The $\exp(-\varphi(\xi))$-expansion method for exact solutions to the nonlinear KdV equation and the (2+1) dimensional Zakharov-Kuznetsov (ZK) equations

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ABSTRACT

The $\exp(-\varphi(\xi))$-expansion method is predominant and useful mathematical tool for constructing exact traveling wave solutions to nonlinear evolution equations interweaved in mathematical physics, engineering, and physical sciences. In this article, we investigate solutions to the Korteweg-de Vries (KdV) equation and the (2+1)-dimensional Zakharov-Kuznetsov (ZK) equations through the $\exp(-\varphi(\xi))$-expansion method. Abundant exact traveling wave solutions with arbitrary parameters have successfully been obtained by this method which are articulated in terms of trigonometric, hyperbolic, and rational functions.

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Introduction

Diverse real world problems in nonlinear science associated with mechanical, structural, aeronautical, ocean, electrical, and control systems can be modeled and control by nonlinear evolution equations (NLEEs). Therefore, the studies of nonlinear differential equations have been a vigorous field of research for the past few decades. One of the central issues for these models is to obtain their traveling wave solutions. The traveling wave solutions of NLEEs play a momentous and very important role to comprehend the internal mechanism of nonlinear phenomena. Therefore, the inquisitiveness of searching traveling wave solutions of NLEEs is increasing day after day and has been a hot issue to the researchers. It is significant to observe that, all sorts of NLEEs cannot be solved by a unique method. Due to this reason, a lot of techniques have been successfully developed by different group of mathematicians, scientists, and engineers. For instance, the Backlund transformation [1], the Jacobi elliptic function method [2], the tanh-function method [3, 4], the auxiliary equation method [5], the trial function method [6], the variational iteration method [7-9], the homogenous balance method [10, 11], the Hirota's bilinear transformation method [12], the homotopy perturbation method [13-15], the inverse scattering method [16], the Miura transformation [17], the Exp-function method [18-20], the F-expansion method [21], the sine-cosine method [22, 23], the truncated Painleve expansion method [24], the generalized Riccati equation [25], the asymptotic method [26], the non-perturbative method [27], the (G'/G)-expansion method [28-34], the extended F-expansion Method [35], the Weierstrass elliptic function method [36], etc.

The $\exp(-\varphi(\xi))$-expansion method is promising for investigating solitary wave solutions of nonlinear differential equations [37]. In this article, we implement the method to the KdV equation and the (2+1) dimensional Zakharov-Kuznetsov (ZK) equations. The main idea of this method is the traveling wave solutions of a nonlinear evolution equation can be presented by a polynomial in $\exp(-\varphi(\xi))$, where $\varphi(\xi)$ satisfies the ordinary differential equation (ODE):

$$\varphi'(\xi) = \exp(-\varphi(\xi)) + \mu \exp(\varphi(\xi)) + \lambda, \qquad (1)$$

The degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms and the coefficients of the polynomial can be obtained by solving a set of simultaneous algebraic equations.

The KdV equation and the (2+1)-dimensional Zakharov-Kuznetsov (ZK) equations have been studied by using different methods. Such as, Zhang [38] employed Exp-function method to investigate KdV equation for constructing exact solutions, Khalfallah [39] implemented homogeneous balance method for obtaining exact travelling wave solutions of the (2+1) dimensional Zakharov-Kuznetsov (ZK) equation.

The aim of this article is to find exact solutions of the KdV equation and the (2+1) dimensional Zakharov-Kuznetsov (ZK) equations using through the $\exp(-\varphi(\xi))$ expansion method.

This article is arranged as follows. In section 2, we give the basic idea of this method. In section 3, we apply this method to solve the KdV equation and (2+1) dimensional Zakharov-Kuznetsov (ZK) equations. Finally in section 4, we have drawn our conclusions.

Basic Idea of the $\exp(-\varphi(\xi))$**-expansion Method**

Suppose the nonlinear partial differential equation for $u(x,t)$ is in the form

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0, \tag{2}$$

were P is the polynomial in $u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In order to determine $u(x,t)$ explicitly by the $\exp(-\varphi(\xi))$ expansion method, we have to carry out the following steps:

Step1: We introduce a traveling wave variable ξ combining the real variable x and t as follows:

$$u(\xi) = u(x,t), \quad \xi = x - Vt, \tag{3}$$

where V is the speed of the travelling wave. Using the traveling wave variable (3), equation (2) changes to an ODE for $u = u(\xi)$:

$$Q(u, u', u'', u''', \ldots) = 0, \tag{4}$$

where Q is a function of $u(\xi)$ and its derivatives.

