Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 78 (2015) 29922-29927

ISSN: 2229-712X

Inner product space in Fourier approximation

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ARTICLE INFO

Article history:

Received: 22 March 2014;

Received in revised form:

11 January 2015;

Accepted: 23 January 2015;

Keywords

Inner product,

Fourier,

Element.

ABSTRACT

Let f be an element and S a subset of a normed linear space x. A basic Problem of approximation theory is to find an element of S, which is as close as possible to f, i.e. seek an element S* of S such that ‖ f - S*‖ ≤ ‖f - s‖ for all S* in S. This work seeks to use the fourier approximation method using the Inner product space to obtain a best approximation of functions. The fourier approximation in calculus is shown to be a special least square approximation. Define an inner product and norm on c[-π, π] by the equation.

$$f.g = \int_{-\pi}^{\pi} f(x)\, g(x) dx \; and \;\; \| f \| = \left[\int_{-\pi}^{\pi} f^2(x) dx\right]^{\frac{1}{2}} L$$

Let J_n be the set of all

trigonometric polynomial of degree at most n. We seek S* of J_n such that the integral $\int_{-\pi}^{\pi}$ [f (x) - S* (x)] ^{2}dx is minimum. The inner product space, or orthogonal projections, least square approximations and Fourier approximations are defined. The inner product space and orthogonal projection are applied to the least square approximation and subsequently to Fourier approximation in calculus.

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Introduction

In real life situation, exact values are needed to solve problems. Efforts are made to obtain values that are close which subsequently yields the"almost" exact results. The usefulness of approximation in mathematics in particular and life situations in general cannot be overemphasized.

Many problems in the physical sciences and engineering involve the approximation of a function f by another function g. If f is in [a,b], then usually g is chosen from a given subspace w of c[a,b] where c[a,b] is the inner product space of all continuous functions of f.

Definition of terms

Vector Space: Let V be a set on which two operations (vector addition and vector multiplication) are defined. If the following axioms are satisfied for every u,v and w in V and every scalar c and d, then V is called a vector space

u+visinV

u+v=v+u

u+(v+w)=(u+v)+w

v has a zero vector O such that for every u in V, u + 0 = u.

For every U, there is a vector in V denoted by –u such that u + (-u) = O

cu is in V

c(u+v)=cu+cv } Distributivity

(c+d)u=cu+dv }

c (du) = (cd)u Associativity

and

I(u)=u Scalar identity

Linear Dependence/Independence

A set of vectors S = {v_1, v_2, . . . v_k) in a vector space V is called linearly independent if the vector equation $c_l v_l$+ C_2V_2 + ….+ C_kV_k = 0 has only the trivial solution, c_l = 0, c_2= 0 ,….,c_k = 0. If there are also non-trivial solution, then S is called linearly dependent.

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Spanning Set of a Vector Space

Let S = $\{v_1, v_2, ..., v_k\}$be a subset of a vector space V, the set S is called a spanning set of V if every vector in V can be written as a linear combination of vectors in S. S spans V.

Basis: A set of vector S ={ v1, v2, ... vn} in a vector space V is called a basis for V if S spans V and S is linearly independent.

Dimension of a vector space:

If a vector space V has a basis consisting of n - vectors, then the number n is called the dimension of V denoted by dim (V) = n. If V consists of the zero vector alone, the dimension of V is defined as zero.

Length of a Vector in R^n: The length of a vector V = $(V_1, V_2, \ldots V_n)$in R^n is given by $\| \mathbf{V} \| = \sqrt{(\mathbf{V}_1^2 + \mathbf{V}_2^2 + \cdots + \mathbf{V}_n^2)}$ and

The distance between two vectors u and v in R^n is d(u,v)=$\| \mathbf{U} - \mathbf{V} \|$.

Furthermore

d(u,v) ≥ \0; d(u,v) = 0 iff u = v and d(u,v) = d(v,u).

Dot Product: The dot product of u = $(u_l, u_2, \ldots, u_n)$

and v = $(v_1, v_2, \ldots, V_n)$is the scalar quantity

u.v = $u_1v_1+u_2v_2+\ldots+u_nv_n$

Two Vectors u and v in R^n are orthogonal if u.v = o.

Normalizing Vectors:

If V is a non-zero vector in R^n, then the vector u = v/ $\| \mathbf{v} \|$.with length 1 and has the same dimension as v.u is called the unit vector in the direction of v. The process of finding the unit vector in the direction of v is called normalizing the vector v.

