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Dynamics of an infinitesimal body with a decreasing mass in the restricted three body problem apart from the equilibrium points

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ABSTRACT

The motion of an infinitesimal body with variable mass is studied. The equations of motion of the restricted three body problem with infinitesimal body of variable mass are introduced. The Jeans' law of mass change is applied. To tackle the dynamical problem, the Hamiltonian of the problem is formed with the time as independent variable. The solution of the equations of motion is formulated based on Lie series developments. Numerical representation of coordinates and momenta are given.

Introduction

Gylden (1884) treated the two-body problem of variable masses. He wrote the differential equations of motion for the problem. Few years later the publishing of the pioneering work by Gylden, Mestschersky (1893) obtained the first integrable case to for a specific mass variation law. This mass variation law, and its following generalization (1902), are known as Mestschersky laws. After Mestschersky's contribution, the physical meaning of the problem became clear and it is known as Gylden-Mestschersky problem.

The binary systems enrich the problem via the mass and luminosity relation of the stars. There are brilliant names who contributes this problem very early, among them, Jeans (1924) studied the orbits of binary stars, found a more general mass variation law that was based on the relation between mass and luminosity of the stars presented by Eddington in the same year. Mestschersky's laws are special cases of Jeans' law. Gelfgat (1959) considered a different mass variation law. Berkovic (1981) investigated the problem using a differential equation transformation method. Also a number of approximate analytic solution were found, e.g., Prieto, Docobo (1997) and Lukyanov (2005) studied the particular problem where the total mass is constant, which can be applied to conservative mass transfer in close binary systems Lukyanov (2008).

The Gylden-Mestschersky problem can also be generalized to include the restricted three-body problem. In this case, it can be assumed that the two primaries have their motion determined by the Gylden-Mestschersky equations. Thus, one have to deal only with the motion of the third body, which does not affect the main bodies motion. It was shown Gelfgat (1973) that this problem presents particular solutions that are analogous to the stationary solutions of the classical problem of constant masses: the three collinear solutions L_1 to L_3 and the two triangular solutions L_4 and L_5. Since then, further characteristics of this problem have been studied, for example, Lukyanov (2009), Singh et al. (2010). Besides the Gylden-Mestschersky problem, there are many different cases of two-body problems with variable mass Razbitnaya (1985). These can be classified according to the presence or not of reactive forces, to the variation of the mass of just one or both of the bodies, to whether the bodies move in an inertial frame or not and so on.

The subject of the perturbed restricted three body problem is drawing scientists' attention since the last decades of the previous century. One of those important considered perturbations is the varying mass due to its application in comets dynamics that burn out while passing their perihelion points (Comet-Jupiter-Sun system), in addition to mass change problems arise for space vehicles (Vehicle-Moon-Earth system).

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Shrivastava and Ishwar (1983), Singh and Ishwar (1984, 1985), Das et al. (1988) and Singh (2008) formulated the restricted problem with variable mass using Jean's law (1924) in which the masses of the primaries are constant and for the third body is a function of time.

Our aim is to concern with the orbit of third body varying mass in the Hamiltonian framework, using a Lie series the equations of motion of the third body are integrated.

Problem Formulation

The classical circular restricted three body problem is defined as two massive primaries $m_2 = m_1$ which are restricted to move in a circular plane, and an infinitesimal body of mass m that is affected by the gravitational field of the two primaries. Also m has no effect on $m_1,\ m_2$. Consider the third body m as a variable mass, e.g. a comet and let the primaries are of constant masses in a barycentric frame ($OXYZ$) with angular velocity ω around Z -axis, then the equations of the third body motion are given by (Singh, 2008)

$$\left.\begin{aligned} \frac{\dot m}{m}\left(\dot x-\omega y\right)+\ddot x-2\omega\dot y &= -\frac{1}{m}\frac{\partial U}{\partial x} \\ \frac{\dot m}{m}\left(\dot y-\omega x\right)+\ddot y-2\omega\dot x &= -\frac{1}{m}\frac{\partial U}{\partial y} \end{aligned}\right\} \tag{1}$$

where m_1, m_2 and m are located at $(-a,0)$, $(b,0)$ and (x,y) respectively.

