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New simple stationary solution of a fluid queue driven by an M/M/1 Queue

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ABSTRACT

In this paper, the motivation has been to give new explicit expressions for the stationary distribution of the buffer content for a fluid queue driven by an M/M/1 queue. The expression of the buffer content distribution is derived in terms of modified Bessel functions. Finally, numerical assessment is presented to visualize the buffer content distribution.

Introduction

A fluid queue describes the behavior of a fluid level in a storage device (called a buffer). The input and output rates in a fluid queue are regulated by an external stochastic process, which is often referred to as a stochastic environment (or background). The fluid queue is usually called a fluid model driven by the stochastic process. In the last two decades, fluid queues have attracted considerable interest and have been well accepted as a useful mathematical tool for modeling, for example, packet voice and video systems with or without background data, computer networks including call admission control, traffic shaping and modeling of TCP, and production and inventory systems. Readers may refer to Anick, Mitra and Sondhi (1982), Mitra (1988) and Kulkarni (1997) for more details.

Many researchers have studied fluid models driven by finite state space Markov processes that modulate the input rate in the fluid buffer. In addition, the researchers mostly used the classical spectral analytic method to determine the stationary distribution of the buffer content. Kulkarni (1997) gave a complete survey for the early literature of the fluid queue and provided a general framework for analyzing the model with finite state Markov environment by means of spectral analysis.

The case where the state space is infinite has been discussed by researchers Virtamo and Norros (1994). They analyzed a fluid queue driven by an M/M/1 queue and proposed a spectral decomposition method. Adan and Resing (1996) considered the background process as an alternating renewal process, corresponding to the successive idle and busy period of the M/M/1 queue. Van Doorn and Scheinhardt (1997) studied a fluid queue fed by an infinite state birth-death process. They solved the infinite differential system by the use of orthogonal polynomials with respect to a signed measure which explicitly given in the case of M/M/1 queue and leads the integral expression obtained by Adan and Resing (1996), and Virtamo and Norros (1994). Sericola and Tuffin (1999) considered case that is more general where the fluid queue is driven by a Markovian queue with the hypothesis that only one state has a negative drift. Using the differential system, the fluid level distribution is obtained in terms of a series and coefficients computed by means of recurrence relations.

Moreover, this study is extended to the finite buffer case by Sericola (2001). The transient distribution of that fluid queue and the convergence to the stationary distribution has been analyzed by Barbot and Sericola (2001). Barbot and Sericola (2002) obtained an analytic expression for the joint stationary distribution of the buffer level and the state of the M/M/1queue. Finally, based on continued fraction approach, Parthasarathy and Vijayashree (2002) presented solutions of a fluid queue driven by an M/M/1 queue with a general boundary condition when compared to the model discussed by Adan and Resing (1996) and Virtamo and Norros (1994) .

Recently, many authors have introduced a new class of fluid queue with vacations and disasters (see, Ammar (2014 a,b), Mao et al (2010, 2011, 2012), and Baek et al (2013)).

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As mentioned above, many author have discussed the stationary solution of the fluid queue driven by an M/M/1 queue but there is no comparison among the results for it. Thus, in this paper our motivation for this study is to derive a new explicit expression for the buffer content distribution and compared the obtained results with some previous results to clarify the efficiency of our results. We investigated the solution by defining the generating function in a special way which results in a simple differential equation. Using the properties of Bessel functions in the solution of this differential equation, the complete solution can be extracted in a direct way.

The reset of this paper is organized as follows: Section 2 gives a description of the fluid queue model. In Section 3, transient solution of buffer content distribution is derived. We perform sensitivity analysis through numerical experiments in Section 4.

Model Description

Consider a fluid model driven by a single server queueing process with state-dependent arrival and service rates. It consists of an infinitely large buffer in which the fluid flow is regulated by the state of the background queueing process. Denote the background queueing process by $\{X(t), t \geq 0\}$ taking values in $\{0,1,2,...\}$ where $X(t)$ denotes the number of customers in the system at time t. Let λ_n and μ_n denote the mean arrival and service rates, respectively, when there are n customers in the queue. The arrivals are of Poisson fashion and the service times are exponentially distributed.

During the busy period of the server, a fluid commodity which we refer as credit accumulates in an infinite fluid buffer at a constant rate $r > 0$. The credit buffer depletes the fluid during the idle periods of the server at a constant $r_0 < 0$ as long as the buffer is nonempty.

We denote by $C(t)$, the content of the buffer at time t. Clearly, the 2-dimensional process $\{X(t), C(t), t \geq 0\}$ constitutes a Markov process which possesses a unique stationary distribution under a suitable stability condition.

In order that a limit distribution for $C(t)$ exists as $t \to \infty$, the stationary net input rate should be negative, that is,

$$r_0 + r\sum_{i=1}^{\infty}\pi_i < 0, \tag{1}$$

where $\pi_i = \lambda_0\lambda_1...\lambda_{i-1}/(\mu_1\mu_2...\mu_i), i = 1,2,3,...,$ and $\pi_0 = 1$ are called the potential coefficients.

