := 1/r_{ab} \left((-p_{ab} f_{ab}')' + q_{ab} f_{ab} \right)$$

defined on

$$\mathbf{dom}(A_{\min -}) = \{ f_- \in D_{\max}^- : f_-(a) = p_- f_-'(a) = 0 \},$$

$$\mathbf{dom}(A_{\min +}) = \{ f_+ \in D_{\max}^+ : f_+(b) = p_+ f_+'(b) = 0 \}$$

and

$$\mathbf{dom}(S_{ab}) = \{ f_{ab} \in D_{\max}^{ab} : f_{ab}(a) = p_{ab} f_{ab}'(a) = f_{ab}(b) = p_{ab} f_{ab}'(b) = 0 \},$$

with

$$D_{\max}^- = \{ f_- \in L^2_{|r_-|}((-\infty, a)) : f_-, p_- f_-' \in AC_{loc}((-\infty, a)), l f_- \in L^2_{|r_-|}((-\infty, a)) \},$$

$$D_{\max}^+ = \{ f_+ \in L^2_{r_+}((b, +\infty)) : f_+, p_+ f_+' \in AC_{loc}((b, +\infty)), l f_+ \in L^2_{r_+}((b, +\infty)) \}$$

and

$$D_{\max}^{ab} = \{ f_{ab} \in L^2_{|r_{ab}|}((a, b)) : f_{ab}, p_{ab} f_{ab}' \in AC_{loc}((a, b)), l f_{ab} \in L^2_{|r_{ab}|}((a, b)) \}$$

are closed symmetric operators in the anti-Hilbert space $L^2_{r_-}((-\infty, a))$, Hilbert space $L^2_{r_+}((b, +\infty))$ and Krein space $L^2_{r_{ab}}((a, b))$, respectively. The adjoint operators $A^*_{\min -}$, $A^*_{\min +}$ and S^*_{ab} are the usual maximal operators defined on $D_{\max}^-$, $D_{\max}^+$ and $D_{\max}^{ab}$, respectively.

Let $\mathbf{dom}(S) = \mathbf{dom}(A)_{\min -} \oplus \mathbf{dom}(S)_{ab} \oplus \mathbf{dom}(A)_{\min +}$ and let the operator S be defined on $\mathbf{dom}(S)$,

$$S = \begin{pmatrix} A_{\min -} & 0 & 0 \\ 0 & S_{ab} & 0 \\ 0 & 0 & A_{\min +} \end{pmatrix} \tag{2.6}$$

with respect to the Krein space $L^2_{r_-}((-\infty, a)) \oplus L^2_{r_{ab}}((a, b)) \oplus L^2_{r_+}((b, +\infty))$. Then S is a closed symmetric operator in the Krein space $L^2_r(\mathbb{R})$ with finite defect 4. Moreover, we have

$$S = A \restriction_{\mathrm{dom}(S)}, \quad A = S^* \restriction_D,$$

where

$$D = \mathbf{dom}(A) = \{ f \in \mathrm{dom}(A_{\min -})^* \oplus \mathrm{dom}(S_{ab})^* \oplus \mathrm{dom}(A_{\min +})^* :$$

$$f_-(a) = f_{ab}(a), \; p_- f_-'(a) = p_{ab} f_{ab}'(a), \; f_+(b) = f_{ab}(b), \; p_+ f_+'(b) = p_{ab} f_{ab}'(b) \}. \tag{2.7}$$

Theorem 2.1. If the operator L is semibounded from below, then $\rho(A) \neq \emptyset$. (see Th. 4.5 in [18])

If the operator L is nonnegative, $L \neq 0$, Theorem 2.1 together with spectral properties of self-adjojnt operators in Krein spaces implies the following theorem:

Theorem 2.2. If the operator L is nonnegative, then the spectrum of A is real, $\sigma(A) \subset \mathbf{R}$.

Weyl-Titchmarsh m-coefficients

Let $c(x,\lambda)$ and $s(x,\lambda)$ denote the linearly independent solutions of equation (2.4) satisfying the following initial conditions at a

$c(a,\lambda) = ps'(a,\lambda) = 1$, $pc'(a,\lambda) = s(a,\lambda) = 0$

Since equation (2.4) is limit point at $+\infty$, the Weyl-Titchmarsh theorem (see [9]) states that there exists a unique holomorphic function $m_+(\cdot) \in \mathbb{C}\setminus\mathbb{R} \to \mathbf{C}$, such that the function $s_+(x,\lambda) - m_+(\lambda)c_+(x,\lambda)$ belongs to $L^2_{r_+}((b,+\infty))$. Similarly, the limit point case at $-\infty$ yields the fact that there exists a unique holomorphic function $m_-(\cdot) \in \mathbb{C}\setminus\mathbb{R} \to \mathbf{C}$, such that $s_-(x,\lambda) - m_-(\lambda)c_-(x,\lambda)$ belongs to $L^2_{-r_-}((-\infty,a))$.

The functions m_+ and m_- are called the Weyl-Titchmarsh m-coefficients for (2.4) on $(b,+\infty)$ and on $(-\infty,a)$, respectively. We put

$$M_\pm(\lambda) \coloneqq \pm m_\pm(\pm\lambda), \qquad (2.8)$$

$$\psi_+(x,\lambda) = s_+(x,\lambda) - M_+(\lambda)c_+(x,\lambda), [\psi_-(x,\lambda) = -(s]_-(x,\lambda) - M_-(\lambda)c_+(x,\lambda)). \qquad (2.9)$$

By the definition of $m_\pm$, the functions $\psi_+(x,\lambda)$ and $\psi_-(x,\lambda)$ belong to $L^2_{r_+}((b,+\infty))$ and $L^2_{-r_-}((-\infty,a))$ for all $\lambda \in \mathbb{C}\setminus\mathbf{R}$, respectively. The function $M_+(\cdot)$ ($M_-(\cdot)$) is said to be the Weyl-Titchmarsh m-coefficient for equation (2.1) on $(b,+\infty)$ (on $(-\infty,a)$).