and

$$U = -\,Gm\left(\frac{m_1}{\rho_1}+\frac{m_2}{\rho_2}\right)-\frac{1}{2}m\omega^2\left(x^2+y^2\right)$$

with

$$\rho_1=\left[\left(x+a\right)^2+y^2\right]^{1/2} \quad \text{and} \quad \rho_2=\left[\left(x-b\right)^2+y^2\right]^{1/2}$$

are the distances from the masses m_1 and m_2 to m respectively and G is the gravitational constant.

Using Jeans' law to describe the mass change, (Jeans, 1924) for the star of the main sequence.

$$\frac{dm}{dt}=-\alpha m^n, \qquad 1.4\le n\le 4.4, \qquad \alpha\in\mathbb{R} \quad \text{(the real number)} \tag{2}$$

Introducing a space-time transformation $(x,y,t)\to(\xi,\eta,\Gamma)$ scaled by the ratio of the mass of the third body m at $t=t_1$ and m_0 at $t=0$, i.e. $\gamma=m/m_0$. This transformation preserves the dimensions of the space and time. Considering the values suggested by Shrivastava and Ishwar (1983) yields

$$x=\frac{\xi}{\sqrt{\gamma}} \quad y=\frac{\eta}{\sqrt{\gamma}}, \quad dt=\frac{d\Gamma}{\sqrt{\gamma}}, \quad \rho_1=\frac{r_1}{\sqrt{\gamma}}, \quad \rho_2=\frac{r_2}{\sqrt{\gamma}} \tag{3}$$

using Jeans' law, we have

$$\frac{d\gamma}{dt}=-\beta\gamma^n, \qquad \beta=\alpha m_0^{n-1} \tag{4}$$

So the system of equations (1) can be rewritten as

$$\left.\begin{aligned} \xi''-2\omega\eta' &= -\frac{1}{m_0}\frac{\partial U^*}{\partial\xi}+\frac{\beta^2}{4}\xi \\ \eta''+2\omega\xi' &= -\frac{1}{m_0}\frac{\partial U^*}{\partial\eta}+\frac{\beta^2}{4}\eta \end{aligned}\right\} \tag{5}$$

where

$$U^* = -Gm_0\gamma^{3/2}\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right)-\frac{1}{2}m_0\omega^2\left(\xi^2+\eta^2\right)$$

with

$$r_1=\left[\left(\xi+a\gamma^{1/2}\right)^2+\eta^2\right]^{1/2} \quad \text{and} \quad r_2=\left[\left(\xi-b\gamma^{1/2}\right)^2+\eta^2\right]^{1/2}$$

And the primes indicate differentiation with respect to Γ.

The system of equations (5) can be rewritten as, (Singh and Ishwar, 1984),

$$\xi''-2\eta''=\Omega_\xi, \qquad \eta''+2\xi'=\Omega_\eta \tag{6}$$

where

$$\Omega=\frac{1}{2}B\left(\xi^2+\eta^2\right)+\gamma^{3/2}\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right), \qquad B=\frac{\beta^2}{4}+1$$

$$r_1^2=\left(\xi+\mu\gamma^{1/2}\right)^2+\eta^2, \qquad r_2^2=\left(\xi-(1-\mu)\gamma^{1/2}\right)^2+\eta^2$$

Choosing the normalizing variables and units as

$$m_1=1-\mu,\quad m_2=\mu,\quad \mu=\frac{m_2}{m_1+m_2}\le\frac{1}{2},\quad G=1,\quad \omega=1,\quad a=\mu,\quad b=1-\mu$$

