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# Using Double ARA Integral Transform in Solving Integral-Differential Equations

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**ABSTRACT**

The goal objective of this study is to propose a new double ARA transform (DARAT). We employ this transform to prove the convolution, existence and other relevant theorems as well as derivatives properties. Subsequently, this transform is applied to solve some illustrative integral- differential equations. This approach was shown to be a powerful and efficient means to tack integral- differential equations.



## 1. Introduction

Integral transform methods are regarded among the basic and most widely used approach in solving partial differential equations. For instance, partial differential equations (PDEs) are utilized to model many phenomena in disciplines related to Mathematics such as Physics, Engineering and other scientific fields. These models are written using a variety of partial differential equations [1-9]. Integral transforms can be equally utilized to solve both integral and integral- differential equations. Thus, this study primarily seeks to solve the aforementioned types through double ARA transform. The double integral transform provides us with an efficient means where by integral- differential equations can be easily transformed into algebraic equation so as to obtain the exact solutions. This study is rooted in the advances made by Scholars who exerted great efforts to develop the above method and apply them to a wide range of Mathematical problems. Example, include carried out the double Laplace transform[10-13] , double Sumudu transform [13–17], the double Laplace–Sumudu transform [18,19] , the double Elzaki transform [20] , the double Kamal transform [21], the double Shehu transform [22], ARA transform [23-25] , among other.

Among the previous method is double Laplace-ARA Transform which has been increasingly used recently to solve partial differential equations (PDEs), integral equation (IEs) and integral- partial differential equations (IPDEs).

This article is organized as following: In Section 2 , Fundamental Facts of new integral ARA transformation .In Section 3, we introduce a new double integral transform and present some properties of this transform. In Section 4, we apply the DARAT to IDEs. In Section 5, some examples are presented and solved with the DARAT. Lastly, In Section 6, the conclusion.

## 2. Fundamental Facts of the ARA Transform of order $n$:

A new integral transform known as the ARA integral transform of order $n$ of the continuous function $f(t)$ on the interval $[0,\infty)$ is defined by:

$$\mathcal{G}_n[f(t)] = Q(n,s) = s\int_0^\infty t^{n-1}e^{-st}f(t)\,dt, s>0, n=1,2,3,\ldots \quad (1)$$

And; the inverse of the ARA transform is given by:

$$\mathcal{G}_{n+1}^{-1}[\mathcal{G}_{n+1}[f(t)]] = \frac{(-1)^{2n}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,Q(s)ds = f(t) \quad (2)$$

*2.1.ARA integral transform of order $n$ of some basic functions*

i. $\mathcal{G}_n[1] = \frac{\Gamma(n)}{s^{n-1}}$,

ii. $\mathcal{G}_n[t^m] = \frac{\Gamma(m+n)}{s^{m+n-1}}$,

iii. $\mathcal{G}_n[e^{at}] = \frac{s\Gamma(n)}{(s-a)^n}$,

iv. $\mathcal{G}_n[\sin(at)] = \left(1+\frac{a^2}{s^2}\right)^{\frac{-n}{2}} s^{1-n}\Gamma(n)\sin\left(n\tan^{-1}\left(\frac{a}{s}\right)\right)$,

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v. $\mathcal{G}_n[sinh(at)] = \frac{s}{2}\Gamma(n)\frac{1}{s^n}\left(\frac{1}{\left(1-\frac{a}{s}\right)^n} + \frac{1}{\left(1+\frac{a}{s}\right)^n}\right)$.

Theorem 2.1.(*Existence conditions*). If the function $f(t)$ is a piecewise continuous in every finite interval $0 \le t \le \alpha$ and satisfies$|t^{n-1}f(t)| \le Te^{\alpha t}$.

where$T$ is positive constant, then, the ARA transform of order $n$ of the function $f(t)$ exists for all $s > \alpha$.

Proof of Theorem 2.1.Using the definition of ARA transform, we get$|F(n,s)| = \left|s\int_0^\infty t^{n-1}e^{-st}f(t)\,dt\right|$.

Using the property of improper integral, we get

$$|F(n,s)| = \left|s\int_0^\infty t^{n-1}e^{-st}f(t)\,dt\right| \le s\left|\int_0^\infty t^{n-1}e^{-st}f(t)\,dt\right|$$

$$\le s\int_0^\infty e^{-st}|t^{n-1}f(t)|\,dt \le s\int_0^\infty e^{-st}Te^{\alpha t}\,dt$$

$$= sT\int_0^\infty e^{-(s-\alpha)t}dt = \frac{sT}{s-\alpha}.$$

This improper integral converges for all $s > \alpha$. Thus, $\mathcal{G}_{n+1}[f(t)]$ exists.

**3. Double ARA transform of order one (DARAT):**

In this section, we define a new integral transform DARAT. We present basic properties concerning the existence conditions, linearity and the inverse of this transform. Moreover, some essential properties and results are used to compute the DARAT for some basic functions. We introduce the convolution theorem and the derivatives properties of the new transform. Recall that the ARA transform of order one of a piecewise continuous function $f(t)$ on $[0,\infty)$ is given as:

$$\mathcal{G}_1[f(t)] = Q(s) = s\int_0^\infty e^{-st}f(t)\,dt, s > 0 \quad (3)$$

For simplicity, let us denote $\mathcal{G}_1[f(t)]$by $\mathcal{G}[f(t)]$.