Definition 2.1. The class (R) consists of all holomorphic functions $G:\mathbf{C}_+ \cup \mathbf{C}_- \to \mathbf{C}$ such that $G(\bar\lambda) = \overline{G(\lambda)}$, and $\mathrm{Im}\lambda \cdot \mathrm{Im}G(\lambda) \geq 0$ for $\lambda \in \mathbf{C}_+ \cup \mathbf{C}_-$ (see [10]).

It is well known that

$$\int_b^{+\infty} [[\square|\psi]_+(x,\lambda)|\square]^2 r(x)\,dx = \frac{\mathrm{Im}M_+(\lambda)}{\mathrm{Im}\lambda}, \int_{-\infty}^{a} [[\square|\psi]_-(x,\lambda)|\square]^2 r(x)\,dx = \frac{\mathrm{Im}M_-(\lambda)}{\mathrm{Im}\lambda} \qquad (2.10)$$

for all $\lambda \in \mathbb{C}\setminus\mathbf{R}$ (see [9]). These formulae imply that the functions M_+ and M_- (as well as m_+ and m_-) belong to the class (R). Moreover (see [11,12]) the functions M_+ and M_- admit the following integral representation

$$M_\pm(\lambda) = \int_{-\infty}^{+\infty} \frac{d\tau_\pm(s)}{s-\lambda}, \in \mathbb{C}\setminus\mathbf{R}. \qquad (2.11)$$

Here $\tau_\pm : \mathbb{R} \to \mathbf{R}$ are nondecreasing functions on $\mathbf{R}$ with the following properties:

$$\int_{-\infty}^{+\infty} \frac{d\tau_\pm(s)}{1+|s|} < +\infty,$$

$\tau_+(b) = \tau_-(a) = 0$, $\tau_\pm(s) = \tau_\pm(s-0)$.

The functions τ_+ and τ_- are uniquely determined by the Stieltjes inversion formulae

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_b^s \mathrm{Im}M_+(t+i\varepsilon)\,dt = \frac{\tau_+(s+0)+\tau_+(s-0)}{2}, \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_a^s \mathrm{Im}M_-(t+i\varepsilon)\,dt = \frac{\tau_-(s+0)+\tau_-(s-0)}{2}.$$

The functions τ_+ and τ_- are called spectral functions of the operators

$$\square \quad A\square_\downarrow(0-) \coloneqq A_\downarrow(\min-)^\uparrow * |\{y \in \mathrm{dom}(A_\downarrow(\min-)^\uparrow *): p_\downarrow - \square y_\downarrow - \square^{\uparrow\prime}(a) = 0\} \qquad (2.12)$$

and

$$\square \quad A\square_\downarrow(0+) \coloneqq A_\downarrow(\min+)^\uparrow * |\{y \in \mathrm{dom}(A_\downarrow(\min+)^\uparrow *): p_\downarrow + \square y_\downarrow + \square^{\uparrow\prime}(b) = 0\}, \qquad (2.13)$$

respectively.

Boundary Triplets and Abstract Weyl Functions

Let $(H,[\cdot,\cdot])$ be a Krein space and let $\mathcal{H}$ be a separable Hilbert space. Let S be a closed symmetric operator in H with equal and finite deficiency indices $n_+(S) = n_-(S) = n < \infty$.

Recall the concepts of boundary triplets and abstract Weyl functions (see [13, 14]).

Definition 2.2. A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of an auxiliary Hilbert space $\mathcal{H}$ and linear mappings $\Gamma_j:\mathrm{dom}(S^*)\to\mathcal{H}$, $(j = 0,1)$, is called a boundary triplet for S^* if the following two conditions are satisfied:

(i) $(S^*f, g)_H - (f, S^*g)_H = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}$, $f, g \in \mathrm{dom}(S^*)$;

(ii) the linear mapping $\Gamma = \{\Gamma_0 f, \Gamma_1 f\}: \mathrm{dom}(S^*) \to \mathcal{H} \oplus \mathcal{H}$ is surjective.

The mappings Γ_0 and Γ_1 naturally induce two extensions S_0 and S_1 of S given by

$$S_j := S^* \restriction_{\mathrm{dom}(S_j)}, \ \mathrm{dom}(S_j) = \mathrm{Ker}\Gamma_j, (j = 0,1).$$

It turns out that S_0 and S_1 are self-adjoint operators in H, $S_j^* = S_j$, $(j = 0,1)$.

The $\gamma - field$ of the operator S corresponding to the boundary triplet Π is the operator function $\gamma(\cdot):\rho(S_0) \to [\mathcal{H}, \mathfrak{N}_\lambda(S)]$ defined by $\gamma(\lambda) := (\Gamma_0 \restriction \mathfrak{N}_\lambda(S))^{-1}$, where $\mathfrak{N}_\lambda(S) := \mathrm{Ker}(S^* - \lambda I)$. The function γ is well-defined and holomorphic on $\rho(S_0)$.