Singh (2008) formulated the lagrangian corresponding to system of equations (6) as

$$\mathrm{L}=\frac{1}{2}\left(\xi'^2+\eta'^2\right)+\left(\xi\eta'-\eta\xi'\right)+\frac{1}{2}B\left(\xi^2+\eta^2\right)+\gamma^{3/2}\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right) \tag{7}$$

and the corresponding Hamiltonian can be formulated as

$$\mathrm{H}=\frac{1}{2}\left(P_\xi^2+P_\eta^2\right)+\frac{1}{2}\left(\xi^2+\eta^2\right)+\left(\eta P_\xi-\xi P_\eta\right)-\frac{1}{2}B\left(\xi^2+\eta^2\right)-\gamma^{3/2}\left(\frac{1-\mu}{r_1}+\frac{\mu}{r_2}\right) \tag{8}$$

where P_ξ, P_η are the conjugate momentum corresponding to ξ, η respectively.

Perturbation Approach

A suitable differential operator D, the Lie operator, was introduced by Delva (1984) and Hanslmeier (1984) produces a convergent Lie series like Taylor series. Let the Hamiltonian $\mathrm{H}\left(q,P_q,t\right)$ be function q be the coordinates, P_q be the momenta, and t be the time. The equations of motion can be written as

$$\dot{q}=\frac{dq}{dt}=\frac{\partial \mathrm{H}}{\partial P_q}, \qquad \dot{P}_q=\frac{dP_q}{dt}=-\frac{\partial \mathrm{H}}{\partial q}$$

The linear Lie operator has the general form

$$D=\frac{dq}{dt}\frac{\partial}{\partial q}+\frac{dP_q}{dt}\frac{\partial}{\partial P_q}+\frac{\partial}{\partial t} \tag{9}$$

The solution $\vec{q}\left(q,P_q,t\right)$, $\vec{P}_q\left(q,P_q,t\right)$ can be computed using Lie operator as

$$\vec{q}\left(q,P_q,t\right)=\sum_{j=0}\left(D^j\vec{q}\right)_{\vec{q}_0}\frac{\left(t-t_0\right)^j}{j\,!}, \qquad \vec{P}_q\left(q,P_q,t\right)=\sum_{j=0}\left(D^j\vec{P}_q\right)_{\vec{P}_{q0}}\frac{\left(t-t_0\right)^j}{j\,!}$$

(10)

Where $D^j\vec{q}$, $D^j\vec{P}_q$ are evaluated at initial conditions $\vec{q}_0\left(q_0,\ P_{q_0},\ t_0\right)$ and $\vec{P}_{q_0}\left(q_0,\ P_{q_0},\ t_0\right)$.

The Equations of Motion

This section is aimed to find the equation of motion of third body of variable mass in the restricted three body problem. The analytical solution can be described in term of Lie operator, which utilizing the Hamiltonian for the motion in the neighborhood of a fixed points. The Hamiltonian equations can be written as

$$\dot{\xi} = \frac{\partial \mathrm{H}}{\partial P_\xi} = \eta + P_\xi \tag{11.1}$$

$$\dot{\eta} = \frac{\partial \mathrm{H}}{\partial P_\eta} = -\xi + P_\eta \tag{11.2}$$

$$\dot{P}_\xi = -\frac{\partial \mathrm{H}}{\partial \xi} = \xi\left(B-1\right) + P_\eta - \gamma^{3/2}\left(\frac{\left(1-\mu\right)\left(\mu\gamma^{1/2}+\xi\right)}{r_1^3} + \frac{\mu\left(-\gamma^{1/2}\left(1-\mu\right)+\xi\right)}{r_2^3}\right) \tag{11.3}$$

$$\dot{P}_\eta = -\frac{\partial \mathrm{H}}{\partial \eta} = \eta\left(B-1\right) - P_\xi - \gamma^{\frac{3}{2}}\eta\left(\frac{\left(1-\mu\right)}{r_1^3} + \frac{\mu}{r_2^3}\right) \tag{11.4}$$