Step 2: Suppose the solution of (4) can be expressed by a polynomial in $\exp(-\varphi(\xi))$:

$$u(\xi) = \alpha_n \left(\exp(-\varphi(\xi)) \right)^n + \alpha_{n-1} \left(\exp(-\varphi(\xi)) \right)^{n-1} + \Lambda \tag{5}$$

where $\alpha_n, \alpha_{n-1}, \Lambda$ and V are constants to be determined later such that $\alpha_n \neq 0$ and $\varphi(\xi)$ satisfies (1). The unwritten part in (5) is also a polynomial in $\exp(-\varphi(\xi))$.

Step 3: The positive integer n can be determined by considering the homogeneous balance between the highest order linear terms and nonlinear terms of the highest order appearing in equation (4). Our solutions of (1) depend on the parameters involved.

When $\lambda^2 - 4\mu > 0, \ \mu \neq 0,$

$$\varphi(\xi) = \ln \left\{ \frac{1}{2\mu} \left(-\sqrt{\lambda^2 - 4\mu} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + c_1) \right) - \lambda \right) \right\}. \tag{6}$$

When $\lambda^2 - 4\mu < 0, \ \mu \neq 0,$

$$\varphi(\xi) = \ln \left\{ \frac{1}{2\mu} \left(-\lambda + \sqrt{4\mu - \lambda^2} \tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + c_1) \right) \right) \right\}. \tag{7}$$

When $\lambda^2 - 4\mu = 0, \ \mu \neq 0,$ and $\lambda \neq 0,$

$$\varphi(\xi) = \ln \left\{ -\frac{2(\lambda(\xi + c_1) + 2)}{\lambda^2(\xi + c_1)} \right\}. \tag{8}$$

When $\mu = 0, \ \lambda \neq 0$,

$$\varphi(\xi) = -\ln\left\{\frac{\lambda}{\exp(\lambda(\xi + c_1)) - 1}\right\} \cdot \tag{9}$$

When $\mu = 0, \ \lambda = 0$,

$$\varphi(\xi) = \ln(\xi + c_1) \cdot \tag{10}$$

Step 4: Substituting (5) in (4) and using (1), the left hand side of (4) is converted into a polynomial in $\exp(-\varphi(\xi))$. Equating each coefficient of this polynomial to zero, we obtain a set of nonlinear algebraic equations for $\alpha_n, \Lambda, V, \lambda$ and μ.

Step 5: Solving the algebraic equations obtained in Step 4 with the aid of a computer algebra system, we obtain the values of the constants $\alpha_n, \Lambda, V, \lambda$ and μ. The solutions of (1) together with the values of α_n, Λ, V; constitute traveling wave solutions for the nonlinear evolution equation (2).

Applications of the Method

In this section, we will put forth the $\exp(-\varphi(\xi))$-expansion method to construct traveling wave solutions of the KdV equation and the (2+1) dimensional Zakharov-Kuznetsov (ZK) equations.

The KdV Equation

Let us consider the KdV equation

$$u_t + uu_x + \beta u_{xxx} = 0 \cdot \tag{11}$$

Using the traveling wave variable $\xi = x - Vt$, (11) is converted into the following ODE for $u = u(\xi)$:

$$-Vu' + uu' + \beta u''' = 0 \cdot \tag{12}$$

Integrating (12) with respect to ξ once, we obtain

$$C - Vu + \frac{u^2}{2} + \beta u'' = 0 \tag{13}$$

where the primes denote the derivatives with respect to ξ and C is an integration constant to be determined. Considering the homogeneous balance between the highest-order derivative u'' and the nonlinear term u^2, we obtain $n = 2$. Therefore, the solution of (13) is given by

$$u = \alpha_2(\exp(-\varphi(\xi)))^2 + \alpha_1 \exp(-\varphi(\xi)) + \alpha_0, \tag{14}$$

where $\alpha_2 \neq 0 \cdot \alpha_0$ and α_1 are constants to be determined.