Definition 2.3. Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the operator S^*. The operator valued function $M(\cdot):\rho(S_0) \to [\mathcal{H}]$ defined by

$$M(\lambda) := \Gamma_1 \gamma(\lambda), \ \lambda \in \rho(S_0)$$

is called the Weyl function of S corresponding to the boundary triplet Π.

Let $C, D \in [H]$. Considering the followingextension $\tilde{S}$ of S, $S \subset \tilde{S}$,

$$\tilde{S} = S_1(C, D) := S^* \restriction \mathrm{dom}(S_1(C, D)),$$

$$\mathrm{dom}(S_1(C, D)) = \{f \in \mathrm{dom}(S^*): C\Gamma_1 f + D\Gamma_0 f = 0\}. \quad (2.14)$$

Notice that each proper extension $\tilde{S}$ of S has the form (2.14), i.e., if $S \subset \tilde{S} \subset S^*$, then there exist $C, D \in [H]$ such that $\tilde{S} = S_{C,D}$.

Theorem 2.3. ([22]) Suppose $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the operator S^*, $M(\cdot)$ is the corresponding Weyl function, and $\tilde{S} = S_{C,D}$, where $S_{C,D}$ is defined by (2.14). Assume also that $C, D, (CC^* + DD^*)^{-1} \in [\mathcal{H}]$. Then:

(i) $\lambda \in \rho(S_0) \cap \rho(\tilde{S})$ if and only if $0 \in \rho(D + CM(\lambda))$.

(ii) For each $\lambda \in \rho(S_0) \cap \rho(\tilde{S})$ the following equality holds true

$$(\tilde{S} - \lambda)^{-1} = (S_0 - \lambda)^{-1} - \gamma(\lambda)\big(D + CM(\lambda)\big)^{-1} C\gamma^*(\bar{\lambda}). \quad (2.15)$$

Boundary triplets for Sturm-Liouville operator A

1. Let $A_{\min+}$ and $A_{\min-}$ be the operators defined in Subsection 2.1. Since equation (1.1) is in the limit point case at $+\infty$ and $-\infty$, then the deficiency indices of the symmetric operator are (1,1) and for all $f_\pm, g_\pm \in \mathrm{dom}(A^*_{\min\pm})$ we have

$$(A^*_{\min+} f_+, g_+) - (f_+, A^*_{\min+} g_+) = (p_+ f_+{}')(b)\overline{g_+(b)} - f_+(b)\overline{(p_+ g_+{}')(b)}, \quad (3.1)$$

$$(A^*_{\min-} f_-, g_-) - (f_-, A^*_{\min-} g_-) = (p_- f_-{}')(a)\overline{g_-(a)} - f_-(a)\overline{(p_- g_-{}')(a)}. \quad (3.2)$$

Hence the triplets $\Pi^+ = \{\mathbb{C}, \Gamma_0^+, \Gamma_1^+\}$ and $\Pi^- = \{\mathbb{C}, \Gamma_0^-, \Gamma_1^-\}$, where

$$\Gamma_0^+ f_+ := (p_+ f_+{}')(b), \ \Gamma_1^+ f_+ := -f_+(b), \ f_+ \in \mathrm{dom}(A^*_{\min+}),$$

$$\Gamma_0^- f_- := (p_- f_-{}')(a), \ \Gamma_1^- f_- := -f_-(a), \ f_- \in \mathrm{dom}(A^*_{\min-}),$$

are the boundary triplets for $A^*_{\min+}$ and $A^*_{\min-}$, respectively. By the definition of the functions $\psi_+(\cdot,\lambda)$ and $\psi_-(\cdot,\lambda)$ (see Subsection 2.2), we obtain

$$\mathfrak{N}_\lambda(A_{\min\pm}) := \mathrm{Ker}(A^*_{\min\pm} - \lambda) = \{c\psi_\pm(\cdot,\lambda), c \in \mathbb{C}\}, \ \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (3.3)$$

Denote by γ^+ and γ^- the $\gamma - fields$ corresponding to the boundary triplets Π^+ and Π^-. By (2.9) and (3.3), we get

$$\gamma^{\pm}(\lambda)c = \left(\Gamma_0^{\pm} \upharpoonright \mathfrak{N}_\lambda(A_{\min\pm})\right)^{-1} c = c \cdot \psi_{\pm}(x,\lambda), \; c \in \mathbb{C}, \quad \lambda \in \mathbb{C} \setminus \mathbf{R} . \tag{3.4}$$

Further, the self-adjoint extension $A_{\min\pm}^{*} \upharpoonright \mathrm{Ker}(\Gamma_0^{\pm})$ of $A_{\min\pm}$ coincides with the operator $A_{0\pm}$. The Weyl function $\widetilde{M}_{\pm}(\cdot)$ of $A_{\min\pm}$ corresponding to the boundary triplets $\Pi^{\pm}$ is defined by

$$\widetilde{M}_{\pm}(\lambda) \coloneqq \Gamma_1^{\pm}\gamma^{\pm}(\lambda), \; \lambda \in \rho(A_{0\pm}).$$

Combining (3.4) with (2.8) and (2.9), one obtains $\widetilde{M}_{\pm}(\lambda)c = cM_{\pm}(\lambda)$, $c \in \mathbb{C}$, $\lambda \in \mathbb{C} \setminus \mathbf{R}$. In the sequel we will write $M_{\pm}$ instead of $\widetilde{M}_{\pm}(\cdot)$.

