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Semi smooth graceful graph and construction of new graceful trees

V J Kaneria[1,*] and M M Jariya[2]

[1]Department of Mathematics, Saurashtra University, RAJKOT – 360 005.

[2]V V P Engineering College, RAJKOT – 360 005, India.

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ABSTRACT

In this paper we paper we define smooth graceful labeling and semi smooth graceful labeling for a graph. We also prove that a grid graph P_nXP_m is smooth graceful graph and a star $K_{1,n}$ is semi smooth graceful graph. Using this we proved that star of a star, star of smooth graceful tree and path union of a smooth graceful tree are graceful trees. We also get graceful labeling for comb graph and simple lobster graph.

Introduction

The graceful labeling introduced by A. Rosa [7] in 1967. He proved that cycle C_n is graceful, when $n\equiv 0, 3 \pmod 4$. Bloom and Golomb [1] proved that the complete graph K_n is graceful, when $n\leq 4$. For detail survey of graph labeling one can refer Gallian [2]. Jin et al. [4] proved that join sum of graceful trees is graceful. Liu et al. [6] proved that radical product of graceful trees is graceful.

Kaneria and Jariya [5] define smooth graceful labeling for a graph and proved that cycle C_n ($n\equiv 0 \pmod 4$), path P_n and complete bipartite graph $K_{2,n}$ are smooth graceful graphs. Using this they also proved a graph obtained by C^+_m ($m\equiv 2 \pmod 4$), and C_n ($n\equiv 0 \pmod 4$) with a path of arbitrary length is graceful as well as a graph obtained by joining C_m ($m\equiv 0 \pmod 4$) and W_n with a path of arbitrary length is also graceful.

We begin with a simple undirected finite graph $G = (V,E)$ on p vertices and q edges. For all terminology and notations we follows Harary [3]. First of all we shall recall some definitions which are useful for this paper.

Definition – 1.1 : A function f is called *graceful labeling* of a graph $G = (V,E)$ if $f : V \longrightarrow \{0,1,2, \dots \dots , q\}$ is injective and the induce function $f^*: E \longrightarrow \{1,2, \dots , q\}$ defined as $f^*(e) = |f(u) – f(v)|$ is bijective for every edge $e=(u,v) \in E$. A graph G is called *graceful graph* if it admits a graceful labeling.

Definition – 1.2 : Let G be a graph and $G_1, G_2, \dots , G_n$, $n\geq 2$ be n copies of graph G. Then the graph obtained by adding an edge from G_i to G_{i+1} ($i=1,2, \dots , n–1$) is called *path union* of n copies of the graph G.

Definition – 1.3 : A graph obtained by replacing each vertex of star $K_{1,n}$ by a graph G of n vertices is called star of G and it is denoted by G^*. The graph G which replaced at the center of $K_{1,n}$ we call the central copy of G^*.

Definition – 1.4 : A bipartite graceful graph G with graceful labeling f is said to be *smooth graceful graph* if it admits an injective map $g : V \longrightarrow \{0,1, \dots , \lfloor\frac{(q-1)}{2}\rfloor, \lfloor\frac{(q+1)}{2}\rfloor+l, \dots , q+l\}$ such that its induce edge labeling map $g^*: E \longrightarrow \{1+l, 2+l, \dots , q+l\}$ defined as $g^*(e)=|g(u) – g(v)|$, for every edge $e = (u,v) \in E$, for every $l \in N$ is a bijection.

Definition – 1.5 : A bipartite graceful graph G with graceful labeling f is said to be *semi smooth graceful graph* if it admits an injective map $g : V \longrightarrow \{0,1, \dots , t –1, t+l,t+1+l, \dots , q+l\}$ such that its induce edge labeling map $g^*: E \longrightarrow \{1+l, 2+l, \dots , q+l\}$ defined as $g^*(e)=|g(u) – g(v)|$, for every edge $e = (u,v) \in E$, for every $t, l \in N$ is a bijective map.

Definition – 1.6 : A *comb tree* means a graph obtained by joining (t+1) copies of path P_t of length t with vertices of one more additional copy of P_t.