By using (1), from (14), we obtain

$$\begin{aligned}u'' &= 6\alpha_2(\exp(-\varphi(\xi)))^4 + (2\alpha_1 + 10\alpha_2\lambda)(\exp(-\varphi(\xi)))^3 \\ &\quad + (4\alpha_2\lambda^2 + 8\alpha_2\mu + 3\alpha_1\lambda)(\exp(-\varphi(\xi)))^2 \\ &\quad + (2\alpha_1\mu + 6\alpha_2\mu\lambda + \alpha_1\lambda^2)(\exp(-\varphi(\xi))) + 2\alpha_2\mu^2 + \alpha_1\lambda\mu.\end{aligned} \tag{15}$$

$$\begin{aligned}u^2 &= \alpha_2^2(\exp(-\varphi(\xi)))^4 + 2\alpha_1\alpha_2(\exp(-\varphi(\xi)))^3 \\ &\quad + (2\alpha_0\alpha_2 + \alpha_1^2)(\exp(-\varphi(\xi)))^2 + 2\alpha_1\alpha_0(-\varphi(\xi)) + \alpha_0^2.\end{aligned} \tag{16}$$

Substituting (14)-(16) into (13) and collecting all terms with the same power of $\exp(-\varphi(\xi))$ together, the left hand is transformed to a polynomial in $\exp(-\varphi(\xi))$. Equating the coefficients of this polynomial to zero, we obtain an over-determined set of algebraic equations for $\alpha_1, \ \alpha_0, \ \lambda, \ \mu, \ C$ and V as follows:

$$\frac{1}{2}\alpha_2^2 + 6\beta\alpha_2 = 0,$$

$$\alpha_1\alpha_2 + 2\beta\alpha_1 + 10\beta\alpha_2\lambda = 0,$$

$$4\beta\alpha_2\lambda^2 + 8\beta\alpha_2\mu + 3\beta\alpha_1\lambda - V\alpha_2 + \frac{1}{2}\alpha_1^2 + \alpha_0\alpha_2 = 0,$$

$$\alpha_1\alpha_0 - V\alpha_1 + \beta\alpha_1\lambda^2 + 2\beta\alpha_1\mu + 6\beta\alpha_2\lambda\mu = 0,$$

$$C + 2\beta\alpha_2\mu^2 + \beta\alpha_1\lambda\mu - V\alpha_0 + \frac{1}{2}\alpha_0^2 = 0.$$

Solving the set of simultaneous algebraic equations by using the symbolic computation systems, such as Maple 13, we obtain the following solutions:

$$C = 24\beta^2\mu^2 + 12\beta^2\lambda^2\mu + \frac{1}{2}\alpha_0^2 + 8\mu\beta\alpha_0 + \lambda^2\beta\alpha_0, \ \alpha_0 = \alpha_0, \ \alpha_1 = -12\beta\lambda, \ \alpha_2 = -12\beta \qquad (17)$$

where λ and μ are arbitrary constants.

By using (17) into 14, we obtain

$$u = -12\beta\exp(-\varphi(\xi))^2 - 12\beta\lambda\exp(-\varphi(\xi)) + \alpha_0, \qquad (18)$$

where $\xi = x - (\alpha_0 + 8\mu\beta + \beta\lambda^2)t$.

Now making use of solutions (6)-(10) into (18), we obtain the following traveling wave solutions of the KdV equation (11):

Type 1: When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$, we obtain hyperbolic function solution

$$u_1 = -48\beta\mu^2\left\{\lambda + \sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + c_1)\right)\right\}^{-2}$$

$$+ 24\beta\lambda\mu\left\{\lambda + \sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + c_1)\right)\right\}^{-1} + \alpha_0,$$

where $\xi = x - (\alpha_0 + 8\mu\beta + \beta\lambda^2)t$, c_1 is an arbitrary constant.

Type 2: When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$, we obtain trigonometric function solution

$$u_2 = -48\beta\mu^2\left\{\lambda - \sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + c_1)\right)\right\}^{-2}$$

$$+ 24\beta\lambda\mu\left\{\lambda - \sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + c_1)\right)\right\}^{-1} + \alpha_0.$$

Type 3: When $\lambda^2 - 4\mu = 0$ but $\mu \neq 0, \lambda \neq 0$, we obtain rational function solution

$$u_3 = -\frac{3\beta\lambda^4(\xi + c_1)^2}{\{\lambda(\xi + c_1) + 2\}^2} + \frac{6\beta\lambda^3(\xi + c_1)}{\{\lambda(\xi + c_1) + 2\}} + \alpha_0.$$

Type 4: When $\mu = 0$ and $\lambda \neq 0$, we obtain exponential function solution

$$u_4 = -\frac{12\beta\lambda^2\exp\{\lambda(\xi + c_1)\}}{\{1 - \exp(\lambda(\xi + c_1))\}^2} + \alpha_0.$$

Type 5: When $\mu = 0$ and $\lambda = 0$, we obtain rational function solution

$$u_5 = -12\beta\left(\xi + c_1\right)^{-2} + \alpha_0,$$

where $\xi = x - (\alpha_0 + 8\mu\beta + \beta\lambda^2)t$, c_1 is an arbitrary constant.

