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Comparison in between differential transformation method and variation of parameter method for higher order boundary value problem

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ABSTRACT

We have to make comparison among differential transformation method (DTM) and Variation of Parameter method (VOPM). We provide two examples in order to compare our results and find exact solutions also. The numerical examples show that the DTM is a good method compared to the VOPM since it is effective, uses less time in computation, easy to implement and achieve high accuracy. In addition, DTM has many advantages compared to VOPM since the calculation of Adomian polynomial is tedious. From the numerical results, DTM is suitable to apply for nonlinear problems.

Introduction

The differential transformation method is a numerical method based on a Taylor expansion. This method constructs an analytical solution in the form of a polynomial. Differential Transform Method (DTM) is one of the analytical methods for differential equations. The basic idea was initially introduced by Zhou [16] in 1986. Its main application therein is to solve both linear and nonlinear initial value problems in electrical circuit analysis. This method develops a solution in the form of a polynomial. Though it is based on Taylor series, still it is totally different from the traditional higher order Taylor series method. The DTM is an alternative procedure for getting Taylor series solution of the differential equations. This method reduces the size of computational domain and is easily applicable to many problems. Large list of methods, exact, approximate and purely numerical are available for the solution of differential equations. Most of these methods are computationally intensive because they are trial-and error in nature, or need complicated symbolic computations. The differential transformation technique is one of the numerical methods for ordinary differential equations. This method constructs a semi-analytical numerical technique that uses Taylor series for the solution of differential equations in the form of a polynomial. It is different from the high-order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally time-consuming especially for high order equations. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. The main advantage of this method is that it can be applied directly to nonlinear ODEs without requiring linearization, perturbation. This method will not consume too much computer time when applying to nonlinear or parameter varying systems. This method gives an analytical solution in the form of a polynomial. But, it is different from Taylor series method that requires computation of the high order derivatives. The differential transform method is an iterative procedure that is described by the transformed equations of original functions for solution of differential equations. Chen and Liu have applied this method to solve two-boundary-value problems [2]. Jang, Chen and Liu apply the two-dimensional differential transform method to solve partial differential equations [3]. Yu and Chen apply the differential transformation method to the optimization of the rectangular fins with variable thermal parameters [4,5]. Jang, M.J., C.L. Chen and Y.C. Liy,[12-13-14] On solving the initial value problems using the differential transformation method Applied Mathematics and Computation. I.H. Hassan, N. Bildik, A. Konuralp, F. Bek, S. Kucukarslan [9,19] Comparison differential transformation technique with Adomian decomposition method for linear and nonlinear initial value

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problems and for PDE also. Unlike the traditional high order Taylor series method which requires a lot of symbolic computations, the differential transform method is an iterative procedure for obtaining Taylor series solutions.

The method of variation of parameters was introduced by the Swiss-born mathematician Leonard Euler (1707 - 1783) and completed by the Italian-French mathematician Joseph-Louis Lagrange (1736 - 1813).Variation of Parameter method is free from round off errors, calculation of the so-called Adomian's polynomials, perturbation, linearization and uses only the initial conditions which are easier to implement and reduces the computational work while still maintaining a higher level of accuracy. It is worth mentioning that Ma et al. [20-22] used Variation of Parameters Method (VOPM) for solving involved non-homogeneous partial differential equations and obtained solution formulas helpful in constructing the existing solutions coupled with a number of other new solutions.

Description of the Method DTM

If $\omega(\mathbf{x})$ is a given function, its differential transform is defined as

$$\omega(\mathbf{n}) = \frac{1}{\mathbf{n}!}\left[\frac{\mathbf{d}^{\mathbf{n}}\omega(\mathbf{x})}{\mathbf{dx}^{\mathbf{n}}}\right]_{\mathbf{x}=\mathbf{x}_0} \qquad (1)$$

and the differential inverse transformation of $\omega(\mathbf{x})$ is defined as

$$\omega(\mathbf{x}) = \sum_{\mathbf{n}=0}^{\infty} \omega(\mathbf{x})(\mathbf{x}-\mathbf{x}_0)^{\mathbf{n}} \qquad (2)$$

For practical application, the function $\omega(\mathbf{n})$ is expressed by a finite series

$$\omega(\mathbf{x}) = \sum_{\mathbf{n}=0}^{\mathbf{m}} \omega(\mathbf{x})(\mathbf{x}-\mathbf{x}_0)^{\mathbf{n}} \qquad (3)$$

to put equation (2) into (3), we get

$$\omega(\mathbf{x}) = \sum_{\mathbf{n}=0}^{\mathbf{m}} (\mathbf{x}-\mathbf{x}_0)^{\mathbf{n}} \frac{1}{\mathbf{n}!}\left[\frac{\mathbf{d}^{\mathbf{n}}\omega(\mathbf{x})}{\mathbf{dx}^{\mathbf{n}}}\right]_{\mathbf{x}=\mathbf{x}_0} \qquad (4)$$

Which is the Taylor's series for $\omega(\mathbf{x})$ at $x = x_0$. Now, the fundamental operations of the DTM are given in table 1.