2. Consider the regular Strum-Liouville operator s_{ab}, s_{ab} is a densely defined closed symmetric operator in the Krein space $L^2_{r_{ab}}((a,b))$ and has defect two, its adjoint S^{*}_{ab} is given by

$$s_{ab} f_{ab} \coloneqq 1/r_{ab}\left((-p_{ab} f_{ab}')' + q_{ab} f_{ab}\right), \; \mathrm{dom}(S^{*}_{ab}) = D_{\max,ab} .$$

For $f, g \in \mathrm{dom}(S^{*}_{ab})$, we have

$$(S^{*}_{ab}f, g) - (f, S^{*}_{ab}g) = \begin{pmatrix} f_{ab}(a)\overline{(p_{ab}g_{ab}')(a)} + (p_{ab}f_{ab}')(a)\overline{g_{ab}(a)} \\ f_{ab}(b)\overline{(p_{ab}g_{ab}')(b)} - (p_{ab}f_{ab}')(b)\overline{g_{ab}(b)} \end{pmatrix}.$$

Hence $\Pi^{ab} = \{\mathbf{C}, \Gamma_0^{ab}, \Gamma_1^{ab}\}$ is a boundary triplet for S^{*}_{ab}, where

$$\Gamma_0^{ab} f_{ab} \coloneqq \begin{pmatrix} -p_{ab} f_{ab}'(a) \\ p_{ab} f_{ab}'(b) \end{pmatrix}, \; \Gamma_1^{ab} \coloneqq \begin{pmatrix} f_{ab}(a) \\ f_{ab}(b) \end{pmatrix}.$$

Let $\varphi_\lambda, \psi_\lambda \in L^2_{r_{ab}}((a,b))$ be the fundamental solutions of $(-p_{ab}h')' + q_{ab}h = \lambda r_{ab}h$, $\lambda \in \mathbf{C}$, satisfying the initial conditions

$\varphi_\lambda(a) = 1$, $(p_{ab}\varphi_\lambda')(a) = 0$ and $\psi_\lambda(a) = 0$, $(p_{ab}\psi_\lambda')(a) = 1$.

Since

$$\mathfrak{N}_\lambda(S_{ab}) = \mathrm{Ker}(S^{*}_{ab} - \lambda) = \mathrm{sp}\{\varphi_\lambda, \psi_\lambda\}, \quad \lambda \in \mathbf{C} \setminus \mathbf{R} . \tag{3.5}$$

Denote by γ^{ab} the $\gamma - field$ corresponding to the boundary triplets Π^{ab}. By (3.5), we get

$$\gamma^{ab} c = (\Gamma_0^{ab} \upharpoonright \mathfrak{N}_\lambda(s_{ab}))^{-1} c = \mathrm{sp}\{\varphi_\lambda, \psi_\lambda\} .$$

Furthermore

$$x \mapsto \varphi_\lambda(x)(p_{ab}\psi_\lambda')(x) - (p_{ab}\varphi_\lambda')(x)\psi_\lambda(x) = 1$$

has the constant value 1, we find that the Weyl function m_{ab} (see [12]) is given by

$$m_{ab}(\lambda) = \frac{1}{(p_{ab}\varphi_\lambda')(b)} = \begin{pmatrix} (p_{ab}\psi_\lambda')(b) & 1 \\ 1 & \varphi_\lambda(b) \end{pmatrix}, \lambda \in \mathbb{C} \setminus \mathbf{R} .$$

3. The operator $S = A_{\min-} \oplus S_{ab} \oplus A_{\min+}$ is a closed densely defined symmetric operator of defect 4 in the Krein space $L^2_{r_-}((-\infty,a)) \oplus L^2_{r_{ab}}((a,b)) \oplus L^2_{r_+}((b,+\infty))$ and it is straightforward to check that $\{\mathbf{C}^4, \Gamma_0, \Gamma_1\}$, where

$$\Gamma_0 f \coloneqq \begin{pmatrix} \Gamma_0^- f_- \\ \Gamma_0^+ f_+ \\ \Gamma_0^{ab} f_{ab} \end{pmatrix}, \; \Gamma_1 f \coloneqq \begin{pmatrix} \Gamma_1^- f_- \\ \Gamma_1^+ f_+ \\ \Gamma_1^{ab} f_{ab} \end{pmatrix}, \tag{3.6}$$

$\{f_-, f_+, f_{ab}\} \in \mathrm{dom}(A^{*}_{\min-}) \oplus \mathrm{dom}(A^{*}_{\min+}) \oplus \mathrm{dom}(S^{*}_{ab})$ is a boundary triple for the adjoint operator $A^{*}_{\min-} \oplus S^{*}_{ab} \oplus A^{*}_{\min+}$.

Further, we put

$$S_0 \coloneqq S^{*} \upharpoonright \mathrm{Ker}(\Gamma_0) = A_{0-} \oplus A_{0+} \oplus A_{0ab}, \tag{3.7}$$

where

$A_{0ab} := S_{ab}^{*} \upharpoonright \{y \in \text{dom}(S_{ab}^{*}) : (p_{ab}\, y_{ab}')(a) = (p_{ab}\, y_{ab}')(b) = 0\}$.

Therefore, the operator function $\gamma(\cdot): \rho(S_0) \to [\mathbb{C}^4, \mathfrak{N}_\lambda(S)]$ defined by

$$\gamma(\cdot)\begin{pmatrix} c_- \\ c_+ \\ c_{ab} \end{pmatrix} := \gamma^+(\lambda)c_+ + \gamma^-(\lambda)c_- + \gamma^{ab}(\lambda)c_{ab} = c_+\psi_+ + c_-\psi_- + c_1\varphi_\lambda + c_2\psi_\lambda$$

is the $\gamma - field$ corresponding to the boundary triplet $\Pi = \{\mathbb{C}^4, \Gamma_0, \Gamma_1\}$. Moreover, the operator Weyl function (see [13]) has the following form

$M(\lambda) = (M_-(\lambda) \; 0 \; 0 \; M_+(\lambda)) \quad (0\; 0) \quad (0\; 0) \quad (0 \; 0 \; 0 \; 0) \quad ((p_{ab}\,\psi_\lambda')(b)/(p_{ab}\,\varphi_\lambda')(b) \; 1/(p$ [illegible]

$, \lambda \in \rho(S_0)$.