The inverse ARA transform is given by

$$\mathcal{G}^{-1}[F(s)] = \frac{(-1)^2}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,F(s)ds = f(t). \quad (4)$$

In the following arguments, we state some basic properties of ARA transform of order 1. Assume that $F(s) = \mathcal{G}[f(t)]$ and $G(s) = \mathcal{G}[g(t)]$and $a, b \in \mathcal{R}$. Then, we have

$\mathcal{G}[a\,f(t) + b\,g(t)] = a\,\mathcal{G}[f(t)] + b\,\mathcal{G}[g(t)]$.(5)

$\mathcal{G}^{-1}[a\,F(s) + b\,G(s)] = a\,\mathcal{G}^{-1}[F(s)] + b\,\mathcal{G}^{-1}[G(s)]$.(6)

$\mathcal{G}[t^\alpha] = \frac{\Gamma(\alpha+1)}{s^\alpha}, \quad \alpha > 0$. (7)

$\mathcal{G}[e^{at}] = \frac{s}{s-a}$. (8)

$\mathcal{G}[\sin at] = \frac{as}{s^2+a^2}$. (9)

$\mathcal{G}[\cos at] = \frac{s^2}{s^2+a^2}$. (10)

$\mathcal{G}[\sinh at] = \frac{as}{s^2-a^2}$.(11)

$\mathcal{G}[\cosh at] = \frac{s^2}{p^2-a^2}$.(12)

$\mathcal{G}\left[f^{(n)}(t)\right] = s^nF(s) - \sum_{k=0}^{n-1} p^{n-k}f^{(k)}(0)$. (13)

The above results can be obtained from the definition of ARA transform with simple calculations.

**Definition 3.1.**Let $u(x,t)$ be continuous function of two positive variables $x$and $t$.Then the DARAT of $u(x,t)$ is defined as:

$$\mathcal{G}_t[u(x,t)] = Q(v,s) = vs\int_0^\infty\int_0^\infty e^{-(vx+st)}[u(x,t)]\,dxdt\,, v, s > 0$$

Clearly, the DARAT is a linear integral transformation as shown below:

$$\mathcal{G}_x\mathcal{G}_t[A\,u(x,t) + B\,w(x,t)] = vs\int_0^\infty\int_0^\infty e^{-(vx+st)}[A\,u(x,t) + B\,w(x,t)]\,dxdt$$

$$= Avs\int_0^\infty\int_0^\infty e^{-(vx+st)}[u(x,t)]\,dxdt + Bvs\int_0^\infty\int_0^\infty e^{-(vx+st)}[w(x,t)]\,dxdt$$

$$= A\,\mathcal{G}_x\mathcal{G}_t[\,u(x,t)] + B\,\mathcal{G}_x\mathcal{G}_t[\,w(x,t)]$$

Where $A$ and $B$ are constants.
And, the inverse of the DARAT is given as:

$$\mathcal{G}_x^{-1}\left[\mathcal{G}_t^{-1}[Q(v,s)]\right] = \left(\frac{1}{2\pi i}\right)\int_{c-i\infty}^{c+i\infty}\frac{e^{vx}}{v}dv\left(\frac{1}{2\pi i}\right)\int_{r-i\infty}^{r+i\infty}\frac{e^{st}}{s}Q(v,s)ds = u(x,t) \quad (14)$$

Property 3.1.Let $u(x,t) = f(x)g(t), x > 0, t > 0$. Then

$$\mathcal{G}_x\mathcal{G}_t[u(x,t)] = \mathcal{G}_x[f(x)]\cdot\mathcal{G}_t[g(t)]$$

Proof of Property 3.1. $\mathcal{G}_x\mathcal{G}_t[u(x,t)] = \mathcal{G}_x\mathcal{G}_t[f(x)g(t)] = vs\int_0^\infty\int_0^\infty e^{-(vx+st)}\,[f(x)g(t)]dxdt$

$$= v\int_0^\infty e^{-vx}\,[f(x)]dx\cdot s\int_0^\infty e^{-st}\,[g(t)]dt$$

$$= \mathcal{G}_x[f(x)]\cdot\mathcal{G}_t[g(t).]$$

### 3.1 Double ARAT Transform of some basic functions

i. Let $u(x,t) = 1, x > 0, t > 0$ .Then :

$$\mathcal{G}_x\mathcal{G}_t[1] = vs\int_0^\infty\int_0^\infty e^{-(vx+st)}\,dxdt = v\int_0^\infty e^{-vx}\,dx\cdot s\int_0^\infty e^{-st}\,dt = 1$$

From Eq.(7) ,we get

$$\mathcal{G}_x\mathcal{G}_t[1] = 1.$$

ii. Let $u(x,t) = x^\alpha t^\beta, x > 0, t > 0$ and $\alpha, \beta$ are constants. Then

$$\mathcal{G}_x\mathcal{G}_t[x^\alpha t^\beta] = vs\int_0^\infty\int_0^\infty e^{-(vx+st)}\,[x^\alpha t^\beta]dxdt = v\int_0^\infty e^{-vx}\,[x^\alpha]dx\cdot s\int_0^\infty e^{-st}\,[t^\beta]dt = \mathcal{G}_x[x^\alpha]\cdot\mathcal{G}_t[t^\beta]$$

From Eq.(7) ,we get:

$$\mathcal{G}_x\mathcal{G}_t[x^\alpha t^\beta] = \frac{\alpha!\,\beta!}{v^\alpha s^\beta}.$$

iii Let $u(x,t) = e^{\alpha x+\beta t}, x > 0, t > 0$ and $\alpha, \beta$ are constants. Then

$$\mathcal{G}_x\mathcal{G}_t[e^{\alpha x+\beta t}] = vs\int_0^\infty\int_0^\infty e^{-(vx+st)}\,[e^{\alpha x+\beta t}]dxdt = v\int_0^\infty e^{-vx}\,[e^{\alpha x}]dx\cdot s\int_0^\infty e^{-st}[e^{\beta t}]\,dt$$
$$= \mathcal{G}_x[e^{\alpha x}]\cdot\mathcal{G}_t[e^{\beta t}]$$