Now Lie operator is constructed, using perturbation approach section, see equation (9), by

$$D = \frac{d\xi}{dt}\frac{\partial}{\partial \xi} + \frac{d\eta}{dt}\frac{\partial}{\partial \eta} + \frac{dP_\xi}{dt}\frac{\partial}{\partial P_\xi} + \frac{dP_\eta}{dt}\frac{\partial}{\partial P_\eta} + \frac{\partial}{\partial t} \tag{12}$$

$$D\xi = \dot{\xi} \tag{13.1}$$

$$D\eta = \dot{\eta} \tag{13.2}$$

$$DP_\xi = \dot{P}_\xi \tag{13.3}$$

$$DP_\eta = \dot{P}_\eta \tag{13.4}$$

The series for ξ

The double action of D on ξ can be computed as

$$D^2\xi = \left(\frac{d\xi}{dt}\frac{\partial}{\partial \xi} + \frac{d\eta}{dt}\frac{\partial}{\partial \eta} + \frac{dP_\xi}{dt}\frac{\partial}{\partial P_\xi} + \frac{dP_\eta}{dt}\frac{\partial}{\partial P_\eta} + \frac{\partial}{\partial t}\right)\dot{\xi}$$

$$D^2\xi = 2\left(B-1\right)\xi + 2P_\eta - \gamma^{3/2}\left[\frac{\left(1-\mu\right)\left(\xi+\mu\gamma^{1/2}\right)}{2r_1^3} + \frac{\mu\left(\xi-\left(1-\mu\right)\gamma^{1/2}\right)}{r_2^3}\right] \tag{14}$$

The solution for ξ can be written as

$$\xi\left(t\right) = \left(\xi\right)_{\xi_0} + \left(D\xi\right)_{\vec{\xi}_0}\left(t-t_0\right) + \left(D^2\xi\right)_{\vec{\xi}_0}\frac{\left(t-t_0\right)^2}{2!} + \ldots \tag{15}$$

The series for η

The double action of D on η can be computed as

$$D^2\eta = \left(\frac{d\xi}{dt}\frac{\partial}{\partial \xi} + \frac{d\eta}{dt}\frac{\partial}{\partial \eta} + \frac{dP_\xi}{dt}\frac{\partial}{\partial P_\xi} + \frac{dP_\eta}{dt}\frac{\partial}{\partial P_\eta} + \frac{\partial}{\partial t}\right)\dot{\eta}$$

$$D^2\eta = 2(B-1)\eta - 2P_\xi - \gamma^{\frac{3}{2}}\left[\frac{(1-\mu)\eta}{2r_1^3} + \frac{\mu\eta}{r_2^3}\right] \tag{16}$$

The solution for η can be written as

$$\eta(t) = (\eta)_{\xi_0} + (D\eta)_{\vec{\eta}_0}(t-t_0) + (D^2\eta)_{\vec{\eta}_0}\frac{(t-t_0)^2}{2!} + \ldots \tag{17}$$

The series for P_ξ

The double action of D on P_ξ can be computed as

$$D^2P_\xi = \left(\frac{d\xi}{dt}\frac{\partial}{\partial\xi} + \frac{d\eta}{dt}\frac{\partial}{\partial\eta} + \frac{dP_\xi}{dt}\frac{\partial}{\partial P_\xi} + \frac{dP_\eta}{dt}\frac{\partial}{\partial P_\eta} + \frac{\partial}{\partial t}\right)\dot{P}_\xi$$