Lemma 3.1. Let A be the operator associated with equation (2.1) and let the operator S_0 be defined by (3.7). Then $\sigma(A) \cap \rho(S_0) = \{\lambda \in \rho(S_0) : \Delta = 0\}$, where

$$\Delta = \left(p_{ab}\psi_\lambda'\right)(b)M_+(\lambda) - \varphi_\lambda(b)M_-(\lambda) - \left(p_{ab}\varphi_\lambda'\right)(b)M_-(\lambda)M_+(\lambda) + \psi_\lambda(b).$$

Proof. Let us rewrite (2.6) as follows $\text{dom}(A) = \{f \in \text{dom}(S^*) : C\Gamma_1 f + D\Gamma_0 f = 0\}$, where

$$C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

By Theorem 2.3, $\lambda \in \rho(A) \cap \rho(S_0)$ if and only if $0 \in \rho(D + CM(\lambda)$. Since

$$\det\left(D + CM(\lambda)\right) = \begin{vmatrix} \dfrac{(p_{ab}\psi_\lambda')(b)}{(p_{ab}\varphi_\lambda')(b)} - M_-(\lambda) & \dfrac{1}{(p_{ab}\varphi_\lambda')(b)} \\ \dfrac{1}{(p_{ab}\varphi_\lambda')(b)} & \dfrac{\varphi_\lambda(b)}{(p_{ab}\varphi_\lambda')(b)} + M_+(\lambda) \end{vmatrix}.$$

We see that $\lambda \in \rho(A) \cap \rho(S_0)$ exactly when $\Delta \neq 0$.

Similarity of the operator A

Theorem 4.1. Let A be the operator associated with equation (2.1). Then

(i) If $\lambda \in \mathbb{R}$, then $\lambda \in \sigma(A)$ if and only if $\Delta = 0$.

(ii) If the operator A_{0+} is semibounded from below and the operator A_{0-} is semibounded from above, then $M_+(\lambda) \neq M_-(\lambda)$.

Proof. Statement (i) obviously follows from Lemma 3.1 and the fact that $\rho(S_0) \subset \mathbb{R}$.

Let us prove (ii) The operator A_{0+} and A_{0-} are semibounded, i.e., $A_{0+} \geq \eta_0 I$ and $-\eta_0 I$, $\eta_0 \in \mathbb{R}$. Therefore, there exists $\eta_1 \in (-\infty, \eta_0]$ such that $\sigma(A_{0+}) \subset [\eta_1, +\infty)$ and $\sigma(A_{0-}) \subset (-\infty, -\eta_1]$. On the other hand, the operators $A_{0\pm}$ are unbounded. These facts imply $\sigma(A_{0+}) \neq \sigma(A_{0-})$.

Since $\sigma\left(A_{0\pm}\right) = \text{supp}\, d\tau_\pm$, one immediately gets $\text{supp}\, d\tau_+ \neq \text{supp}\, d\tau_-$. By the stieltjes inversion formula (2.10) we conclude that $M_+(\lambda) \neq M_-(\lambda)$ on $\mathbb{C} \setminus \mathbb{R}$.

Lemma 4.2. Let T be a closed operator in a Hilbert space H and $\sigma(T) \subset \mathbb{R}$. If T is similar to a self-adjoint operator, then there exists a positive constant $C > 0$ such that

$$|\text{Im}\lambda| \cdot \left\|(T-\lambda)^{-1}\right\|_H \leq C \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R} \}. \tag{4.1}$$

Theorem 4.3. If A is similar to a self-adjoint operator, then the functions

$$\frac{\text{Im}M_+(\lambda)}{\Delta} \text{ and } \frac{\text{Im}M_-(\lambda)}{\Delta} \tag{4.2}$$

are well defined and bounded on $\mathbb{C} \setminus \mathbb{R}$.

Proof. Suppose that A is similar to a self-adjoint operator. Then $\sigma(A) \subset \mathbb{R}$. By Lemma 3.1,

$\Delta \neq 0$ for all $\lambda \in \mathbf{R}$. Hence the functions (4.2) are well defined.

Further, by Lemma 4.2, there exists a positive constant $C > 0$ such that

$$|\mathrm{Im}\lambda| \cdot \|(A-\lambda)^{-1}\|_H \leq C \text{ for all } \lambda \in \mathbb{C}\setminus\mathbf{R} \}. \tag{4.3}$$

Since the operator $A_0 = A_0^*$ is self-adjoint, then

$$|\mathrm{Im}\lambda| \cdot \|(A_0-\lambda)^{-1}\|_H \leq 1 \text{ for all } \lambda \in \mathbb{C}\setminus\mathbf{R} \}. \tag{4.4}$$

Combining this inequality with (4.3), we get

$$|\mathrm{Im}\lambda| \cdot \left\|(A-\lambda)^{-1} - (A_0-\lambda)^{-1}\right\|_H \leq C+1, \lambda \in \mathbb{C}\setminus\mathbf{R}. \tag{4.5}$$