Function	Transformed function
$\omega(x) = n_1 g(x) \pm n_2 h(x)$	$\omega(n) = n_1 G(n) \pm n_2 H(n)$
$\omega(x) = \frac{\mathbf{d}^{\mathbf{n}}(\mathbf{x})}{\mathbf{dx}^{\mathbf{n}}}$	$\omega(n) = \frac{(\mathbf{n}+\mathbf{m})!}{\mathbf{m}!} G(n+m)$
$\omega(x) = g(x)h(x)$	$\omega(n) = \sum_{\mathbf{n}_1}^{\mathbf{n}} \mathbf{G}(\mathbf{n}_1)\mathbf{H}(\mathbf{n}-\mathbf{n}_1)$
$\omega(x) = c\, h(x)$	$\omega(n) = c\, H(x)$, here c is constant
$\omega(x) = \mathbf{x}^{\mathbf{m}}$	$\omega(n) = \delta(n-m)$, where $\delta(n-m) = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$
$\omega(x) = g_1(x)\, g_2(x) \ldots\ldots g_{m-1}(x)\, g_m(x)$	$\omega(n) = \sum_{\mathbf{n}_{\mathbf{m}-1=0}}^{\mathbf{n}} \sum_{\mathbf{n}_{\mathbf{m}-2=0}}^{\mathbf{m}-1} \sum_{\mathbf{n}_{2=0}}^{\mathbf{n}3} \sum_{\mathbf{n}_{1=0}}^{\mathbf{n}2} [\mathbf{G}_2(\mathbf{n}_1)\mathbf{G}_1(\mathbf{n}_2 -]\mathbf{n}_1) \times \mathbf{G}_{\mathbf{m}-1}(\mathbf{n}_{\mathbf{m}-1} - \mathbf{n}_{\mathbf{m}-2}$

Description of the Method VOPM

To illustrate the basic concept of the technique, we consider the general nth-order boundary value problem $y^{(n)}(x) = f(x, y)$ (5)

with boundary conditions

$y(a) = B_1, y^{(n)}(a) = B_n$, $y(b) = A_1, y^{(m)}(b) = A_m$ here n, m=1,2,3….. n < m

The variation of parameters method gives the general solution of (5) as

$$y = \sum_{i=1}^{n} A_i y^i(x) + \sum_{i=1}^{n} y^i(x) \int_0^x f(s, y_i)\, g(s) ds. \tag{6}$$

Using the boundary conditions, we have the following iterative method for finding the approximate solution y_{n+1} as $y_{n+1}(x) = h(x)$

$$+\int_a^b f(s, y_n)\, g(s) ds + \int_0^x f(s, y_1) g(s) ds \tag{7}$$

where h(x) is determined using the boundary conditions.

We solve this system of equations to and the value of unknown parameters in form of integrals and hence we have the particular solution as mentioned in the 2nd term of above equation. In particular solution an independent variable inside integral sign is replaced by some dummy variable making function of and is written as. If we move the variable inside the integral sign which was outside in equation, we obtain function of and is represented by in the following equation.

Numerical examples

In this section, two numerical examples will be presented to assess the efficiency of the DTM. For the sake of comparison, we will use the error defined as

Errors = analytical solution − approximate solution

The rest of this paper is organized as follows. In section 2 and 3, we give the analysis of the DTM and HPM. In section 4, we present numerical results to demonstrate the efficiency of DTM as compared to HPM with the help of two examples.

Example 1: We have to take a nonlinear fifth order boundary value problem

$$e^x y^5(x) - y^2(x) = 0, \; 0 < x < 1, \tag{8}$$

with the conditions:

$$y(0) = 1,\; y'(0) = 1,\; y''(0) = 1,\; y(1) = e,\; y'(1) = e \tag{9}$$

Exact solution to this problem is: $y(x) = e^x$

The differential transform of (8) is:

$$Y(r+5) = \frac{1}{\prod_{i=1}^{5}(r+i)} \sum_{l=0}^{r} \sum_{m=0}^{l} \frac{(-1)^m}{m!} Y(l-m) Y(r-l)$$

Differential transformation of the boundary conditions (9) is:

$y(0) = 2, y(1) = 2, y(2) = 0.5$

$$\sum_{r=0}^{N} Y(r) = \sum_{r=0}^{N} rY(r) = e$$

Using (8) (9) and the inverse differential transform, the following series solution of the problem, up to higher terms obtained.