Letting

$$F_j(t,u) \equiv \Pr\{X(t) = j,\, C(t) \leq u\}, \quad j \in S,\, t, u \geq 0,$$

it is not difficult to show that the Kolmogorov forward equations for the Markov process $\{X(t), C(t)\}$ are given by

$$\frac{\partial F_0(t,u)}{\partial t} = -r_0\frac{\partial F_0(t,u)}{\partial u} - \lambda_0 F_0(t,u) + \mu_1 F_1(t,u),$$

$$\frac{\partial F_j(t,u)}{\partial t} = -r\frac{\partial F_j(t,u)}{\partial u} - \left(\lambda_j + \mu_j\right)F_j(t,u) + \lambda_{j-1}F_{j-1}(t,u) + \mu_{j+1}F_{j+1}(t,u),$$

When the process is in equilibrium $\partial F_j(t,u)/\partial t \equiv 0$ and let $F_j(t,u) \equiv F_j(u)$. Hence the above system reduces to a system of ordinary differential equations:

$$\frac{dF_0(u)}{du} = -\frac{\lambda_0}{r_0}F_0(u) + \frac{\mu_1}{r_0}F_1(u), \tag{2}$$

$$\frac{dF_j(u)}{du} = -\frac{\left(\lambda_j + \mu_j\right)}{r}F_j(u) + \frac{\lambda_{j-1}}{r}F_{j-1}(u) + \frac{\mu_{j+1}}{r}F_{j+1}(u), \; u \geq 0, \; j = 1,2,.... \tag{3}$$

When the net input rate of fluid flow into the buffer is positive, the buffer content increases and buffer cannot stay empty. It follows that the solution to (2) and (3) must satisfy the boundary conditions

$$F_j(0)=0, \qquad j=1,2,\ldots,$$
$$F_0(0)=a, \quad \text{for some constant } a \tag{4}$$

Transient solution of fluid queue driven by *M/M/1* queue

We consider the fluid model discussed in previous section with the background process as an *M/M/1*queue with mean arrival and service rates to be λ and μ respectively.

Since $\pi_i=\rho^i$, with $\rho\equiv\lambda/\mu$, it follows from (1) that

$$\rho<\frac{r_0}{r_0-r}$$

Theorem 1

The stationary distribution of the buffer content distribution of an fluid queue driven by *M/M/1* queue is given by

$$\lim_{t\to\infty}\Pr(C(t)>u)=1-\sum_{j=o}^{\infty}F_j(u)$$

where

$$F_0(u)=\frac{r}{r_0}\int_0^u \exp\left\{-\left(\frac{\lambda+\mu}{r}\right)y\right\}q_1(y)\,dy+\delta_{0a}$$

and

$$F_j(u)=\left(\frac{\lambda}{\mu}\right)^j F_0(u)+\frac{r}{\mu}\exp\left\{-\left(\frac{\lambda+\mu}{r}\right)u\right\}\sum_{k=1}^{j}q_k(u)\left(\frac{\lambda}{\mu}\right)^{j-k}, \quad j=1,\,2,\,3,\,\ldots,$$

Proof.

If we substitute $\lambda_j=\lambda$ and $\mu_j=\mu$ in (2), (3) and define

$$q_j(u)=\begin{cases}\exp\left(\frac{\lambda+\mu}{r}\right)\left[\frac{\mu}{r}F_j(u)-\frac{\lambda}{r}F_{j-1}(u)\right], & j=1,2,3,\ldots\\ 0, & j=0,-1,-2,\ldots\end{cases}$$

$$H(z,u)=\sum_{j=-\infty}^{\infty}q_j(u)z^j,\quad H(z,0)=z^a\left[\left(\frac{\mu}{r}-\frac{\lambda z}{r}\right)(1-\delta_{0a})-\frac{\lambda z}{r_0}\delta_{0a}\right].$$

From (2) and (3), for $\lambda_j=\lambda$ and $\mu_j=\mu$

$$\frac{\partial H(z,u)}{\partial u}=\left(\frac{\lambda z}{r}+\frac{\mu}{rz}\right)H(z,u)-\frac{\mu}{r}q_1(u) \tag{5}$$

Solving (5) we find that

$$H(z,u)=H(z,0)\exp\left\{\left(\frac{\lambda z}{r}+\frac{\mu}{rz}\right)u\right\}-\frac{\mu}{r}\int_0^u q_1(y)\exp\left\{\left(\frac{\lambda z}{r}+\frac{\mu}{rz}\right)(u-y)\right\}dy \tag{6}$$

We know that, if $\alpha=\frac{2\sqrt{\lambda\mu}}{r}$ and $\beta=\sqrt{\lambda/\mu}$, then

$$\exp\left\{\left(\frac{\lambda z}{r}+\frac{\mu}{rz}\right)u\right\}=\sum_{j=-\infty}^{\infty}(\beta z)^j I_j(\alpha u)$$