Definition – 1.7 : A *simple lobster* means a path union of t copies of P_4 in which we join these consecutive copies of P_4 by an edge at the middle vertex of P_4.

Main Results

Theorem – 2.1 : P_nXP_m is smooth graceful graph.

Proof : Let $v_{i,j}$ ($1 \leq i \leq n$, $1 \leq j \leq m$) be vertices of the grid graph P_nXP_m. We know that P_nXP_m is a bipartite graceful graph with a graceful labeling function $f : V(P_nXP_m) \longrightarrow \{0,1,2, \dots , q\}$, where $q= 2mn–(m+n)$.

Now we define $g : V(P_nXP_m) \longrightarrow \{0,1, \dots , \lfloor\frac{(q-1)}{2}\rfloor, \lfloor\frac{(q+1)}{2}\rfloor+l, \dots , q+l\}$ such that

$$g(u) = f(u), \text{ when } f(u)< \frac{q}{2}$$
$$= f(u) + l, \text{ when } f(u) \geq \frac{q}{2}$$

and its induce edge labeling function $g^* : E(P_nXP_m) \longrightarrow \{1+l, 2+l, \dots , q+l\}$ defined by

$g^*(e) = |g(u) – g(v)|$, for every edge $e = (u,v) \in E$.

Since for any $e = (u,v) \in E$ in P_nXP_m, one of f(u) and f(v) is less than $\frac{q}{2}$ and another is greater than or equal to $\frac{q}{2}$, we must have

$$g^*(e) = |g(u) – g(v)|$$
$$= |f(u) – f(v)| +l$$
$$= f^*(e)+l, \forall e \in E(P_nXP_m).$$

Therefore $g^*(E(P_nXP_m))= \{1+l, 2+l, \dots , q+l\}$, which gives g^* is bijection. Hence P_nXP_m is a smooth graceful graph.

Tel:

E-mail address: kaneria_vinodray_j@yahoo.co.in

Theorem – 2.2 **:** $K_{1,n}$ is a semi smooth graceful graph.

Proof : Let $v_0, v_1, \dots, v_n$ be vertices of $K_{1,n}$. We know that $K_{1,n}$ is a bipartite graceful graph with graceful labeling $f : V(K_{1,n}) \longrightarrow \{0,1,2, \dots, n\}$ defined by

$f(v_i) = i, \quad \forall\, i=0,1,2, \dots, n$

Now we define $g : V(K_{1,n}) \longrightarrow \{0, 1+l, 2+l, \dots, n+l\}$ such that its induce edge labeling map $g^* : E(K_{1,n}) \longrightarrow \{1+l, 2+l, \dots, n+l\}$ defined by

$g(u) = f(u)$, when $u = v_0$ and

$= f(u)+l$, when $u \in \{ v_1, v_2, \dots, v_n\}$

Now for any $e = (u,v) \in E(K_{1,n})$,

$$\begin{aligned} g^*(e) &= g^*((u,v)) \\ &= |g(u) - g(v)| \\ &= |g(v_0) - g(v_i)| \text{ , for some } i \in \{ 1, 2, \dots, n\} \\ &= |0 - (f(v_i)+l)| \\ &= |f(v_0) - f(v_i)| + l \\ &= f^*((v_0,v_i)) + l \\ &= f^*(e) + l . \end{aligned}$$

Therefore $g^*(E) = \{1+l, 2+l, \dots, n+l\}$ and so it is a bijective map.

Hence $K_{1,n}$ is a semi smooth graceful graph.

Theorem – 2.3 **:** Star of $K_{1,n}$ is a graceful tree.

Proof : Let G be a graph formed by star of $K_{1,n}$. i.e. G is a graph obtained by each vertices of $K_{1,n}$ by $K_{1,n}$ itself. Let $v_{0,0}, v_{0,1}, \dots, v_{0,n}$ be vertices of the central copy of $K^*_{1,n}$. Let $v_{i,0}, v_{i,1}, \dots, v_{i,n}$ be vertices of i^{th} copy $K_{1,n}$ of $K^*_{1,n}$, $\forall$ i=1, 2, … , n+1. We define $f : V(K^*_{1,n}) \longrightarrow \{0,1,2, \dots, q\}$ where $q=n^2+3n+1$ as follows.

