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A survey of Haar wavelet methods for the solution of PDEs- A Short Review

Inderdeep Singh and Sheo Kumar

Department of Mathematics, Dr. B.R. Ambedkar National Institute of Technology, Jalandhar-144011(Punjab), India.

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ABSTRACT

Wavelet is a recently developed mathematical tool for many problems related to science and engineering. Wavelet also applied in numerical analysis and estimation. In this paper, A survey of Haar wavelet methods to solve PDEs is presented. Moreover, the use of Haar wavelets is found to be accurate, simple, fast, flexible, convenient, computationally attractive.

Introduction

Wavelet analysis and wavelet transform are recently developed mathematical tool for solving the linear and non-linear ODEs, PDEs. Wavelets also applied in numerous disciplines such as image compression, data compression, denoising data, etc. Wavelet methods have been applied to solve different types of PDEs from beginning of the early 1990s(see e.g [81],[82]). Wavelet methods for the solution of differential equations are discussed in many papers(see e.g. [51], [52], [53], [54], [55], [56], [57], [58], [59]). For the solution of ODEs and PDEs , many different approaches such as Taylor-Galerkin method, Collocation method, Finite Element method, Laplace decomposition method, Adomian decomposition methods, Variational Iterative method, Homotopy Perturbation method, Wavelet – Galerkin method, Filter Bank method, etc. are used. Filter bank method for the solution of hyperbolic PDEs is presented in [53]. Kalman filters are applied for the solution of differential equation in [56]. A multilevel wavelet collocation method for the solution of PDEs in a finite domain is presented in [59]. Comparsion of Galerkin collocation and the method of linse for PDE's is presented in [46]. The wavelets based methods for PDEs can be separated into three categories:

In the first category, wavelets are used in the framework of a classical grid adaptive numerical code, to detect where the grid has to be refined or coarsed to optimally represent the solution. Instead of expanding the solution in terms of wavelets, the wavelet transform is used to determine the adaptive grid [119]. In a second category, multiresolution analysis and their associated scale function bases may be used as alternative bases in Galerkin methods. Such methods have convergence properties similar to spectral methods and partial derivative operators are discretize similar as in finite difference. In the third category, the only one which uses the compression properties of wavelet bases, contain specific wavelet methods for partial differential equations.

The class of compactly supported wavelet bases was introduced by Daubechies in 1988. They are an orthonormal bases for functions in $L^2(R)$. Multiresolution analysis (MRA) is the theory that was used by Daubechies to show that for any non-negative integer n there exists an orthogonal wavelet with compact support such that all the derivatives up to order n exist. MRA describes a sequence of nested approximation spaces V_j in $L^2(R)$ such that closure of their union equals $L^2(R)$. MRA has following axioms:

$$\{0\} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(R)$$

$$\overline{\bigcup_{j=\infty}^{j=-\infty} V_j} = L^2(R)$$

$$\bigcap_{j \epsilon Z} V_j = 0$$

$$f \epsilon V_j \text{ if and only if } f\big(2(.)\big) \,\epsilon\, V_{j+1}$$

$$\varphi(x-k)_{k\epsilon Z} \text{ is an orthonormal basis for } V_0$$

Tele:
E-mail addresses: indermaths12@yahoo.com

We define W_j to be the orthogonal complement of V_j in V_{j+1} such that

$$V_{j+1} = V_j + W_j$$

Here, $\varphi_{j,k}(x) = 2^{\frac{j}{2}}\varphi(2^j x - k), k\epsilon Z$ is an orthonormal basis for V_j and φ is the solution of scaling equation

$$\varphi(x) = \sqrt{2}\sum_{k=0}^{D-1} a_k \varphi(2x - k)$$

Where D is the order of wavelet.