From Eq.(8) ,we get

$$\mathcal{G}_x\mathcal{G}_t[e^{\alpha x+\beta t}] = \frac{vs}{(v-\alpha)(s-\beta)}.$$

Similarly,

$$\mathcal{G}_x\mathcal{G}_t[e^{i(\alpha x+\beta t)}] = \frac{vs}{(v-i\alpha)(s-i\beta)}$$

Using the property of complex analysis, we have:

$$\mathcal{G}_x\mathcal{G}_t[e^{i(\alpha x+\beta t)}] = \frac{vs(vs-\alpha\beta) + ivs(v\beta + s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)}$$

Using Euler's formulas: $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$ , $\cos x = \frac{e^{ix}+e^{-ix}}{2}$

And the formulas: $\sinh x = \frac{e^{x}-e^{-x}}{2}$ , $\cosh x = \frac{e^{x}+e^{-x}}{2}$

Therefore, we conclude the following:

$\mathcal{G}_x\mathcal{G}_t[\sin(\alpha x+\beta t)]=\frac{vs(v\beta+s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)}$,

$\mathcal{G}_x\mathcal{G}_t[\cos(\alpha x+\beta t)]=\frac{vs(sv-\alpha\beta)}{(v^2+\alpha^2)(s^2+\beta^2)}$,

$\mathcal{G}_x\mathcal{G}_t[\sinh(\alpha x+\beta t)]=\frac{vs(v\beta+s\alpha)}{(v^2-\alpha^2)(s^2-\beta^2)}$,

$\mathcal{G}_x\mathcal{G}_t[\cosh(\alpha x+\beta t)]=\frac{vs(sv+\alpha\beta)}{(v^2-\alpha^2)(s^2-\beta^2)}$.

**3.2.Existence conditions for DARAT**

Let $u(x,t)$ be function of exponential order $\alpha$ and $\beta$ as $x\to\infty$ and $t\to\infty$.If there exists a positive $N$ such that $\forall x>X$and $t>T$,we have:

$$|u(x,t)|\le Te^{\alpha x+\beta t}$$

We can write $u(x,t)=O(e^{\alpha x+\beta t})$ as $x\to\infty$and $t\to\infty$, $v>\alpha$ and $s>\beta$.

Theorem **3.1**.Let $u(x,t)$ be a continuous function on the region $[0,X)\times[0,T)$ of exponential order $\alpha$ and $\beta$.Then $\mathcal{G}_x\mathcal{G}_t[u(x,t)]$ exists for $v$ and $s$ provided $Re(v)>\alpha$and $Re(s)>\beta$.

Proof of Theorem **3.1**.Using the definition of DARAT, we get

$$|Q(v,s)|=\left|vs\int_0^\infty\int_0^\infty e^{-(vx+st)}[u(x,t)]\,dxdt\right|\le vs\int_0^\infty\int_0^\infty e^{-(vx+st)}|u(x,t)|\,dxdt$$

$$\le Tv\int_0^\infty e^{-(v-\alpha)x}dx\, s\int_0^\infty e^{-(s-\beta)t}\,dx$$

$=\frac{Tvs}{(v-\alpha)(s-\beta)},\quad Re(v)>\alpha$and$Re(s)>\beta$

**3.3 Some important theorems of DARAT**

**Theorem 3.2.(***Shifting Property***).**Let $u(x,t)$ be a continuous function and $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial t}\right]=Q(v,s)$.Then:

$$\mathcal{G}_x\mathcal{G}_t\left[e^{\alpha x+\beta t}u(x,t)\right]=\frac{vs}{(s-\beta)(v-\alpha)}Q(v-\alpha,s-\beta) \quad (15)$$

**Proof of Theorem 3.2**

$$=\frac{vs}{(s-\beta)(v-\alpha)}(s-\beta)(v-\alpha)\int_0^\infty\int_0^\infty e^{-(v-\alpha)x}e^{-(s-\beta)t}u(x,t)\,dxdt$$

$$=\frac{vs}{(s-\beta)(v-\alpha)}Q(v-\alpha,s-\beta)$$

**Theorem 3.3.(***Periodic Function***). Let $\mathcal{G}_x\mathcal{G}_t[u(x,t)]$ exists, where $u(x,t)$ periodic function of periods $\alpha$ and $\beta$ such that:**

$$u(x+\alpha,t+\beta)=u(x,t)\,,\forall x,y$$

**Then**

$$\mathcal{G}_t[u(x,t)]=\left(\frac{vs\int_0^\alpha\int_0^\beta e^{-(vx+st)}(u(x,t))dxdt}{(1-e^{-(v\alpha+s\beta)})}\right) \quad (16)$$

**Proof of Theorem 3.3.**Using the definition of DARAT, we get

$$\mathcal{G}_x\mathcal{G}_t[u(x,t)]=vs\int_0^\infty\int_0^\infty e^{-(vx+st)}[u(x,t)]\,dxdt \quad (17)$$

Using the property of improper integral, Eq.(17) can be written as

$$\mathcal{G}_x\mathcal{G}_t[u(x,t)] = vs\int_0^\alpha\int_0^\beta e^{-(vx+st)}\big(u(x,t)\big)\,dxdt + vs\int_\alpha^\infty\int_\beta^\infty e^{-(vx+st)}\big(u(x,t)\big)\,dxdt \quad (18)$$