$f(v_{0,0})= 0;$

$f(v_{0,i})= q+1-i, \quad \forall\, i=1, 2, \dots, n;$

$f(v_{1,0})= f(v_{0,n}) -1;$

$f(v_{1,i})= (n+1) -i, \quad \forall\, i=1, 2, \dots, n;$

$f(v_{l,i})= f(v_{l-2,i}) + (n+1), \text{ if } f(v_{l-2,i}) < \frac{q}{2}$

$= f(v_{l-2,i}) - (n+1), \text{ if } f(v_{l-2,i}) > \frac{q}{2}, \forall\, i=0,1, 2, \dots, n, \ \forall\, l=2, 3, \dots, n+1.$

Now we see that the difference of vertex labels for central copy $K^{(0)}_{1,n}$ with its other copies $K^{(i)}_{1,n}$ $(1 \le i \le n+1)$ is precisely following sequence.

$(n+1)^2, (n+1), n(n+1), 2(n+1), (n-1)(n+1), \dots, \lfloor \frac{n}{2} \rfloor (n+1).$

Now join each vertex of central copy $K^{(0)}_{1,n}$ with its other copies $K^{(i)}_{1,n}$ by an edge to corresponding vertices. Above defined labeling function f give rise graceful labeling to $K^*_{1,n}$ and so G is a graceful tree.

Illustration – 2.4 **:** Stat of $K_{1,5}$ and its graceful labeling shown in *figure* – 1

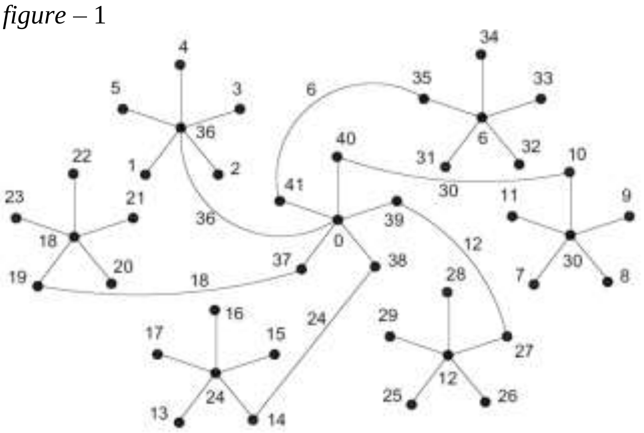

Figure 1. A tree obtained by star of star ($K_{1,n}$) and its graceful labeling

Theorem – 2.5 **:** Path union of a smooth graceful tree is also a graceful tree.

Proof : Let G be a path union of t copies of a smooth graceful tree T, where |V(T)|=p and |E(T)|=q. Let $f : V(T) \longrightarrow \{0,1,2, \dots, q\}$ be a smooth graceful labeling for T. Then its induce edge labeling function is $f^* : E(T) \longrightarrow \{1,2, \dots, q\}$, which is bijection. Let $V(T)= \{v_1, v_2, \dots, v_p\}$. Let $v_{i,1}, v_{i,2}, \dots, v_{i,p}$ be vertices of i^{th} copy of G, $\forall$ i=1, 2, … , t. Then we define $g : V(G) \longrightarrow \{0,1,2, \dots, Q\}$, where Q= tq+t – 1 as follows.