In [1], numerical solution of PDEs with decomposition method is presented. In [3], A laplace decomposition algorithm is applied to class of nonlinear differential equations. In [4], laplace decomposition method is used for the solution of Klein-Gordon equation. In [5], laplace decomposition method is used for the solution of nonlinear coupled partial differential equations. In [6], numerical solution of duffing equation by the laplace decomposition algorithm. Solution of linear and nonlinear fractional diffution and wave equations by adomian decomposition method is presented in [7]. Also, adomian decomposition methods are applied to find the solution of various problems related to science and engineering (see e.g. [8],[9],[10],[11],[15],[19]). In [16],[17],[18], adomian decomposition method is used to find the solution of fractional differential equations. In [20], variational iterative method (VIM) is applied to find the solution of multi-order fractional differential equations. In [35], variational iteration method is applied for the solution of Burger's and coupled Burger's equations. In [21],[22], homotopy perturbation method is applied to find the solution of fractional differential equations. In [14], the decomposition approach is applied to problems related to inverse heat conduction. In [12], wavelet-based method is applied to find the solution of nonlinear evolution equation. In [13], finite-element method for the solution of one dimensional parabolic systems is presented. In [23], [29] Taylor-Galerkin method is applied to find the solution of Burger's equation and convective transport problems. In [32], three- step wavelet Galerkin method for parabolic and hyperbolic partial differential equations is presented. In [33], time accurate solution of Korteweg–de Vries equation using wavelet Galerkin method is presented. Wavelet-Galerkin method is also used for the solutions of quasilinear hyperbolic conservation equations in [34]. In [38], Wavelet-Galerkin method is applied for the solutions of one and two dimensional partial differential equations. In [36], Taylor-Galerkin and Taylor-collocation methods for the numerical solutions of Burgers' equation using B-splines is presented. Wavelet-Galerkin method is also applied for ODEs in [42],[43]. In [40], Wavelet-Galerkin method is applied for the solution of operator equation. In [41], wavelet-Galerkin method is applied for two-point boundary value problems. Collocation methods is applied for the solution of partial differential equations (see e.g.[44],[45],[46],[47],[48],[49],[50]). Nonlinear PDEs has wide variety in scientific applications such as plasma physics, solid state physics, optical fibres, biology, chemical kinetics and fluid dynamics. Many powerful methods are used to find the soliton and multi-soliton solutions of PDEs. Such methods are inverse scattering method [62], bilinear transformation method [63], tanh method [64],[69], extended tanh method [65], tanh-coth method [71], exp-function method [68], homogeneous balance method [67].

In [52], the Gaussian wavelet is applied for the solution of differential equation. In [54], Daubechies wavelet is used. Numerical difficulties appear in the treatment of nonlinearities, where integrals of products of wavelet and their derivatives must be computed. This can be done by introducing the connection coefficients(see e.g.[54],[58]), but this method is applicable only for a narrow class of equations. An adaptive method for computing such integrals is presented in [58]. Among the different wavelet families, mathematically most simple are the Haar wavelets [2]. Due to the simplicity of Haar wavelets, it is used for solving PDEs. The Haar wavelet methods provides better solutions than classical ones. Haar wavelet methods for solving PDE has several advantages:

1. Solution obtained with these methods are very accurate than other known methods.
2. It is very convenient methods for solving the boundary value problems.
3. It has small computational time for solving ODEs, PDEs and Integral equations.
4. It is very simpler than other known methods.

Haar wavelet :

Beginning from 1980s, wavelets have been used for solution of PDE. The wavelet algorithms for solving PDE are based on the Galerkin techniques or on the collocation methods. Evidently all attempts to simplify the wavelet solutions for PDE are welcome. One possibility for this is to make use of Haar wavelet family. Haar wavelets which are Daubechies of order 1, are consists of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. A drawback of Haar wavelets is their discontinuity. Since the derivatives do not exist in the breaking points, it is not possible to apply the Haar wavelets for solving PDEs directly. There are two possibilities to overcome these situations:

(a) One way is to regularize the Haar wavelets with interpolating splines(e.g B-Splines), This approach has been applied in [60].

(b) The other way is to make use of the integral method, which was proposed in [61].

The set of Haar functions is defined as a group of square waves with magnitude ±1. The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. They are defined in the interval [0,1].

$$h_i(x) = \begin{cases} 1 & for\ \alpha \le x < \beta \\ -1 & for\ \beta \le x < \gamma \\ 0 & otherwise\ in\ [0,1). \end{cases} \quad (1)$$

where $\alpha = \frac{k}{m}, \beta = \frac{k+0.5}{m}, \gamma = \frac{k+1}{m}$. Integer $m = 2^j$, $(j = 0, 1, 2, 3, \dots \dots \dots \dots, J)$ indicates the level of the wavelet. $k = 0,1,2, \dots \dots, m-1$ is the translation parameter. Maximal level of resolution is J. The index i is calculated according the formula $i = m + k + 1$, in the case of minimal values $m = 1, k = 0$ we have $i = 2$. The maximal value of i is $i = 2M$. where $M = 2^J$. It is assumed that the value $i = 1$, corresponds to the scaling functionin [0,1].