Putting $x=\alpha+\rho$ and $t=\beta+\tau$ on the second integral in Eq.(18). We obtain

$$Q(v,s) = vs\int_0^\alpha\int_0^\beta e^{-(vx+st)}\big(u(x,t)\big)\,dxdt + vs\int_0^\infty\int_0^\infty e^{-(v(\alpha+\rho)+s(\beta+\tau))}\big(u(\alpha+\rho,\beta+\tau)\big)\,d\rho d\tau \quad (19)$$

Using the periodicity of the function $u(x,t)$, Eq.(19) can be written by

$$Q(v,s) = vs\int_0^\alpha\int_0^\beta e^{-(vx+st)}\big(u(x,t)\big)\,dxdt + e^{-(v\alpha+s\beta)}vs\int_0^\infty\int_0^\infty e^{-(v\rho+s\tau)}\big(u(\rho,\tau)\big)\,d\rho d\tau \quad (20)$$

Using the definition of DARAT, we get

$$Q(v,s) = vs\int_0^\alpha\int_0^\beta e^{-(vx+st)}\big(u(x,t)\big)\,dxdt + e^{-(v\alpha+s\beta)}Q(v,s) \quad (21)$$

Thus, Eq.(21) can be simplified into

$$Q(v,s) = \frac{1}{\left(1-e^{-(v\alpha+s\beta)}\right)}\left(s\int_0^\alpha\int_0^\beta e^{-(vx+st)}\big(u(x,t)\big)\,dxdt\right)$$

**Theorem 3.4.**(*Heaviside Function*)**.** Let $\mathcal{G}_x\mathcal{G}_t[u(x,t)]$ exists and $\mathcal{G}_x\mathcal{G}_t[u(x,t)] = Q(v,s)$, then

$$\mathcal{G}_x\mathcal{G}_t[u(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)] = e^{-v\delta-s\varepsilon}Q(v,s) \quad (22)$$

where $H(x-\delta,t-\varepsilon)$ is the Heaviside unit step function defined as

$$H(x-\delta,t-\varepsilon) = \begin{cases} 1 & , x>\delta, t>\varepsilon \\ 0 & , \quad Ohtherwise \end{cases}$$

**Proof of Theorem 3.4.** Using the definition of DARAT, we get

$$\mathcal{G}_x\mathcal{G}_t[u(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)] = vs\int_0^\infty\int_0^\infty e^{-(vx+st)}\big(u(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)\big)\,dxdt \quad \textbf{(23)}$$

$$= vs\int_0^\infty\int_0^\infty e^{-(vx+st)}\big(u(x-\delta,t-\varepsilon)\big)\,dxdt$$

**Putting $x-\delta=\rho$ and $t-\varepsilon=\tau$ in Eq.(23). We obtain**

$$\mathcal{G}_x\mathcal{G}_t[u(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)] = vs\int_0^\infty\int_0^\infty e^{-v(\delta+\rho)-s(\varepsilon+\tau)}\big(u(\rho,\tau)\big)\,d\rho d\tau \quad \textbf{(24)}$$

**Thus, Eq.(24) can be simplified into**

$$\mathcal{G}_x\mathcal{G}_t[u(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)] = e^{-v\delta-s\varepsilon}\left(vs\int_0^\infty\int_0^\infty e^{-v\rho-s\tau}\big(u(\rho,\tau)\big)\,d\rho d\tau\right) = e^{-v\delta-s\varepsilon}Q(v,s)$$

**Theorem 3.5**.(*Convolution Theorem*). Let $\mathcal{G}_x\mathcal{G}_t[u(x,t)]$ and $\mathcal{L}_x\mathcal{G}_t[w(x,t)]$are exists and $\mathcal{G}_x\mathcal{G}_t[u(x,t)] = Q(v,s)$, $\mathcal{G}_x\mathcal{G}_t[w(x,t)] = W(v,s)$, then

$$\mathcal{G}_x\mathcal{G}_t[u ** w(x,t)] = \frac{1}{vs}Q(v,s)W(v,s) \quad (25)$$

where

$$(u ** w)(x,t) = \int_0^x\int_0^t u(x-\rho,t-\tau)w(\rho,\tau)\,d\rho d\tau$$

Where, the symbol $(**)$ denotes the double convolution with respect to $x$and $t$.

**Proof of Theorem 3.5.** Using the definition of DARAT, we get

$$\mathcal{G}_x\mathcal{G}_t[(u ** w)(x,t)] = vs\int_0^\infty\int_0^\infty e^{-(vx+st)}\big(u ** w(x,t)\big)\,dxdt \quad (26)$$

$$= vs\int_0^\infty\int_0^\infty e^{-(vx+st)}\left(\int_0^x\int_0^t u(x-\rho,t-\tau)w(\rho,\tau)\,d\rho d\tau\right)dxdt$$

Using the Heaviside unit step function, Eq.(26) can be written as

$$\mathcal{G}_x\mathcal{G}_t[u ** w(x,t)] = vs\int_0^\infty \int_0^\infty e^{-(vx+st)}\left(\int_0^\infty \int_0^\infty u(x-\rho,t-\tau)H(x-\rho,t-\tau)w(\rho,\tau)\,d\rho d\tau\right)dxdt \quad (27)$$