$g(v_{1,i}) = f(v_i), \qquad \text{when } f(v_i) < \lfloor \frac{q}{2} \rfloor$

$= (Q - q) + f(v_i), \qquad \text{when } f(v_i) \ge \lfloor \frac{q}{2} \rfloor, \ \forall\, i=1, 2, \dots, p;$

$g(v_{2,i}) = g(v_{1,i}) + (Q - q), \qquad \text{when } g(v_{1,i}) < \frac{Q}{2}$

$= g(v_{1,i}) - (Q - q), \qquad \text{when } g(v_{1,i}) > \frac{Q}{2}, \ \forall\, i=1, 2, \dots, p;$

$g(v_{l,i}) = g(v_{l-2,i}) + (q + 1), \qquad \text{when } g(v_{l-2,i}) < \frac{Q}{2}$

$= g(v_{l-2,i}) - (q + 1), \qquad \text{when } g(v_{l-2,i}) > \frac{Q}{2}, \ \forall\, i=1, 2, \dots, p, \ \forall\, l=3, 4, \dots, t.$

Now choose a vertex v of T and each corresponding vertices with v in each copy of $T^{(i)}$, join by an edge, $\forall$ i=1, 2, … , t. Then above labeling to the path union of t copies of a smooth graceful tree T is give rise graceful labeling to G. Thus G is a graceful tree.

Illustration – 2.6 **:** Path union of 5 copies of smooth graceful tree and its graceful labeling shown in *figure* – 2.

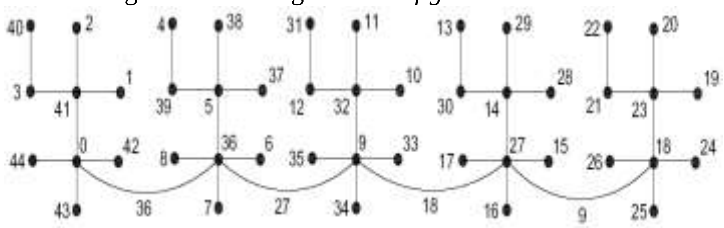

Figure 2. A tree obtained by path union of a tree T and its graceful labeling

Corollary – 2.7 **:** Every comb tree is a graceful tree.

Proof : We know that every path P_t on t + 1 vertices is a smooth graceful graph (Proved by Kaneria and Jariya [5]). Comb tree is the path union of t copies of P_t , so by last *Theorem – 2.5* the comb tree is a graceful tree.

Illustration – 2.8 **:** Comb tree formed by P_4 , a path on five vertices and its graceful labeling shown in *figure* – 3.

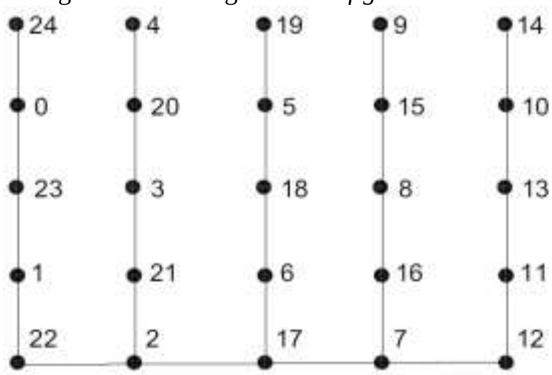

Figure 3. Comb tree obtained by path union of 6 copies of P_4 and its graceful labeling

Corollary – 2.9 **:** Every simple lobster is a graceful tree.

Proof : We know that every path P_4 on 5 vertices is a smooth graceful graph. A simple lobter is the path union of t copies of

P_4 join by the corresponding the middle vertex of each copy. So by last *Theorem – 2.5* a simple lobster is a graceful tree.

Illustration – 2.10 **:** A simple lobster of length 6 and its graceful labeling shown in *figure* – 4.

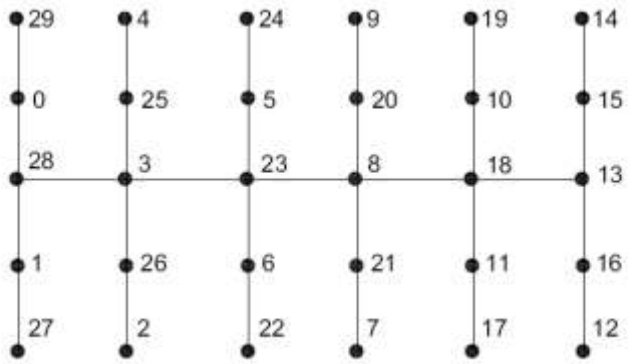

Figure 4. A graph of a simple lobster of six length and its graceful labeling

Theorem – 2.11 **:** Let T be smooth graceful tree. Then star of T is also a graceful tree.