$$\begin{aligned}
D^2P_\xi = {} & 2(B-1)\eta + (B-2)P_\xi - \frac{(1-\mu)\gamma^{1/2}}{2r_1^3}\Big[2(2\eta+P_\xi)\gamma + \left(3\xi + 4\mu\gamma^{\frac{1}{2}}\right)\gamma' \\
& \times - \frac{3\gamma^{1/2}}{r_1^2}\left(\xi+\mu\gamma^{1/2}\right)\left(2(\eta P_\eta + \xi P_\xi)\gamma^{1/2} + 2\mu(\eta+P_\xi)\gamma + \mu\left(\xi+\mu\gamma^{1/2}\right)\gamma'\right)\Big] \\
& - \frac{\mu\gamma^{1/2}}{2r_2^3}\Big[2(2\eta+P_\xi)\gamma + \left(3\xi - 4(1-\mu)\gamma^{1/2}\right)\gamma' - \frac{3\gamma^{1/2}}{r_2^2}\left(\xi-(1-\mu)\gamma^{1/2}\right) \\
& \times\left(2(\eta P_\eta + \xi P_\xi)\gamma^{1/2} - 2(1-\mu)(\eta+P_\xi)\gamma - (1-\mu)\left(\xi-(1-\mu)\gamma^{1/2}\right)\gamma'\right)\Big]
\end{aligned} \tag{18}$$

The solution for P_ξ can be written as

$$P_\xi(t) = (P_\xi)_{\xi_0} + (DP_\xi)_{\vec{P}_{\xi_0}}(t-t_0) + (D^2P_\xi)_{\vec{P}_{\xi_0}}\frac{(t-t_0)^2}{2!} + \ldots \tag{19}$$

Rahoma et al. (2009, 2011) calculated γ' expanding the function $m(t)$ in a Taylor series yields:

$$m(t) = m_0 + \dot{m}_0(t-t_0) + \ldots$$

where m_0 is the value of mass in certain initial instant t_0, $\dot{m}_0$ is the derivative of the function m with respect to t evaluated at $t = t_0$. i.e.

$$\gamma = 1 - \beta(t-t_0) + \ldots\ldots$$

So

$$\gamma' = 1 - \beta \tag{20}$$

The series for P_η

The double action of D on P_η can be computed as

$$D^2P_\eta = \left(\frac{d\xi}{dt}\frac{\partial}{\partial\xi} + \frac{d\eta}{dt}\frac{\partial}{\partial\eta} + \frac{dP_\xi}{dt}\frac{\partial}{\partial P_\xi} + \frac{dP_\eta}{dt}\frac{\partial}{\partial P_\eta} + \frac{\partial}{\partial t}\right)\dot{P}_\eta$$

$$\begin{aligned}
D^2P_\eta = {} & 2(B-1)\xi + (B-2)P_\eta + \frac{(1-\mu)\gamma^{1/2}}{2r_1^3}\Big[2(2\xi-P_\eta)\gamma - 3\eta\gamma' + 2\mu\gamma^{3/2} + \frac{3\gamma^{1/2}\eta}{r_1^2} \\
& \times\left(2(\eta P_\eta + \xi P_\xi)\gamma^{1/2} + 2\mu(\eta+P_\xi)\gamma + \mu\left(\xi+\mu\gamma^{1/2}\right)\gamma'\right)\Big]
\end{aligned}$$

$$+\frac{\mu\gamma^{1/2}}{2r_2^3}\Big[2\left(2\xi-P_\eta\right)\gamma-3\eta\gamma'-2\left(1-\mu\right)\gamma^{3/2}+\frac{3\gamma^{1/2}\eta}{r_2^2}$$

$$\times\Big(2\left(\eta P_\eta+\xi P_\xi\right)\gamma^{1/2}-2\left(1-\mu\right)\left(\eta+P_\xi\right)\gamma-\left(1-\mu\right)\left(\xi-\left(1-\mu\right)\gamma^{1/2}\right)\gamma'\Big)\Big] \qquad (21)$$