Thus, Eq.(27) can be written as

$$\mathcal{G}_x\mathcal{G}_t[u ** w(x,t)] = \int_0^\infty \int_0^\infty w(\rho,\tau)d\rho d\tau\left(vs\int_0^\infty \int_0^\infty e^{-v(x+\rho)-s(t+\tau)}u(x-\rho,t-\tau)H(x-\rho,t-\tau)\right)dxdt$$

$$= \int_0^\infty \int_0^\infty w(\rho,\tau)d\rho d\tau\left(e^{-v\rho-s\tau}Q(v,s)\right)$$

$$= Q(v,s)\int_0^\infty \int_0^\infty e^{-v\rho-s\tau}w(\rho,\tau)d\rho d\tau = \frac{1}{vs}Q(v,s)W(v,s).$$

**Theorem 3.6.(***Derivatives Properties***).** Let $u(x,t)$ be a continuous function and $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial t}\right] = Q(v,s)$.Then, we get the following derivatives properties

a) $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial t}\right] = sQ(v,s) - s\mathcal{G}_x[u(x,0)]$,

b) $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial x}\right] = vQ(v,s) - v\mathcal{G}_t[u(0,t)]$

c) $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = s^2Q(v,s) - s^2\mathcal{G}_x[u(x,0)] - s\mathcal{G}_x\left[\frac{\partial u(x,0)}{\partial t}\right]$

d) $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = v^2Q(v,s) - v^2\mathcal{G}_t[u(0,t)] - v\mathcal{G}_t\left[\frac{\partial u(0,t)}{\partial x}\right]$

e) $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial x\partial t}\right] = vsQ(v,s) - vs\mathcal{G}_x[u(x,0)] - vs\mathcal{G}_t[u(0,t)] + vs\,u(0,0)$

**Proof of Theorem 3.6**

a) $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial t}\right] = vs\int_0^\infty \int_0^\infty e^{-(st+vx)}\left[\frac{\partial u(x,t)}{\partial t}\right]dxdt = v\int_0^\infty e^{-vx}dx \cdot s\int_0^\infty e^{-st}\left(\frac{\partial u(x,t)}{\partial t}\right)dt$

Using integrating by parts, we obtain

Let $u = e^{-st} \quad \Rightarrow \quad du = -se^{-st}dt$,

$$dv = \frac{\partial u(x,t)}{\partial t}dt \Rightarrow \quad v = u(x,t)$$

Thus

$$s\int_0^\infty e^{-st}\left(\frac{\partial u(x,t)}{\partial t}\right)dt = s\left(-u(x,0) + s\int_0^\infty e^{-st}u(x,t)dt\right)$$

$$\therefore \quad \mathcal{G}_x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial t}\right] = s\,Q(v,s) - s\,\mathcal{G}_x[u(x,0)] \quad (28)$$

b) $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial x}\right] = vs\int_0^\infty \int_0^\infty e^{-(st+vx)}\left[\frac{\partial u(x,t)}{\partial x}\right]dxdt = s\int_0^\infty e^{-st}dt \cdot v\int_0^\infty e^{-vx}\left(\frac{\partial u(x,t)}{\partial x}\right)dx$

Using integrating by parts, we obtain

Let $u = e^{-vx} \Rightarrow \quad du = -ve^{-vx}dx$,

$$dV = \frac{\partial u(x,t)}{\partial x}dx \quad \Rightarrow \quad V = u(x,t)$$

Thus

$$v\int_0^\infty e^{-vx}\left(\frac{\partial u(x,t)}{\partial x}\right)dx = v\left(-u(0,t) + v\int_0^\infty e^{-vx}u(x,t)dx\right)$$

$$\therefore \quad \mathcal{G}_x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial x}\right] = v\,Q(v,s) - v\,\mathcal{G}_t[u(0,t)] \quad (29)$$

c) $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = vs\int_0^\infty \int_0^\infty e^{-(st+vx)}\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right]dxdt = v\int_0^\infty e^{-vx}dx \cdot s\int_0^\infty e^{-st}\left(\frac{\partial^2 u(x,t)}{\partial t^2}\right)dt$

Using integrating by parts, we obtain

Let $u = e^{-st} \Rightarrow \quad du = -se^{-st}dt$,

$$dv = \frac{\partial^2 u(x,t)}{\partial t^2} dt \implies v = \frac{\partial u(x,t)}{\partial t}$$

Thus

$$s\int_0^\infty e^{-st}\left(\frac{\partial^2 u(x,t)}{\partial t^2}\right)dt = s\left(-\frac{\partial u(x,0)}{\partial t} + s\int_0^\infty e^{-st}\left(\frac{\partial u(x,t)}{\partial t}\right)dt\right)$$

Using Eq(28), we have

$$\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}\right] = s^2 Q(v,s) - s^2\mathcal{G}_x[u(x,0)] - s\,\mathcal{G}_x\left[\frac{\partial u(x,0)}{\partial t}\right] \quad (30)$$

d) $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = vs\int_0^\infty\int_0^\infty e^{-(st+vx)}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]dxdt = s\int_0^\infty e^{-st}dt \cdot v\int_0^\infty e^{-vx}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)dx$

Using integrating by parts, we obtain

Let $u = e^{-vx} \Rightarrow du = -ve^{-vx}dx$,

$$dV = \frac{\partial^2 u(x,t)}{\partial x^2}dx \Rightarrow V = \frac{\partial u(x,t)}{\partial x}$$

Thus

$$v\int_0^\infty e^{-vx}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)dx = v\left(-\frac{\partial u(0,t)}{\partial x} + v\int_0^\infty e^{-vx}\left(\frac{\partial u(x,t)}{\partial x}\right)dx\right)$$