$$h_1(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & otherwise \end{cases} \quad (2)$$

Let us define the collocation points $x_l = (l - 0.5)/2M$, $(l = 1,2,3, \dots \dots ,2M)$ and discredits the Haar function $h_i(x)$. In the collocation points, the first four Haar functions can be expressed as follow:

$h_1(x) = [1,\ 1,\ 1,\ 1]$, $h_2(x) = [1,\ 1, -1, -1]$, $h_3(x) = [1, -1,\ 0,\ 0]$, $h_4(x) = [0, 0, 1, -1]$.

We introduce the notation:

$$H_4(x) = [h_1(x), h_2(x), h_3(x), h_4(x)]^T$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

where $H_4(x)$ is called Haar coefficient matrix . It is a square matrix of order 4.

In this way we get the coefficient matrix $H(i, l) = (h_i(x_l))$, which has the dimension 2M×2M.

Let us integrate equation (1),

$$q_i = \int_0^x h_i(x)dx, \quad (3)$$

In the collocation points equation (3) gets the form

$$Q(i, l) = q_i(x_l),$$

where Q is a $2M \times 2M$ matrix.

Chen and Hsiao [61] presented this matrix in the form $Q_n = P_n H_n$, where $P_n H_n$ is interpreted as the product of the matrices P_n and H_n, called Haar integration and coefficient matrix, respectively.

The operational matrix of integration P, which is a 2M square matrix, is defined by the equation:

$$P_{i,1}(x) = \int_0^{x_l} h_i(x)dx \quad (4)$$

$$P_{i,v+1}(x) = \int_0^x P_{i,v}(x)dx, \quad (5)$$

where $v = 1, 2, 3, 4, \dots \dots$

These integrals can be evaluated using equation (1) and first four of them are given:

$$P_{i,1}(x) = \begin{cases} x - \alpha & for\ x \in [\alpha, \beta) \\ \gamma - x & for\ x \in [\beta, \gamma) \\ 0 & elsewhere \end{cases} \quad (6)$$

$$P_{i,2}(x) = \begin{cases} \frac{1}{2}(x-\alpha)^2 & for\ x \in [\alpha,\beta) \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma-x)^2 & for\ x \in [\beta,\gamma) \\ \frac{1}{4m^2} & for\ x \in [\gamma,1) \\ 0 & elsewhere \end{cases} \quad (7)$$

$$P_{i,3}(x) = \begin{cases} \frac{1}{6}(x-\alpha)^3 & for\ x \in [\alpha,\beta) \\ \frac{1}{4m^2}(x-\beta) - \frac{1}{6}(\gamma-x)^3 & for\ x \in [\beta,\gamma) \\ \frac{1}{4m^2}(x-\beta) & for\ x \in [\gamma,1) \\ 0 & elsewhere \end{cases} \quad (8)$$

$$P_{i,4}(x) = \begin{cases} \frac{1}{24}(x-\alpha)^4 & for\ x \in [\alpha,\beta) \\ \frac{(x-\beta)^2}{8m^2} - \frac{(\gamma-x)^4}{24} + \frac{1}{192m^4} & for\ x \in [\beta,\gamma) \\ \frac{(x-\alpha)^2}{8m^2} + \frac{1}{192m^4} & for\ x \in [\gamma,1) \\ 0 & elsewhere \end{cases} \quad (9)$$

Chen and Hsiao [61] showed that the following recursive formula for operational matrix of integration holds:

$$P_{\mu\times\mu} = \begin{bmatrix} P_{\mu/2} & -\frac{1}{2\mu}H_{\mu/2} \\ \frac{1}{2\mu}H_{\mu/2}^{-1} & 0 \end{bmatrix}$$

Function approximations:

Any square integrable function $u(x)$ in the interval [0,1] can be expanded by a Haar series of infinite terms:

$$u(x) = \sum_{i=0}^{\infty} c_i h_i(x) \quad , \quad i \in \{0\} \cup N \quad (10)$$

where the Haar coefficients c_i are determined as:

$$c_0 = \int_0^1 u(x) h_0(x) dx, \quad c_n = 2^j \int_0^1 u(x) h_i(x) dx ,$$

$i = 2^j + k\ ,\ j \geq 0,\ 0 \leq k < 2^j\ ,\ x \in [0,1)$ such that the following integral square error ε is minimized:

$$\varepsilon = \int_0^1 [u(x) - \sum_{i=0}^{m-1} c_i h_i(x)]^2 dx, \quad m = 2^j\ ,\ j \in \{0\} \cup N.$$

Usually the series expansion of (10) contains infinite terms for smooth $u(x)$. If $u(x)$ is piecewise constant by itself , or may be approximation as piecewise constant during each subinterval , then $u(x)$ will be terminated at finite m terms, that is:

$$u(x) = \sum_{i=0}^{m-1} c_i h_i(x) = c_{(m)}^T h_{(m)}(x)$$

where the coefficients $c_{(m)}^T$ and the Haar function vector $h_{(m)}(x)$ are defined as:

$$c_{(m)}^T = [c_0, c_1, c_2, \dots\dots, c_{m-1}] \quad \text{and} \quad h_{(m)}(x) = [h_0(x), h_1(x), \dots\dots, h_{m-1}(x)]^T$$

where T is the transpose.

Haar wavelets have been applied extensively for signal processing in communications and research areas in physical science, mathematical science, chemical science and biological science. In [104], Haar wavelet method is applied for the solution of differential equations. Haar wavelet methods applied to many PDEs because the solution obtained by these methods are more accurate than other known methods and these methods take small computational time to obtained the solution of different types of PDEs. In solving ODEs by using Haar related method, an operational matrix of integration based on Haar wavelet had derived in [61]. After it, a simple matrix method to solve ODEs is presented in [72]. After it, A simple method to solve PDEs is presented in [73]. In [74], Haar wavelet method is applied for the solution of wave equation. In [75], Haar wavelet method is applied to solve the convection-diffusion equation. In [76], Haar wavelet method is applied to solve nonlinear stiff equations. In [77], Lepik presented the numerical solution of evolution equation by Haar wavelet method and in [83], Cattani presented the Haar wavelet technique for evolution problems. The Chan-Allen equation arises in many scientific applications such as mathematical biology, quantum mechanics and plasma physics. The Chan-Allen equation serves as a model for the study of phase separation in isothermal, isotropic, binary mixtures such as molten alloys. Haar wavelet method is used to solve the Chan-Allen equation in [88] and compared the result of this method with the exact solution. In [90], Haar wavelet method is used to find the solution of some well known nonlinear parabolic partial differential equations. The equations include the Nowell-whitehead equation, Chan-Allen equation, Fitzhugh-Nagumo equation, Fisher's equation, Burger' equation and Burger-Fisher's equation. All these are nonlinear parabolic equation and compared the results of these equations by this method with the exact solution, which shows that Haar waveet method is more accurate, fast, flexible and convenient than other known methods. In [78], Haar wavelet direct method is applied to solve variational problem. In [79], Haar wavelet approach is proposed to solve the travelling wave equation. In [80], numerical solution of heat equation with Haar wavelet is presented. Three dimensional analysis of functionally graded plate based on the Haar wavelet method is presented in [84]. Solution of PDEs with the aid of two-dimensional Haar wavelet is presented in [85]. In [86], Fitzhugh-Nagumo equation is solved with Haar wavelet method. The Fitzhugh-Nagumo equation describes the dynamical behavior near the bifurcation point for the Rayleigh-Benard convection of binary fluid mixtures [111]. It is an important nonlinear reaction-diffusion equation and applied to model the transmission of nerve impulse. It is also used in biology and the area of population genetics in circuit theory. In [73], Haar wavelet method is applied to solve Fisher's equation. The Fisher's equation combines diffusion with logistic nonlinearity. This equation arises in chemical kinetics and population dynamics, which includes problems such as nonlinear evolution of a population in a one-dimensional habitat, neutron population in a nuclear reaction. The Newell-whitehead equation describes the dynamical behavior near the bifurcation point for the Rayleigh-Benard convection of binary fluid mixtures [111]. The Newell-whitehead equation is solved by Haar wavelet method in [90]. The Burger's equation is a nonlinear PDE. The Burger's equation is used in disciplines as a simplified model for turbulence, shock wave formation, mass transport and boundary layer behavior. The solution of Burger's equation by Haar wavelet is presented in [90]. The solution of Burger-Fisher's equation by Haar wavelet mehod is also presented in [90]. Haar wavelet and related methods are used for Estimating depth profile of soil temperature in [87], Bucking of elastic beams in [93], Optimal control problems in [92], Exploring vibrations of cracked beams in [94], Optimal control problem of time varying state-delayed systems in [95], Solve optimal control and parameter estimation of linear systems in [102], Solve 2D and 3D poisson equations and biharmonic equations in [96], Solve Sine-Gordan and Klein-Gordan equations in [91], Evolution problems in [83], Solve nonlinear oscillatior equations in [97], Solve stiff systems from nonlinear dynamics in [98], Solve Generalized Burger-Huxley equation in [99], Solve Telegraph equation in [100], Solution of nonlinear Korteweg-de Vries-Burger's equation in [101]. In [103], rationalized Haar functions is used to find the solution of linear partial differential equations. In [105], Haar wavelet method is used for solving linear and nonlinear wave-type equations. Numerical solution for nonlinear murray equation using the operational matrices of the Haar wavelets methods in [107]. State analysis of time-varying singular nonlinear systems with the help of Haar wavelets is presented in [108]. State analysis of time varying singular bilinear systems by using Haar wavelets is presented in [109]. In [110], status and achievements of Haar wavelet transform is presented. In [112], a clear procedure for the solution of finite-length beam and convection-diffusion equation by Haar wavelet technique is presented. In [113], Haar wavelet methods is used to solve the two dimensional Burger's equation. The Haar wavelet based discretization techniques for the solution of differential equations is discussed in [114] and also, the linear and nonlinear analysis problems as well as optimal design problems are considered. In [115], discretization method based on Haar wavelets is presented. In [106], numerical method to solve partial differential equations with operational matrices is presented. Aziz et al. [116] presented numerical solution of second-order boundary value problems by collocation method using Haar wavelets. In [23], Wavelet-Taylor Galerkin method for the Burgers equation is presented. In [28], Time-accurate solution of advection-diffusion problems by wavelet-Taylor-Galerkin method is presented. A three- step wavelet Galerkin method for parabolic and hyperbolic partial differential equations is presented in [32]. Also, Time accurate solution of Korteweg–de Vries equation using wavelet Galerkin method is presented in [33]. In [46], Comparsion of Galerkin collocation method and the method of linse for PDE's is presented. In [12], wavelet-based method for numerical solution of nonlinear evolution equations is presented. Fast wavelet based algorithms for linear evolution equations is presented in [30]. Numerical resolution of non-linear partial differential equations using wavelet approach is presented in [31]. In [117], Haar and Legendre wavelets are applied to find the numerical solution of parabolic partial differential equations. In [118], Wavelet collocation methods are applied to find the numerical solution of elliptic boundary value problems.