Substituting $f(\cdot) = \psi_\pm\left(\cdot, \overline{\lambda}\right)$ in (3.8), we obtain from (4.5) the following inequality

$$|\mathrm{Im}\lambda| \cdot \frac{\|\psi_+\|(|L|\,\|\psi_+\| + \|\psi_-\| + |L|\,\|\psi_\lambda\| + \|\varphi_\lambda\|)}{|\Delta|} \leq 2(C+1), \lambda \in \mathbf{C}\setminus\mathbf{R}$$

and

$$|\mathrm{Im}\lambda| \cdot \frac{\|\psi_-\|(|L|\,\|\psi_+\| + \|\psi_-\| + |L|\,\|\psi_\lambda\| + \|\varphi_\lambda\|)}{|L\Delta|} + \frac{|K|}{|L|}\|\psi_-\|(\|\psi_-\| + \|\varphi_\lambda\|) \leq 2(C+1), \lambda \in \mathbb{C}\setminus\mathbf{R},$$

where

$K = (p_{ab}\varphi_\lambda')(b)$, $L = (p_{ab}\psi_\lambda')(b)M_+(\lambda) - \varphi_\lambda(b)M_-(\lambda)$.

Therefore, using (2.9), one immediately gets

$$\frac{\sqrt{(|ImM_+(\lambda)|)(|L|\sqrt{|ImM_+(\lambda)|} + \sqrt{|ImM_-(\lambda)|})}}{|\Delta|} \leq 2(C+1), \lambda \in \mathbf{C}\setminus\mathbf{R},$$

and

$$\frac{\sqrt{(|\mathrm{Im}M_-(\lambda)|)(|L|\sqrt{|\mathrm{Im}M_+(\lambda)|} + \sqrt{|\mathrm{Im}M_-(\lambda)|})}}{|L\Delta|} \leq 2(C+1), \lambda \in \mathbf{C}\setminus\mathbf{R}.$$

Thus, for $\lambda \in \mathbb{C}\setminus\mathbf{R}$, we have

$$\frac{|L||\mathrm{Im}M_+(\lambda)|}{|\Delta|} \leq 2(C+1)$$

and

$$\frac{|\mathrm{Im}M_-(\lambda)|}{|L\Delta|} \leq 2(C+1).$$

Here $L \neq 0$, then

$$\frac{|\mathrm{Im}M_+(\lambda)|}{|\Delta|} \leq \frac{2(C+1)}{|L|}, \quad \frac{|\mathrm{Im}M_-(\lambda)|}{|\Delta|} \leq 2|L|(C+1).$$

This concludes the proof of Theorem 4.3.

Corollary 4.4. Let $a = b$, if A is similar to a self-adjoint operator in Hilbert space, then the functions

$$\frac{\mathrm{Im}M_+(\lambda)}{M_+(\lambda)-M_-(\lambda)} \text{ and } \frac{\mathrm{Im}M_-(\lambda)}{M_+(\lambda)-M_-(\lambda)}$$

are well defined and bounded on $\mathbb{C}\setminus\mathbf{R}$.

Some examples

The main object of this subsection is to present several explicit examples of indefinite Sturm-Liou ville operator of the form (2.1) with the singular critical point.

1. Consider the following operator

$$(A_1 y)(x) = \frac{1}{r(x)(-y'')}, \quad \mathrm{dom}(A_1) = L^2_{|r|}(\mathbf{R}), \tag{5.1}$$

where

$$r(x) = \begin{cases} \dfrac{\mathrm{sgn}(x+1)}{(1-3x)^{-\frac{4}{3}}}, x \le -1, \\ \mathrm{sgn}(x), -1 < x < 1 \\ \dfrac{\mathrm{sgn}(x-1)}{(1+3x)^{-\frac{4}{3}}}, x \ge 1. \end{cases}$$

Lemma 5.1. The differential equation

$$(-y'')(x) = \lambda(1+3x)^{-\frac{4}{3}} y(x), \quad x > 1 \tag{5.2}$$

is in the limit point case at $+\infty$. Moreover, the function

$$m(\lambda) = \frac{-\frac{1}{\lambda} + \frac{1}{\sqrt{-\lambda}} - \frac{2}{\pi}\frac{1}{\sqrt{-\lambda}}\arctan 1}{\sqrt{-\lambda}}, \quad \lambda \in \mathbf{R}_+ \tag{5.3}$$

is the Weyl-Titchmarsh m-coefficient for (5.2).

By Lemma 5.1, we obviously obtain

$$M_+(\lambda) = \frac{-\frac{1}{\lambda} + \frac{1}{\sqrt{-\lambda}} - \frac{2}{\pi}\frac{1}{\sqrt{-\lambda}}\arctan 1}{\sqrt{-\lambda}}, \quad \lambda \in \mathbb{C} \setminus \mathbf{R} \tag{5.4}$$

and

$$M_-(\lambda) = \frac{-\frac{1}{\lambda} - \frac{1}{\sqrt{\lambda}} + \frac{2}{\pi}\frac{1}{\sqrt{\lambda}}\arctan 1}{\sqrt{\lambda}}, \quad \lambda \in \mathbb{C} \setminus \mathbf{R}. \tag{5.5}$$

Lemma 5.2. If A_1 is similar to a self-adjoint operator in Hilbert space, then

$$\lim_{\varepsilon \to +0} \frac{\mathrm{Im}M_+(i\varepsilon)}{\mathrm{Re}M_+(i\varepsilon) + 1} < +\infty.$$