The solution for P_η can be written as

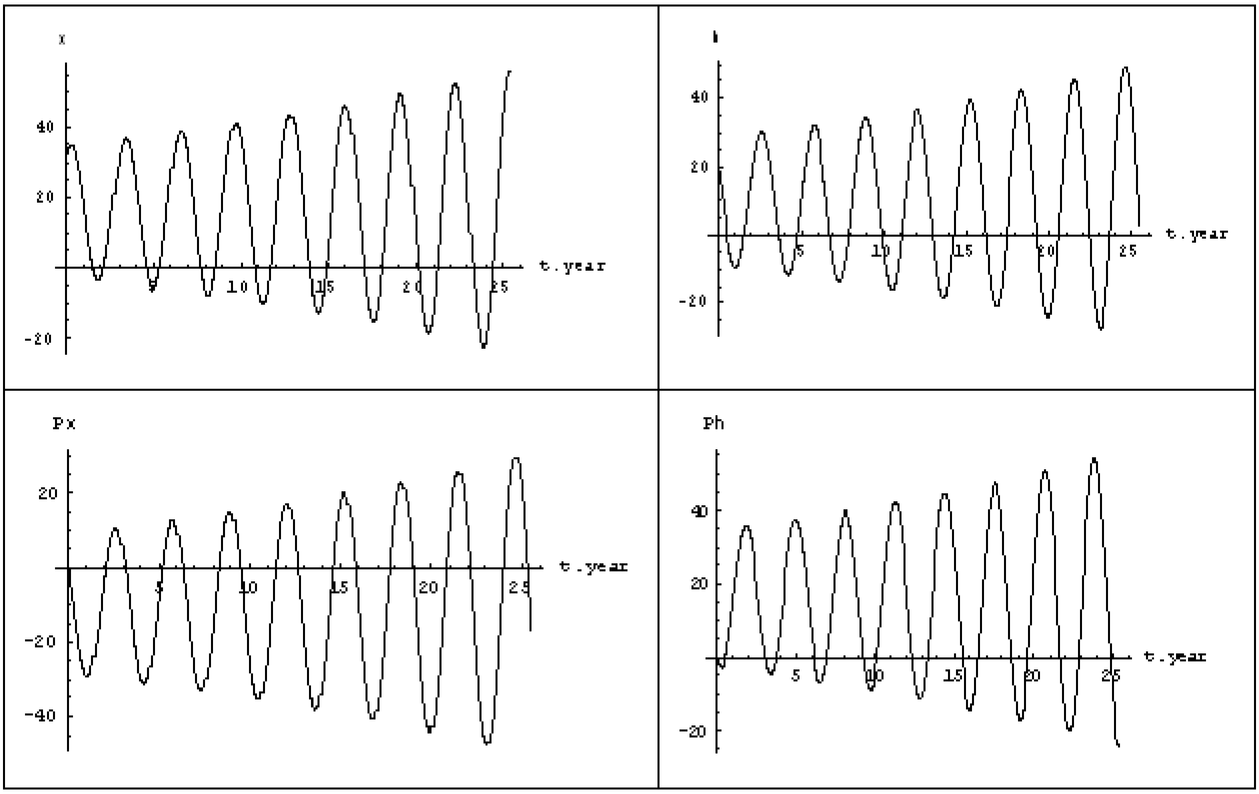
$$P_\eta\left(t\right)=\left(P_\eta\right)_{\eta_0}+\left(DP_\eta\right)_{\vec{P}_{\eta 0}}\left(t-t_0\right)+\left(D^2P_\eta\right)_{\vec{P}_{\eta 0}}\frac{\left(t-t_0\right)^2}{2!}+\dots$$

(22)

Coordinates and Momenta Numerical Representation

The adopted initial conditions are taken as;

$$\xi=32\ AU, \qquad \eta=19.5\ AU,$$
$$P_\xi=78\times10^{-12}m_e AU\ /\ year, \qquad P_\eta=45\times10^{-12}m_e AU\ /\ year$$



Conclusion

The study is concerned with the restricted three body problem with varying mass for the third body. The Hamiltonian of the problem and equations of motions are formulated. The equations of motions are integrated using Lie series. The obtained solution is given as an explicit solution of coordinates and conjugate momenta as functions of time.

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