Cross Product of two Vectors:

Let u = u_{li}; + u_{2j} + u_{3k} and v = v_{1i} + v_{2j} + v_{3k} be vectors in R^3. The cross product of u and v is the vector u x v = $(u_2v_3 - u_3v_2)$i - $(u_lv_3 - u_3v_1)$j + $(u_lv_2 - u_2v_1)$ki.e.

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Inner Product Space

Let u,v and w be vectors in a vector space V and let c be any scalar. An inner product on V is a function that associates a real number <u,v> with each pair of vectors u and v and satisfies the following axioms

<u,v> = <v,u>

<u,v + w> = <u,v> x <u,w>

c <u,v> = <cu, v> and

<u,v>>0 and <v,v> = 0iff v = 0

A vector space V with an inner product is called an inner product space.

Properties of inner Products

Let u, v and w be vectors in an inner product space V and let c be any real number. Then

<0,v> = <v,0> = 0

<u + v, w> = <u, w> + <v,w>

<u,cv> = c <u,v> .

Let u and v be vectors in an inner product space V,

The norm (length) of u is $\| \mathbf{U} \|$= $\sqrt{< \boldsymbol{u}, \boldsymbol{u} >}$

The distance between u and v is d (u,v)= $\| \mathbf{u} - \mathbf{v} \|$. The angle between two nonzero vectors u and v is given by

Cos $\boldsymbol{\theta}$ = <u><u,v></u> , $0 \leq \boldsymbol{\theta} \leq \boldsymbol{\pi}$ and u and v are orthogonal if <u,u> = 0.

|| u |||| v ||

Least Square Approximation

Let f be continuous *on [a,b],* and let w be a subspace of c[ab]. A function *g in w is* called a least square approximation of f with respect to w if the value of I= $\int_a^b [\mathbf{f(x) - g(x)}]^2]\mathbf{dx}$ is a minimum with respect to all other functions w. If the subspace w is the entire space c[a,b] then g(x) = f(x) and hence I = 0.

Now to approximate a function f by another function g where f is in c[a,b], We choose g from a given subspace w of c[a,b]. One way to define how one Function best approximates another function would be that "the area bounded by the graph of f and g in [a,b] i.e.

Area = $\int_a^b [\mathbf{f(x) - g(x)}\ ]^2 dx$

The function g is the least square approximation of f with respect to the inner product space w. Certain functions may be represented as the sum of a converged infinite series.. If the function f(x) is periodic say f(x +2l) = f(x) for all x, one very good way of representing the function is by a trigonometric series. Such a series is of the form

$\frac{a_0}{2} + \sum_{n=0}^{\infty}\left(a_n \cos\frac{n\pi x}{l}\right) bn \sin\frac{n\pi x}{l}$ - - - - - - - - - - - - - - - - - - - (i)

Or

$\frac{a_0}{2} + \sum_{n=0}^{\infty} \cos nx + bn \sin nx$ - (ii)

Where the period is 2π

Suppose that the following integrals exist

$a_n = \frac{1}{l}\int_{-1}^{1} \mathbf{if(x)}\ \cos\frac{n\pi x dx}{1}$, n = o, 1,2 _ _ _ _ _ _ _ _ _ _ _ _ _ _ (iii)

$b_n \frac{1}{l}\int_{-1}^{1} \mathbf{if(x)}\ \text{Sin}\ \frac{n\pi x dx}{l}$,, n = o, 1,2, _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ (iv)

The numbers a_o, a_1, ---, b_1, b_2, …, are called the Fourier coefficients of the function f(x) and the series a_o+ ∞ [a_n, cos nx + bn sin nx] i.e. (ii) is called Fourier series of f.

In Fourier approximation, we consider the functions of the form

$g(x)=a_o/2+a_l \cos x + \dots + a_n \cos nx + b_1 \sin x + b_n \sin nx$

in the subspace W of c[0,2π] spanned by the basis

s = { 1, Cos x, Cos 2x, …, Cos nx, sin x, sin 2x,… sin nx

where the 2n +1 vectors are orthogonal in the inner product space c[0,2π] because (f. g) = $\int_0^{2\pi} \mathbf{f(x)}$ g(x) dx = 0, f ≠ g.

Normalizing each function in the basis, gives the orthogonal basis.