(2+1) dimensional Zakharov-Kuznetsov (ZK) equation

Now we would like to construct the traveling wave solutions for the (2+1) dimensional Zakharov-Kuznetsov (ZK) equation by the proposed method. Let us consider the (2+1) dimensional Zakharov-Kuznetsov (ZK) equation

$$u_t + a(u^2)_x + (bu_{xx} + Ku_{yy})_x = 0. \tag{19}$$

Utilizing the wave transformation $\xi = x + y - Vt$, (19) is converted into the following ODE for $u = u(\xi)$:

$$-Vu' + a(u^2)' + (b+K)u''' = 0. \tag{20}$$

Equation (20) is integrable, therefore integrating once with respect to ξ, we obtain

$$C - Vu + au^2 + (b+K)u'' = 0, \tag{21}$$

where prime denotes derivatives with respect to ξ and C is an integration constant.

Balancing the highest order linear term u'' and the nonlinear term of highest order u^2 in (21), we obtain $n = 2$. Therefore, the solution of (21) is given by

$$u = \alpha_2(\exp(-\varphi(\xi)))^2 + \alpha_1\exp(-\varphi(\xi)) + \alpha_0, \tag{22}$$

where $\alpha_2 \neq 0$. α_0 and α_1 are constants to be determined.

Using (1) from (22), we obtain

$$\begin{aligned}
u'' &= 6\alpha_2\left(\exp(-\varphi(\xi))\right)^4 + \left(2\alpha_1 + 10\alpha_2\lambda\right)\left(\exp(-\varphi(\xi))\right)^3 \\
&\quad + \left(4\alpha_2\lambda^2 + 8\alpha_2\mu + 3\alpha_1\lambda\right)\left(\exp(-\varphi(\xi))\right)^2 \\
&\quad + \left(2\alpha_1\mu + 6\alpha_2\mu\lambda + \alpha_1\lambda^2\right)\left(\exp(-\varphi(\xi))\right) + 2\alpha_2\mu^2 + \alpha_1\lambda\mu.
\end{aligned} \tag{23}$$

$$\begin{aligned}
u^2 &= \alpha_2^2\left(\exp(-\varphi(\xi))\right)^4 + 2\alpha_1\alpha_2\left(\exp(-\varphi(\xi))\right)^3 \\
&\quad + \left(2\alpha_0\alpha_2 + \alpha_1^2\right)\left(\exp(-\varphi(\xi))\right)^2 + 2\alpha_1\alpha_0(-\varphi(\xi)) + \alpha_0^2.
\end{aligned} \tag{24}$$

Substituting (22)-(24) into (21) and collecting all terms with the same power of $\exp(-\varphi(\xi))$ together, the left hand is converted into a polynomial in $\exp(-\varphi(\xi))$. Equating each coefficient of this polynomial to zero, we obtain an over-determined set algebraic equation for α_1, α_0, C and V as follows:

$$6K\alpha_2 + a\alpha_2^2 + 6b\alpha_2 = 0. \quad 2K\alpha_1 + 10K\alpha_2\lambda + 2a\alpha_1\alpha_2 + 2b\alpha_1 + 10b\alpha_2\lambda = 0.$$

$$\begin{aligned}
&3b\alpha_1\lambda + 8K\alpha_2\mu - V\alpha_2 + 4b\alpha_2\lambda^2 + 3K\alpha_1\lambda + 2a\alpha_0\alpha_2 \\
&+ 4K\alpha_2\lambda^2 + a\alpha_1^2 + 8b\alpha_2\mu = 0
\end{aligned}$$

$$\begin{aligned}
&2a\alpha_0\alpha_1 + 6K\alpha_2\lambda\mu - V\alpha_1 + 2b\alpha_1\mu + b\alpha_1\lambda^2 + 2K\alpha_1\mu. \\
&+ K\alpha_1\lambda^2 + 6b\alpha_2\lambda\mu = 0
\end{aligned}$$