Proof. $\lim_{\varepsilon \to 0} \varphi_{i\varepsilon}(x) = 1$ and $\lim_{\varepsilon \to 0} \psi_{i\varepsilon}(x) = x + 1$. So $\lim_{\varepsilon \to 0} \varphi_{i\varepsilon}(1) = 1$, $\lim_{\varepsilon \to 0} \varphi'_{i\varepsilon}(1) = 0$, $\lim_{\varepsilon \to 0} \psi_{i\varepsilon}(1) = 2$ and $\lim_{\varepsilon \to 0} \psi'_{i\varepsilon}(1) = 1$. By Theorem 4.3, if A_1 is similar to a self-adjoint operator in Hilbert space then the limit functions

$$\lim_{\varepsilon \to +0} \frac{\mathrm{Im}M_+(i\varepsilon)}{M_+(i\varepsilon) - M_-(i\varepsilon) + 2} < \infty \quad \text{and} \quad \lim_{\varepsilon \to +0} \frac{\mathrm{Im}M_+(i\varepsilon)}{M_+(i\varepsilon) - M_-(i\varepsilon) + 2} < \infty. \tag{5.6}$$

Since the functions $r(\cdot)$ and $p(\cdot)$ are even, one can easily show that $m_-(\lambda) = m_+(\lambda)$. It follows from $M_\pm(\lambda) = \pm m_\pm(\pm\lambda)$ that $M_-(\lambda) = -M_+(-\lambda)$. Moreover

$M_+(i\varepsilon) - M_-(i\varepsilon) = M_+(i\varepsilon) + M_+(-i\varepsilon) = M_+(i\varepsilon) + \overline{M_+(i\varepsilon)}, \varepsilon > 0$.

Combining the inequality (5.6) and Theorem 4.3, we complete the proof.

Theorem 5.3. Let A_1 be the operator of the form (5.1). Then

(i) The spectrum of A_1 is real, $\sigma(A_1) \subset \mathbf{R}$.

(ii) A_1 is not similar to a self-adjoint operator.

Proof. (i) By Lemma 5.1, the differential expression (5.1) is in the limit point case at both $+\infty$ and $-\infty$. Hence the operator A_1 is self-adjoint in Krein space $L^2_r(\mathbb{R})$. Evidently, the operator L is nonnegative. It follows from Theorem 2.2 that the spectrum of A_1 is real, $\sigma(A_1) \subset \mathbf{R}$.

To prove (ii) we use Lemma 5.2.

Simple calculation show that

$$\mathrm{Im}M_{+}(i\varepsilon)=\frac{1}{\varepsilon}+\frac{1}{\sqrt{2\varepsilon}}+\frac{1}{(\pi\sqrt{2\varepsilon})\left(\ln\frac{\sqrt{\varepsilon^2+1}}{\varepsilon+\sqrt{2\varepsilon}+1}+\frac{\arctan\sqrt{2\varepsilon}}{\varepsilon-1}\right)},$$

$$\mathrm{Re}M_{+}(i\varepsilon)=\frac{1}{\sqrt{2\varepsilon}}+\frac{1}{(\pi\sqrt{2\varepsilon})\left(\ln\frac{\sqrt{\varepsilon^2+1}}{\varepsilon+\sqrt{2\varepsilon}+1}-\frac{\arctan\sqrt{2\varepsilon}}{\varepsilon-1}\right)}$$

and

$$\frac{\mathrm{Im}M_{+}(i\varepsilon)}{\mathrm{Re}M_{+}(i\varepsilon)+1}=\frac{\frac{1}{\varepsilon}+\frac{1}{\sqrt{2\varepsilon}}+\frac{1}{(\pi\sqrt{2\varepsilon})\left(\ln\frac{\sqrt{\varepsilon^2+1}}{\varepsilon+\sqrt{2\varepsilon}+1}+\frac{\arctan\sqrt{2\varepsilon}}{\varepsilon-1}\right)}}{\frac{1}{\sqrt{2\varepsilon}}+\frac{1}{(\pi\sqrt{2\varepsilon})\left(\ln\frac{\sqrt{\varepsilon^2+1}}{\varepsilon+\sqrt{2\varepsilon}+1}-\frac{\arctan\sqrt{2\varepsilon}}{\varepsilon-1}\right)}}$$

$$=\frac{1}{\sqrt{\varepsilon}}\left(\frac{\sqrt{2}+\sqrt{\varepsilon}+\frac{1}{\pi}\left(\ln\frac{\sqrt{\varepsilon^2+1}}{\varepsilon+\sqrt{2\varepsilon}+1}+\frac{\arctan\sqrt{2\varepsilon}}{\varepsilon-1}\right)}{1+\frac{1}{\pi}\left(\ln\frac{\sqrt{\varepsilon^2+1}}{\varepsilon+\sqrt{2\varepsilon}+1}-\frac{\arctan\sqrt{2\varepsilon}}{\varepsilon-1}\right)}\right)\to+\infty,\varepsilon\to+0.$$

Thus A_1 is not similar to a self-adjoint operator.