By comparing the coefficients of z^j on both sides of (6) for $j=1, 2, 3, \ldots$, we see that

$$q_j(u)=\frac{\mu}{r}\beta^{j-a}(1-\delta_{0a})I_{j-a}(\alpha u)-\frac{\lambda}{r}\beta^{j-a-1}(1-\delta_{0a})I_{j-a-1}(\alpha u)$$
$$-\frac{\lambda}{r_0}\beta^{j-a-1}\delta_{0a}I_{j-a-1}(\alpha u)-\frac{\mu}{r}\beta^{j}\int_0^u q_1(y)I_j(\alpha(u-y))dy$$

The above holds for $j=-1, -2, -3, \ldots$ with the left-hand side replaced by 0. Using $I_j(u)=I_{-j}(u)$ for $j=1, 2, 3, \ldots$, we find that

$$q_j(u)=\frac{\mu}{r}\beta^{j-a}(1-\delta_{0a})[I_{j-a}(\alpha u)-I_{j+a}(\alpha u)]+\frac{\lambda}{r}\beta^{j-a-1}(1-\delta_{0a})[I_{j+a+1}(\alpha u)-I_{j-a-1}(\alpha u)]$$
$$+\frac{\lambda}{r_0}\beta^{j-a-1}\delta_{0a}[I_{j+a+1}(\alpha u)-I_{j-a-1}(\alpha u)]. \quad (7)$$

Thus, for $j=1, 2, 3, \ldots,$ we have

$$F_j(u)=\left(\frac{\lambda}{\mu}\right)^j F_0(u)+\frac{r}{\mu}\exp\left\{-\left(\frac{\lambda+\mu}{r}\right)u\right\}\sum_{k=1}^{j}q_k(u)\left(\frac{\lambda}{\mu}\right)^{j-k} \quad (8)$$

and

$$F_0(u)=\frac{r}{r_0}\int_0^u \exp\left\{-\left(\frac{\lambda+\mu}{r}\right)y\right\}q_1(y)dy+\delta_{0a} \quad (9)$$

In this way, we analytically obtain closed form expressions for $F_j(u)$ for both the models as given by (8) and (9).

Numerical Example

In this section, we provide numerical example obtained by employing the above formulas in theorem 1 to study the behavior of the fluid queue driven by a single server queue discussed in the earlier sections.

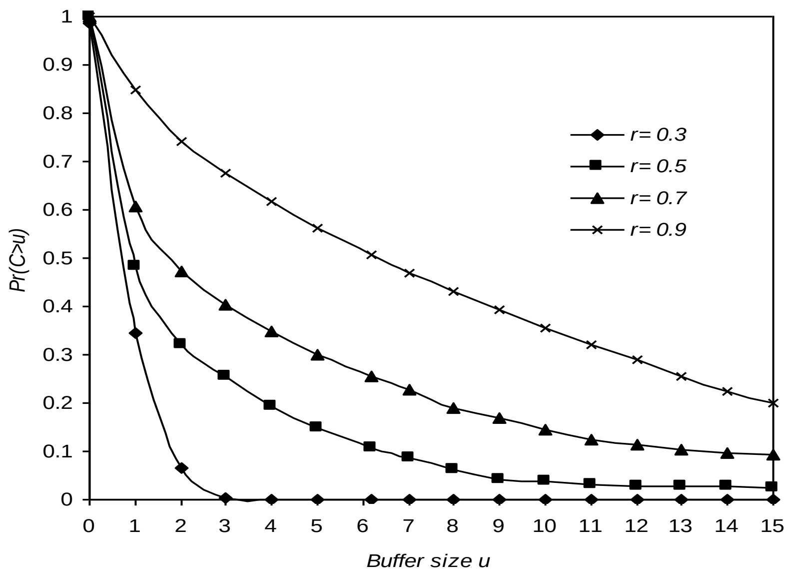

Figure 1. Buffer content distribution with $r_0=-1$ and $a\neq 0$

In Figure 1. Show the variation of the stationary buffer content with buffer size for parameter values $\lambda=0.2$, $\mu=1$ and $a=1-2\rho$ (see Parthasarathy and Vijayashree (2002)). It is noted from the graph that for all different values of r the complementary stationary buffer content distribution is monotonically decreasing to zero when buffer size tends to infinity.

To confirm the efficiency of this solution, we add this comparison with another methods Barbot, Sericola (2002) and Parthasarathy and Vijayashree (2002). To compute three solutions we give the CPU time in seconds taken in computing buffer content distribution using three methods.

Table 2: Comparison between the given three solutions in CPU

Buffer size	Pr(C > u)	CPU Time in Seconds		
		Our Solution	Solution [13]	Solution 16]
5	0.35180	0.0212	0.1153	0.0749
15	0.20711	0.0261	0.1741	0.0752
20	0.07592	0.0311	0.1952	0.0791
30	0.01843	0.0370	0.2037	0.0811

As can be seen, the above results compare quite favorably with those obtained by another solutions.

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