Solving the resulting algebraic system, we get following solutions:

$$\begin{aligned}
C &= \frac{1}{a}(12\mu^2b^2 + 24\mu^2Kb + 12\mu K\lambda^2 b + 6\mu K^2\lambda^2 + \alpha_0^2a^2 + \lambda^2ba\alpha_0 \\
&\quad + 8ab\alpha_0\mu + 8\mu Ka\alpha_0 + a\alpha_0\lambda^2 K + 6\lambda^2b^2\mu + 12\mu^2K^2)
\end{aligned}$$

$$V = \lambda^2 b + 8\mu b + 8\mu K + 2a\alpha_0 + \lambda^2 K , \alpha_0 = \alpha_0 ,$$

$$\alpha_1 = -\frac{6(b+K)\lambda}{a} , \alpha_2 = -\frac{6(b+K)}{a} , \tag{25}$$

where λ and μ are arbitrary constants.

Substituting (25) into (22), we obtain

$$u = -\frac{6(b+K)}{a}(\exp(-\varphi(\xi)))^2 - \frac{6(b+K)\lambda}{a}\exp(-\varphi(\xi)) + \alpha_0 , \tag{26}$$

where $\xi = x - (\lambda^2 b + 8\mu b + 8\mu K + 2a\alpha_0 + \lambda^2 K)\, t$

By using (6)-(10), from (26) we obtain the following traveling wave solutions for the (2+1) dimensional Zakharov-Kuznetsov (ZK) equation.

Category 1: When $\lambda^2 - 4\mu > 0$, and $\mu \neq 0$, we obtain the following solution

$$u_1 = -\frac{24\mu^2(b+K)}{a}\left\{\lambda + \sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + c_1)\right)\right\}^{-2}$$

$$+ \frac{12\lambda\mu(b+K)}{a}\left\{\lambda + \sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + c_1)\right)\right\}^{-1} + \alpha_0 ,$$

where $\xi = x - (\lambda^2 b + 8\mu b + 8\mu K + 2a\alpha_0 + \lambda^2 K)\, t , c_1$ is an arbitrary constant.

Category 2: When $\lambda^2 - 4\mu < 0$, and $\mu \neq 0$, we obtain the following solution

$$u_1 = -\frac{24\mu^2(b+K)}{a}\left\{\lambda - \sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + c_1)\right)\right\}^{-2}$$

$$+ \frac{12\lambda\mu(b+K)}{a}\left\{\lambda - \sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + c_1)\right)\right\}^{-1} + \alpha_0 .$$

Category 3: When $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, and $\lambda \neq 0$, the following solution is obtained

$$u_3 = -\frac{3(b+K)\lambda^4(\xi + c_1)^2}{2a\{\lambda(\xi + c_1) + 2\}^2} + \frac{3(b+K)\lambda^3(\xi + c_1)}{a\{\lambda(\xi + c_1) + 2\}} + \alpha_0 .$$

Category 4: When $\mu = 0$ but $\lambda \neq 0$, the following solution is obtained

$$u_4 = -\frac{6(b+K)\lambda^2}{a}\frac{\exp(\lambda(\xi + c_1))}{\{1 - \exp(\lambda(\xi + c_1))\}^2} + \alpha_0 .$$

Category 5: When $\mu = 0$ and $\lambda = 0$, the following solution is obtained

$$u_5 = -\frac{6(b+K)}{a(\xi + c_1)^2} - \frac{6(b+K)\lambda}{a(\xi + c_1)} + \alpha_0 ,$$

where $\xi = x - (\lambda^2 b + 8\mu b + 8\mu K + 2a\alpha_0 + \lambda^2 K)\, t , c_1$ is an arbitrary constant.

Conclusions

In this article, we have successfully obtained exact traveling wave solutions for the Korteweg-de Vries equation and the (2+1) dimensional Zakharov-Kuznetsov (ZK) equation by means of the $\exp(-\varphi(\xi))$-expansion method. The obtained solutions are

presented through the hyperbolic, trigonometric, exponential, and rational functions. From this study, we observe that the performance of this method is useful, convincing, and reliable. The method is constructive, direct, and simple thanks to a computer algebra system. It is also adequate to find solitary wave solutions for NLEEs. The solutions we have achieved here exhibit the efficacy of the $\exp(-\varphi(\xi))$-expansion method. The method seems to be straightforward and applicable to many other nonlinear evolution equations.

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