Using Eq(29), we have

$$\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = v^2 Q(v,s) - v^2\mathcal{G}_t[u(0,t)] - v\,\mathcal{G}_t\left[\frac{\partial u(0,t)}{\partial x}\right] \quad (31)$$

e) $\mathcal{G}_t\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial x\partial t}\right] = vs\int_0^\infty\int_0^\infty e^{-(st+vx)}\left(\frac{\partial^2 u(x,t)}{\partial x\partial t}\right)dxdt = s\int_0^\infty e^{-st}\left(\frac{\partial^2 u(x,t)}{\partial x\partial t}\right)dt \cdot v\int_0^\infty e^{-vx}dx$

Using integrating by parts , we obtain

$$vs\int_0^\infty\int_0^\infty e^{-(st+vx)}\left(\frac{\partial^2 u(x,t)}{\partial x\partial t}\right)dxdt = \left(-vs\int_0^\infty e^{-st}\left(\frac{\partial u(0,t)}{\partial t}\right)dt + v^2 s\int_0^\infty\int_0^\infty e^{-(st+vx)}\left(\frac{\partial u(x,t)}{\partial t}\right)dxdt\right).$$

And, using Eq.(28) and Eq.(10) , we have

$$\mathcal{L}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial x\partial t}\right] = \mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial x\partial t}\right] = vsQ(v,s) - vs\,\mathcal{G}_x[u(x,0)] - vs\,\mathcal{G}_t[u(0,t)] + vs\,u(0,0). \quad (32)$$

**The previous results of DARAT to some basic functions, some theorems and basic derivatives are summed up in the Table below**

| $u(x,t)$ | $\mathcal{G}_x\mathcal{G}_t[u(x,t)] = Q(v,s)$ |
|---|---|
| $1$ | $1$ |
| $x^\alpha t^\beta$ | $\frac{\alpha!\,\beta!}{v^\alpha s^\beta}$ |
| $e^{\alpha x+\beta t}$ | $\frac{vs}{(v-\alpha)(s-\beta)}$ |
| $\sin(\alpha x+\beta t)$ | $\frac{vs(v\beta+s\alpha)}{(v^2+\alpha^2)(s^2+\beta^2)}$ |
| $\cos(\alpha x+\beta t)$ | $\frac{vs(sv-\alpha\beta)}{(v^2+\alpha^2)(s^2+\beta^2)}$ |
| $\sinh(\alpha x+\beta t)$ | $\frac{vs(v\beta+s\alpha)}{(v^2-\alpha^2)(s^2-\beta^2)}$ |
| $\cosh(\alpha x+\beta t)$ | $\frac{vs(sv+\alpha\beta)}{(v^2-\alpha^2)(s^2-\beta^2)}$ |
| $e^{\alpha x+\beta t}u(x,t)$ | $\frac{vs}{(s-\beta)(v-\alpha)}Q(v-\alpha,s-\beta)$ |
| $u(x-\delta,t-\varepsilon)H(x-\delta,t-\varepsilon)$ | $e^{-v\delta-s\varepsilon}Q(v,s)$ |
| $u ** w(x,t)$ | $\frac{1}{vs}Q(v,s)W(v,s)$ |
| $u_t(x,t)$ | $sQ(v,s) - s\,\mathcal{L}[u(x,0)]$ |

| | |
|---|---|
| $u_x(x,t)$ | $vQ(v,s) - v\mathcal{G}[u(0,t)]$ |
| $u_{tt}(x,t)$ | $s^2Q(v,s) - s^2\mathcal{G}_x[u(x,0)] - s\,\mathcal{G}_x\left[\frac{\partial u(x,0)}{\partial t}\right]$ |
| $u_{xx}(x,t)$ | $v^2Q(v,s) - v^2\mathcal{G}_t[u(0,t)] - v\,\mathcal{G}_t\left[\frac{\partial u(0,t)}{\partial x}\right]$ |
| $u_{xt}(x,t)$ | $vsQ(v,s) - vs\mathcal{G}_x[u(x,0)] - vs\mathcal{G}_t[u(0,t)] + vs\,u(0,0)$ |

## 4. Presentation of Double ARA transform Method in Solving IDEs

To illustrate the basic idea of this method for solving integral partial differential equations. Instance of these are Volterra integral equations, Volterraintegro-partial differential equation and integro-partial differential equations:

### 4.1 Volterra Integral Equation

Consider the following Volterra-integral equation

$$u(x,t) = f(x,t) + \gamma \int_0^x \int_0^t u(x-\rho, t-\tau) w(\rho,\tau) d\rho d\tau \quad (33)$$

where $u(x,t)$ is unknown function, $f(x,t)$ and $w(x,t)$ are two known functions and $\gamma$ is constant.