B = {W_0, W_I, ... W_n, W_{n-1}, ... W_{2n} = }

{ $\frac{1}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}}\cos x, \dots, \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin x, \dots \frac{1}{\sqrt{\pi}}\sin nx$

With theoren 4.1 below together with the orthogonal basis, we write

g(x) = <f, w_o>w_o+ <f, w_1>w_1 +... + <f, w_{2n}> w_{2n}.

The coefficients a_o, a_1, … a_n, b_1, ...,$b_n$for g(x) is the equation

$g(x) = a_o/2 + a_l \cos x + \dots + a_n \cos nx + b_1 \sin x + \dots + b_n \sin nx$ are given by the following integrals.

$a_0 = \langle f, w_0\rangle \frac{2}{2\pi} = \frac{2}{2\pi}\int_0^{2\pi} f(x)\frac{2}{2\pi}dx = \frac{x}{\pi}\int_0^{2\pi} f(x)dx$

$a_1 = \langle f, w_0\rangle \frac{1}{\pi} = \frac{1}{\pi}\int_0^{2\pi} f(x)\frac{1}{\pi}\cos x dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x dx$

$a_1 = \langle f, w_0\rangle \frac{1}{\pi} = \frac{1}{\pi}\int_0^{2\pi} f(x)\frac{1}{\pi} sinnxdx = \frac{1}{\pi}\int_0^{2\pi} f(x) sinnxdx$ g(x) is the nth order Fourier approximation of f on the interval [o, 2π].

Theorems

Theorem: Least square approximation.

Let f be continuous on [a,b] and let w be a finite dimensional subspace of c[a,b]. The least square approximating function of f with respect to w is given by g = < f, w_l>w_l+<f,w_2>w_2+... +<f,w_n,>w_n

Where B = { w_l, w_2, ..., w_n) is an orthogonal basis for w.

Fourier Approximation

On the interval [o, 2π], the least square approximation of a continuous function f with respect to V, the vector space spanned by { 1, cos x, ..., cos nx, sin x, ... sin nx is given by}

g(x)=a_o+a_1 cos x+... +a,,cosnx+bi ... bnare

$a_i = \frac{1}{\pi}\int_0^{2\pi} f(x)$dx, $a_j = \frac{1}{\pi}$ $\int_0^{2\pi} f(x)$f (x)cos jx dx,

$b_j = \frac{1}{\pi}$ $\int_0^{2\pi} f(x)$f(x)sinjxdx, j = 1,2, ... n

Proposition: Suppose that f(x) is piecewise continuous in a $\leq$x $\leq$ b, thenf (x) is bounded there.

If the function g(x) is piecewise continuous in the interval a <x < b. Then the integral I(λ) = $\int_a^b g$ (.£) sin λ t dt tends to zero as $\lambda \to \infty$

i.e. lim I(λ) = $\int_a^b g$ (t) sin λt dt.

Illustration

Obtain the approximation of g(x) = a_o+a_lx+a_2 x^2 for f(x) = e^x, 0$\leq$x $\leq$1 To obtain this approximation g(x), we need to find the value of a_o, a_l and a_2 that minimizes the value of

I = $\int_0^1 [f(x)$ - g(x)$]^2$ dx.

ie I = $\int_0^1 e^x$ [- (a_o+a_1x+a_2x^2)] ^{2}dx

I = $\int_0^1 (e^x - a_o - a_l x - a_2 x^2)^2$ dx

Integrating and setting the partial derivation of I with respect to a_o, a_l, a_2 equal to zero ie.

$6a_o$+ $3a_1$ +$2a_2$ - 6e + 6 -------------------------- (i)

$6a_o$+ $4a_1$ +3_{a2} - 12 ------------------------------- (ii)

$20a_0$ + $15a_1$ + 12,'12- 60e + 120 --------------- (iii)

i.e. (i), (ii) and (iii) becomes

$6a_o$+ $3a_1$ +$2a_2$ - 6 (e -1) -------------------------- (iv)

$6a_o$+ $4a_1$, +$3a_2$ = 12 ------------------------------- (v)

$20a_o$+ $15a_1$ + $12a_2$ = 60 (e -12) ------------------ (vi)

The solution of this system *is*

a_o= - l05 + 39e = 1.013 -------------------------- (vii)

a_1= 588 - 216e = 0.851 --------------------------- (viii)

a_2= -570 + 210e = 0.839 ------------------------- (ix)

Hence g(x) = 1.013 + 0.851x +0.839x^2 -------- (x)

Now expressing I in vector form is done using inner product.