Review shows that the Haar wavelet method is efficient and powerful in solving wide class of linear and nonlinear PDEs. The main advantages of Haar wavelet method are sprase representation, fast transformation and possibility of implementation of fast algorithms if the matrix representation is used. The comparison of the solution obtained by Haar wavelet method with the solution of other known methods shows that the results obtained by Haar wavelet method are more accurate than other solutions.

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Inderdeep Singh received his Masters (M.Sc.) degree in Mathematics from Guru Nanak Dev University, Amritsar, Punjab, India in July 2003. At present, he is pursuing his Ph.D.under the supervision of Professor Sheo Kumar, Department of Mathematics, Dr. B. R. Ambedkar National Institute of Technology, Jalandhar- 144011, Punjab, India. His research interests are numerical analysis, partial differential equations, integral equations and is working on numerical solution of different types of partial differential and integral equations using Haar wavelet.

Dr. Sheo. Kumar is Professor, Department of Mathematics, Dr. B.R. Ambedkar National Institute of Technology, Jalandhar 144011, Punjab, India. He has received M.Sc. degree in Mathematics in 1973, from Indian Institute of Technology, Kanpur, India and Ph.D. degree in 1981, from Indian Institute of Technology, Delhi, India. His research interests are numerical analysis, finite element methods, wavelet methods for partial differential and integral equations.