The main idea of this method is to apply the DARAT to Eq.(33) as the following

$$\mathcal{G}_x\mathcal{G}_t[u(x,t)] = \mathcal{G}_x\mathcal{G}_t\left[f(x,t) + \gamma \int_0^x \int_0^t u(x-\rho, t-\tau) w(\rho,\tau) d\rho d\tau\right] \quad (34)$$

Using the differentiation property of the DARAT of Eq.(34) and *Theorem **3.3.4***, we have

$$Q(v,s) = F(v,s) + \gamma\left(\frac{1}{vs}Q(v,s)W(v,s)\right) \quad (35)$$

Eq.(35) can be simplifying as the following

$$Q(v,s) = \frac{F(s,v)}{1-\frac{\gamma}{vs}W(v,s)} \quad (36)$$

Operating with the inverse of DARAT on both sides of Eq.(36) gives

$$u(x,t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{F(s,v)}{1-\frac{\gamma}{vs}W(v,s)}\right] \quad (37)$$

Where $u(x,t)$ represents the term arising from the known functions $f(x,t)$ and $w(x,t)$

### 4.2 VolterraIntegro - Partial Differential Equations

Consider the following VolterraIntegro-partial differential equations

$$\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t} = f(x,t) + \gamma \int_0^x \int_0^t u(x-\rho, t-\tau) w(\rho,\tau) d\rho d\tau \quad (38)$$

With the following initial condition: $u(x,0) = h(x)$ (39)

and the following boundary condition: $u(0,t) = g(t)$ (40)

where $u(x,t)$ is unknown function, $f(x,t)$ and $w(x,t)$ are two known functions and $\gamma$ is constant.

The main idea of this method is to apply ARA transform to the initial condition in Eq.(39) and to boundary condition in Eq.(40) and the DARAT to Eq.(38),as the following:

The ARA transform to the initial condition in Eq.(39) and to the boundary condition in Eq.(42),we get

$$\mathcal{G}_x[u(x,0)] = \mathcal{G}_x[h(x)] = H(v)$$

$$\mathcal{G}_t[u(0,t)] = \mathcal{G}_t[g(t)] = G(s)$$

The DARAT on both sides of Eq.(38), to get:

$$\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t}\right] = \mathcal{G}_x\mathcal{G}_t\left[f(x,t) + \gamma \int_0^x \int_0^t u(x-\rho, t-\tau) w(\rho,\tau) d\rho d\tau\right] \quad (41)$$

Using the differentiation property of the DARAT of Eq.(41) , above initial and boundary condition and *Theorem **3.3.4***, we have:

$$v\,Q(v,s) - v\,G + s\,Q(v,s) - s\,H = F(v,s) + \frac{\gamma}{vs}Q(v,s)W(v,s) \quad (42)$$

Eq.(42) can be simplifying as the following

$$Q(v,s)=\frac{F(s,v)+vG+sH}{v+s-\frac{\gamma}{vs}W(v,s)} \quad (43)$$

Operating with the inverse of DARAT on both sides of Eq.(43) gives

$$u(x,t)=\mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{F(s,v)+vG+sH}{v+s-\frac{\gamma}{vs}W(v,s)}\right] \quad (44)$$

Where $u(x,t)$ represents the term arising from the known functions $f(x,t), w(x,t), h(x)$ and $g(t)$

**4.3 Integro - Partial Differential Equations**

Consider the following Integro - differential equation

$$\frac{\partial^2 u(x,t)}{\partial t^2}-\frac{\partial^2 u(x,t)}{\partial x^2}+u(x,t)+\int_0^x\int_0^t u(x-\rho,t-\tau)w(\rho,\tau)d\rho d\tau=f(x,t) \quad (45)$$

With the following initial condition: $u(x,0)=h_1(x)$ ,$u_t(x,0)=h_2(x)$(46)

and the following boundary condition: $u(0,t)=g_1(t)$ , $u_x(0,t)=g_2(t)$ (47)

where $u(x,t)$ is unknown function, $f(x,t)$and $w(x,t)$ are two known functions.

The main idea of this method is to apply ARA transform to the initial conditions in Eq.(46) and to boundary condition in Eq.(47) and the DARAT to Eq.(45),as the following:

The ARA transform to the initial condition in Eq.(46) and to the boundary condition in Eq.(47),we get

$\mathcal{G}_x[u(x,0)]=\mathcal{G}_x[h_1(x)]=H_1(v)$ , $\mathcal{G}_x[u_t(x,0)]=\mathcal{G}_x[h_2(x)]=H_2(v)$

$$\mathcal{G}_t[u(0,t)]=\mathcal{G}_t[g_1(t)]=G_1(s),\mathcal{G}_t[u_x(0,t)]=\mathcal{G}_t[g_2(t)]=G_2(s)$$

The DARAT on both sides of Eq.(45), to get

$$\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2 u(x,t)}{\partial t^2}-\frac{\partial^2 u(x,t)}{\partial x^2}+u(x,t)+\int_0^x\int_0^t u(x-\rho,t-\tau)w(\rho,\tau)d\rho d\tau\right]=\mathcal{G}_x\mathcal{G}_t[f(x,t)] \quad (48)$$

Using the differentiation property of the DARAT of Eq.(48) , above initial and boundary conditions and *Theorem **3.3.4*** , we have

$$[s^2Q(v,s)-s^2H_1-sH_2]-[v^2Q(v,s)-vG_1-vG_2]+Q(v,s)+\frac{1}{vs}Q(v,s)W(v,s)=F(v,s) \quad (49)$$

Eq.(49) can be simplifying as the following

$$Q(v,s)=\frac{s\left(F(s,v)+s^2H_1+sH_2-vG_1-vG_2\right)}{vs^3-sv^3+vs+W(v,s)} \quad (50)$$

Operating with the inverse of DARAT on both sides of Eq.(50) gives

$$u(x,t)=\mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{s\left(F(s,v)+s^2H_1+sH_2-vG_1-vG_2\right)}{vs^3-sv^3+vs+W(v,s)}\right] \quad (51)$$

Where $u(x,t)$ represents the term arising from the known functions $f(x,t), h_1(x), g_1(t), h_2(x), g_2(t)$,and $w(x,t)$.

**5. Application of double ARA transform in solving IEs and IPDEs**

In this section we introduce the solution of some familiar IEs and IPDEs such as Volterra integral equation and VolterraIntegro - partial differential equations.