(f.g) = $\int_0^1 f(x)g(x)$dx

Then with the inner product

I = $\int_0^1 [f(x)$ - g(x)$]^2$dx = < f - g, f -g >

= | f - g $|^2$

This is the least square approximation.

Now on the interval approximation square approximation of the continuous function f with respect to the vector space spanned by

{1, cos x ... , cos nx, sin x, ... sin nx} -------------------------- (xi)

Is given by

a

$g(x) = \frac{a_0}{2} + a_1 \cos x + \ldots + a_n \cos nx + b_1 \sin x + \ldots + b_n \sin nx$ ------ (xii)

i.e $(gx) = \frac{a_0}{2} + a_1 \cos x$ + $a_2 \cos 2x$ + a_2 cos2 x in this problem

But $a_o = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$,$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x dx$ and $a_2 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos 2x$ d x

i.e $a_o = \frac{1}{\pi}\int_0^{2\pi} f(x)e^x$,$a_1 = \frac{1}{\pi}\int_0^{2\pi} e^x \cos x\, dx$ and $a_2 = \frac{1}{\pi}\int_0^{2\pi} e^x \cos 2x$ d x

Then,

$a_o = \frac{1}{\pi}\int_0^{2\pi} e^x dx = \frac{2}{\pi}$ sin bx

$a_1 = \frac{1}{\pi}\int_0^{2\pi} e^x \cos 2x dx = \frac{2}{\pi}$ sin bx

$a_2 = \frac{1}{\pi}\int_0^{2\pi} e^x \cos 2x dx = \frac{2}{5\pi}$ sin bx

$b_1 = \frac{1}{\pi}\int_0^{2\pi} e^x \cos 2x dx = \frac{2}{\pi}$ sin x

$b_2 = \frac{1}{\pi}\int_0^{2\pi} e^x \sin 2x dx = \frac{-4}{5\pi}$ sinb$x$ …………….*(xiii)*

Now using the results in (xiii) above with (xii) results to

T* (x) = $\underline{2\sin h\pi}$ $[\frac{1}{2} - \frac{1}{2}\cos x + \frac{1}{-2}\sin x + \frac{1}{5}\cos 2x - \frac{2}{5}\sin 2x]$ ------ (xiv)

(xiv) is the best approximation for the problem.

Summary/ Conclusion

If the function f is an element and S is a subset of a normed linear space X , then given an element s^* of S, the approximation problem states that we need find an element of S that is as close as possible to f such that

$\|f - s^*\| \leq \| f - S\| \quad \forall \; s^* \in S$

The Fourier approximation has been applied using the inner product space to obtain the best approximate of a given polynomials in a given interval.

Relevant terms in vector space, inner product and least square approximations are defined and related to the Fourier approximation. Theorems/propositions are stated and an illustrative example included. It has been shown that the best approximation to problem solutions can be obtained by the method discussed above.

References

Beyer, W:H. (1986): Standard Mathematical Tables. 26th Edition CRC Press Florida.

Cody, W.J. et al. (1988) : Rational Approximation Using Linear Equations "Numerical Mathematics, Volume 12, pages 242 - 251.

Dahlquist G. and Bjorck A. (1990). Introduction to Discrete Mathematics. Brooks/Cole Publishing Company California.

Iyahen, S.O. (2001): Lecture Notes Series - National Mathematical Centre, Abuja.

Jensen, A.J. (1995): Methods of Computation (Linear Space Approach to Numerical Analysis). Scolt Foresons and Company Dakar.

Larson, R.E. and Edward, B.H. (1988): Elementary Linear Algebra. D.C. Health and Company Letington.

Lue, A.S.T. (1974): Basic Pure Mathematics 11 Van Nostrand Rainhold Company London.

Morris A.O. (1982) Linear Algebra: An Introduction. Van Nostrand Rainhold Company London.

Rilvin T.J. (1989): An Introduction to the Approximation of Functions. Blusdell.

Rilvin T.J. (1969): Polynomicals of best Uniform Approximation to Certain Functions. Numerishe Mathematic Vol V. Pages 345-349.

Chikwendu, C.R and Emenonye ,C.E. (2013) Chebyshevs polynomial approximation of an n- warehouse stoch allocation model. International journal of Engineering Innovation Research.