2. Consider the following operator

$$(A_2y)(x)=\frac{1}{r(x)(-y'')},\ \mathrm{dom}(A_2)=L^2_{|r|}(\mathbb{R}), \tag{5.7}$$

where

$$r(x)=\begin{cases}\mathrm{sgn}(x+1), & x\le-1,\\ \mathrm{sgn}x(x^2+2x-4), & -1<x<1,\\ \mathrm{sgn}(x-1), & x\ge 1.\end{cases}$$

Using the method of WKB (see[21]), we can get uniformly valid asymptotic solutions of the equation (5.7), i.e.,

$$\varphi_1\lambda(x)\sim\{([r(-1)]^{1/4}\cos\sqrt{\lambda}\int_{-1}^{x}\sqrt{r(\tau)}\,d\tau)/[r(x)]^{1/4},\qquad -1\le x\le 0,|\lambda|\to\infty, \tag{5.8}$$

(5.9)

satisfying the initial conditions:

$\varphi_\lambda(-1)=1$, $\varphi'_\lambda(-1)=0$ and $\psi_\lambda(-1)=1$, $\psi'_\lambda(-1)=0$,

here

$$k_1=\frac{[r(-1)]^{\frac14}\cos\sqrt{\lambda}\int_{-1}^{0}\sqrt{r(\tau)}d\tau}{\left[\exp\left(\sqrt{\lambda}\int_0^1\sqrt{-r(\tau)}d\tau\right)+\exp\left(-\sqrt{\lambda}\int_0^1\sqrt{-r(\tau)}d\tau\right)\right]},$$

$$k_2=\frac{\sin\sqrt{\lambda}\int_{-1}^{0}\sqrt{r(\tau)}d\tau}{\sqrt{\lambda}[r(-1)]^{\frac14}\left[\exp\left(\sqrt{\lambda}\int_0^1\sqrt{-r(\tau)}d\tau\right)+\exp\left(-\sqrt{\lambda}\int_0^1\sqrt{-r(\tau)}d\tau\right)\right]}.$$

Then it is easy to obtain that

$\varphi_\lambda(1)=2k_1$, $\varphi'_\lambda(1)=-2k_1$ and $\psi_\lambda(1)=2k_2$, $\psi'_\lambda(1)=-2k_2$.

Lemma 5.4. The differential equation

$$-y''(x)=\lambda\,\mathrm{sgn}(x-1)y(x),x>1 \tag{5.10}$$

is in the limit point case at $+\infty$. Moreover, the function

$$m(\lambda)=\frac{\frac{1}{\sqrt{-\lambda}}-\frac{2}{\pi}\frac{1}{\sqrt{-\lambda}}\arctan 1}{\sqrt{-\lambda}},\ \lambda\in \mathbf{R}_+ \qquad (5.11)$$

is the Weyl-Titchmarsh m-coefficient for (5.10).

By Lemma 5.4, we obviously obtain

$$M_+(\lambda)=\frac{\frac{1}{\sqrt{-\lambda}}-\frac{2}{\pi}\frac{1}{\sqrt{-\lambda}}\arctan 1}{\sqrt{-\lambda}},\ \lambda\in \mathbb{C}\setminus\mathbf{R} \qquad (5.12)$$

and

$$M_-(\lambda)=\frac{-\frac{1}{\sqrt{\lambda}}+\frac{2}{\pi}\frac{1}{\sqrt{\lambda}}\arctan 1}{\sqrt{\lambda}},\ \lambda\in \mathbb{C}\setminus\mathbf{R}\,. \qquad (5.13)$$

Theorem 5.5. Let A_2 be the operator of the form (5.7). Then

(i) The spectrum of A_2 is real, $\sigma(A_2)\subset\mathbf{R}$;

(ii) A_2 is not similar to a self-adjoint operator.

Proof. (i) It is similar to the proof of Theorem 5.3 (i).

To prove (ii) we use Theorem 4.3.

Simple calculation show that

$$\mathrm{Im}M_+(i\varepsilon)=\frac{1}{\sqrt{2\varepsilon}}+\frac{1}{(\pi\sqrt{2\varepsilon})\left(\ln\frac{\sqrt{\varepsilon^2+1}}{\varepsilon+\sqrt{2\varepsilon}+1}+\frac{\arctan\sqrt{2\varepsilon}}{\varepsilon-1}\right)},$$

$$\mathrm{Re}M_+(i\varepsilon)=\frac{1}{\sqrt{2\varepsilon}}-\frac{1}{(\pi\sqrt{2\varepsilon})\left(\ln\frac{\sqrt{\varepsilon^2+1}}{\varepsilon+\sqrt{2\varepsilon}+1}-\frac{\arctan\sqrt{2\varepsilon}}{\varepsilon-1}\right)}$$

and

$$\Delta=-2k_2M_+(\lambda)-2k_1M_-(\lambda)+2k_1M_-(\lambda)M_+(\lambda)+2k_2\,.$$

So

$$\frac{\mathrm{Im}M_+(i\varepsilon)}{\Delta}=\frac{\mathrm{Im}M_+(i\varepsilon)}{-2k_2M_+(i\varepsilon)-2k_1M_-(i\varepsilon)+2k_1M_-(i\varepsilon)M_+(i\varepsilon)+2k_2}$$

$$=\left[\exp\left(\sqrt{i\varepsilon}\int_0^{-1}\sqrt{-r(\tau)}d\tau\right)+\exp\left(-\sqrt{i\varepsilon}\int_0^{-1}\sqrt{-r(\tau)}d\tau\right)\right]f(i\varepsilon),$$

and we can easily get that $f(i\varepsilon)=O(1)$ as $\varepsilon\to+\infty$. From this, it follows that

$$\frac{\mathrm{Im}M_+(i\varepsilon)}{\Delta}=\left[\exp\left(\sqrt{i\varepsilon}\int_0^{-1}\sqrt{-r(\tau)}d\tau\right)+\exp\left(-\sqrt{i\varepsilon}\int_0^{-1}\sqrt{-r(\tau)}d\tau\right)\right]O(1)\to+\infty,\ \varepsilon\to+\infty\,.$$

Thus A_2 is not similar to a self-adjoint operator.

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