***Example 5.1***

Let's consider the following Volterra integral equation

$$u(x,t)=\lambda-\gamma\int_0^x\int_0^t u(\rho,\tau)d\rho d\tau. \quad (52)$$

Appling the DARAT to Eq.(52),we have

$$Q(v,s)=\lambda-\mathcal{G}_x\mathcal{G}_t[1 ** u(x,t)]=\lambda-\gamma\left[\frac{1}{vs}Q(v,s)\right] \quad (53)$$

Eq.(53) can be simplifying as the following

$$Q(v,s)=\frac{\lambda vs}{vs+\gamma}. \quad (54)$$

Taking the inverse of DARAT to Eq.(54) , then the solution of Eq.(52) is

$$u(x,t)=\mathcal{L}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{\lambda vs}{vs+\gamma}\right]=\lambda J_0(2\sqrt{\gamma xt}).$$

***Example 5.2***

Let's consider the following Volterra integral equation

$\boldsymbol{4t = \gamma \int_0^x \int_0^t u(x-\rho, t-\tau)u(\rho,\tau)d\rho d\tau}$.(55)
Appling the DARAT to Eq.(55),we have

$\frac{4}{s} = \frac{\gamma}{vs} Q(v,s) \cdot Q(v,s) = \frac{\gamma}{vs}\left(Q(v,s)\right)^2$.(56)

Eq.(56) can be simplifying as the following

$$Q(v,s) = 2\sqrt{\frac{v}{\gamma}} \quad (57)$$

Taking the inverse of DARAT to Eq.(57),we get the solution of Eq.(55)

$$u(x,t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[2\sqrt{\frac{v}{\gamma}}\right] = \frac{2}{\sqrt{\pi\gamma x}}.$$

***Example 5.3***

Let's consider the following Volterra integral equation

$$xe^x - xe^{x-t} = \int_0^x \int_0^t e^{\rho-\tau} u(x-\rho, t-\tau) d\rho d\tau \quad (58)$$

Appling the DARAT to Eq.(58),we have

$$\frac{v}{(v-1)^2} - \frac{vs}{(v-1)^2(s+1)} = \frac{Q(v,s)}{(v-1)(s+1)} \quad (59)$$

Eq.(59) can be simplifying as the following

$$Q(v,s) = \frac{v}{v-1} \quad (60)$$

Taking the inverse of DARAT to Eq.(60),we get the solution of Eq.(58)

$$u(x,t) = \mathcal{L}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{v}{v-1}\right] = e^x$$

***Example 5.4***

Let's consider the following VolterraIntegro - partial differential equations

$$\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t} = -1 + e^x + e^t + e^{x+t} + \int_0^x \int_0^t u(x-\rho, t-\tau) d\rho d\tau \quad (61)$$

With the following initial condition$u(x,0) = e^x$
and the following boundary condition: $u(0,t) = e^t$

By substituting the values of functions $H = \frac{v}{v-1}$, $G = \frac{s}{s-1}$, $F = -1 + \frac{v}{v-1} + \frac{s}{(s-1)} + \frac{vs}{(s-1)(v-1)}$ and $W = 1$ in the general form in Eq.(43), we obtain:

$$Q(v,s) = \frac{-1 + \frac{v}{v-1} + \frac{s}{(s-1)} + \frac{vs}{(v-1)(s-1)} + \frac{vs}{(s-1)} + \frac{s^2}{(v-1)}}{v + s - \frac{1}{s}} \quad (62)$$

Eq.(62) can be simplifying as the following

$$Q(v,s) = \frac{sv}{(v-1)(s-1)} \quad (63)$$

Now , applying the inverse of DARAT to Eq.(63) , then the solution of Eq.(61) is

$$u(x,t) = \mathcal{L}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{vs}{(v-1)(s-1)}\right] = e^{x+t}$$

***Example 5.5***

Let's consider the following Integro - partial differential equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) + \int_0^x \int_0^t e^{x-\rho+t-\tau} u(x-\rho, t-\tau) d\rho d\tau = e^{x+t} + xte^{x+t} \quad (64)$$

With the initial conditions: $u(x,0) = e^x$ , $u_t(x,0) = e^x$

and the boundary conditions: $u(0,t) = e^t$ , $u_x(0,t) = e^t$

Applying the ARA transform to the initial conditions and to the boundary conditions, we have

$$H_1 = H_2 = \frac{V}{v-1}, G_1 = G_2 = \frac{S}{S-1}$$

By substituting the values of functions $H_1 = H_2 = \frac{v}{v-1}, G_1 = G_2 = \frac{s}{s-1}, F = \frac{vs}{(v-1)(s-1)} + \frac{vs}{(v-1)^2(s-1)^2}$ and $W = \frac{v^2 s^2}{(v^2-1)(s^2-1)}$ in the general form in Eq.(50), we obtain

$$Q(v,s) = \frac{vs}{(s-1)(v-1)} \tag{65}$$

Applying the inverse of DARAT to Eq.(65) , then the solution of Eq.(64) is

$$u(x,t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{vs}{(s-1)(v-1)}\right] = e^{x+t}$$

**6. Conclusion**

This paper presents a new double transform method, namely double ARA transform for solving IDEs and IPDEs. The theorems and general properties of the DARAT are discussed and illustrated through examples .The novel approach has revealed great efficiency in tackling integral differential equations and obtaining much higher degrees of accuracy compared with alternative methods. Our future prospects include nonlinear integral- partial differential equations and nonlinear partial differential equations.

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