Proof : Let G be a star of given smooth graceful tree T. Let $|V(T)|=p$, $|E(T)|=q$ and $V(T)= \{v_1 , v_2 , \dots , v_p\}$. Let $f : V(T) \longrightarrow \{0,1,2, \dots , q\}$ be a smooth graceful labeling for T. Let $u_{0,i}$ $(1 \leq i \leq p)$ be vertices of central copy $T^{(0)}$ of G and $u_{l,i}$ $(1 \leq i \leq p)$ be vertices of l^{th} copy of T in G, $\forall$ l=1, 2, … , p. We define a labeling function $g : V(G) \longrightarrow \{0,1,2, \dots , Q\}$, where $Q=(p+1)q+p$ as follows.

$g(u_{0,i}) = f(v_i)$, when $f(v_i) < \lfloor \frac{q}{2} \rfloor$

$= (Q – q) + f(v_i)$, when $f(v_i) \geq \lfloor \frac{q}{2} \rfloor$, $\forall$ i=1, 2, … , p;

$g(u_{1,i}) = g(u_{0,i}) + (Q – q)$, if $g(u_{0,i}) < \frac{Q}{2}$

$= g(u_{0,i}) – (Q – q)$, if $g(u_{0,i}) > \frac{Q}{2}$, $\forall$ i=1, 2, … , p;

$g(u_{l,i}) = g(u_{l-2,i}) + (q + 1)$, if $g(u_{l-2,i}) < \frac{Q}{2}$

$= g(u_{l-2,i}) – (q + 1)$, if $g(u_{l-2,i}) > \frac{Q}{2}$, $\forall$ i=1, 2, … , p, $\forall$ l=2, 3, 4, … , p.

Now we see that the difference of vertex labels for the central copy $T^{(0)}$ with its other copies $T^{(i)}$ $(1 \leq i \leq p)$ is precise the sequence

$p(q+1), (q+1), (p – 1)(q+1), 2(q+1), (p – 2)(q+1), \dots \dots , \lfloor \frac{q}{2} \rfloor (q+1)$.

Now join each vertex of central copy $T^{(0)}$ with it other copy $T^{(i)}$ by an edge to corresponding vertices. Above labeling function g give rise graceful labeling to G and so it is a graceful tree.

Illustration – 2.12 **:** A star of a smooth graceful tree and its graceful labeling shown in *figure* – 5.

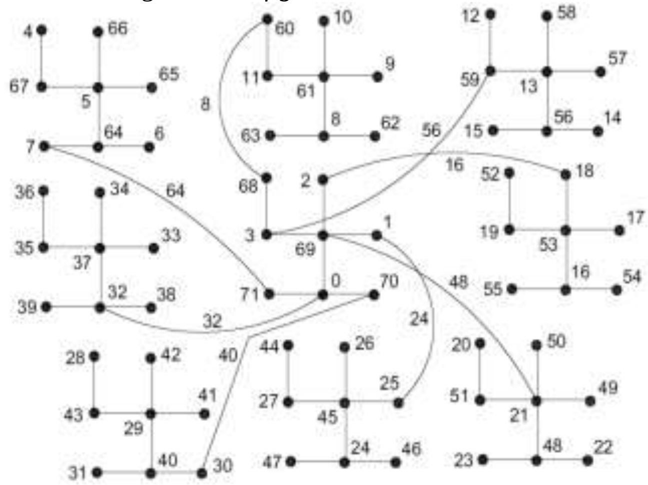

Figure 5. A star of a smooth graceful tree and its graceful labeling

Concluding Remarks

We have introduce a new graceful labeling namely semi smooth graceful labeling and proved that star $K_{1,n}$ $(n \in N)$ is a semi smooth graceful graph. Using this we have constructed some graceful trees by making star or path union of given graceful tree. Present work contribute few new results to the theory of graceful trees. The labeling pattern is demonstrated by suitable illustrations.

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