$$y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + 0.24801\times10^{-4}x^8 + 2.75573\times10^{-6}x^9 +$$

$$2.75573\times10^{-7}x^{10} + 2.50521\times10^{-8}x^{11} + 2.08768\times10^{-9}x^{12} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{10}$$

Now, applying the method of variation of parameter, we get

$$y_{n+1}(x) = B_1 + B_2 X + B_3\frac{x^2}{2!} + B_4\frac{x^3}{3!} + B_5\frac{x^4}{4!} + \int_0^x [e^{-s} y^2{}_n(s) \left\{ \frac{s^4}{4!}] - \frac{xs^3}{3!} + \frac{x^2 s^2}{4} - \frac{x^3 s}{3!} + \frac{x^4}{4!} \right\} ds$$

Using the boundary conditions, we have

$B_1= B_2 =1, B_3=0.5,$

$B_4= -8+3e - \frac{1}{6}\int_0^1 [e^{-s}y^2{}_n(s)\{s^4 - 3s^3 + 3s^2 - s\}]ds$

$B_5 = \frac{11}{2} - 2e + \frac{1}{24}\int_0^1 [e^{-s}y^2{}_n(s)\{3s^4 - 8s^3 + 6s^2 - 1\}]ds$

Consequently we have the following approximations

$y_0(x) = 1+x+0.5x^2+ 0.154845484x^3+ 0.063436344x^4$

$y_1(x)=1+x+0.5x^2+0.1666632966x^3+ 0.04167128061x^4 +0.008333333304x^5 + 0.001388888888x^6 + 1.984126956 \times 10^{-4}x^7 + 0.0000212833763x^8 + 5.635317533 \times 10^{-6}x^9 - 2.755730839 \times 10^{-7}x^{10} -2.253154699 \times 10^{-8}x^{11} -8.973483263 \times 10^{-9}x^{12} +7.968750687 \times 10^{-9}x^{13} -5.345982752 \times 10^{-9}x^{14} + 2.138012899 \times 10^{-12}x^{15} + o(x^{16})$ $y_2(x) =1+x+0.5x^2+0.166666666641x^3+ 0.041666666917x^4 + 0.008333333304x^5 + 0.0013888888x^6 + 1.984126985 \times 10^{-4}x^7 + 2.480058488 \times 10^{-5}x^8 +2.756342416 \times 10^{-6}x^9 +2.755732336 \times 10^{-7}x^{10} + 2.505210694 \times 10^{-8}x^{11} +2.087674295 \times 10^{-9}x^{12} +1.150299398 \times 10^{-10}x^{13} +3.544310482 \times 10^{-11}x^{14} - 2.294103056 \times 10^{-12}x^{15} + o(x^{16})$

Table Error Estimation 1.1

X	Exact Solution	DTM Solution	VOPM Solution	Error Estimate of DTM Solution	Error Estimate of VOPM Solution
0	1.000000000	1.000000000	1.000000000000000	0.0	0.0
0.1	1.105170918	1.105170918	1.105170918075648	1.4×10^{-12}	1.3×10^{-12}
0.2	1.221402758	1.221402758	1.221402758160170	9.8×10^{-12}	1.0×10^{-11}
0.3	1.349858808	1.349858808	1.349858807576003	2.8×10^{-11}	3.2×10^{-11}
0.4	1.491824698	1.491824698	1.491824697641270	5.6×10^{-11}	7.0×10^{-11}
0.5	1.648721271	1.648721271	1.648721270700128	8.7×10^{-11}	1.2×10^{-10}
0.6	1.822118800	1.822118800	1.822118800390509	1.1×10^{-10}	1.9×10^{-10}
0.7	2.013752707	2.013752707	2.013752707470477	1.2×10^{-10}	2.8×10^{-10}
0.8	2.225540928	2.225540928	2.225540928492468	9.9×10^{-11}	3.7×10^{-10}
0.9	2.459603111	2.459603111	2.459603111156950	4.4×10^{-11}	4.7×10^{-10}
1.0	2.718281828	2.7182818288	2.718281828459045	0.8×10^{-12}	5.6×10^{-10}

It shows comparison of the DTM solution and VOIM solution (10) with the exact solution and the error estimates for the same problem. The accuracy of the proposed method can be improved further by adding more terms of the Taylor's series.

Example 2: We have to take a nonlinear sixth order boundary value problem

$$e^x y^6(x) - y^2(x) = 0, \quad 0 < x < 1, \qquad (11)$$

with the conditions:

$$y(0) = 1, y^{(2)}(0) = 1, y^{(4)}(0) = 1, y^{(2)}(1) = e, y^{(4)}(1) = e \qquad (12)$$

Exact solution to this problem is: $y(x) = e^x$

Now, applying the method of variation of parameter, we get

$$y_{n+1}(x)=B_1+B_2x+B_3\frac{x^2}{2!}+B_4\frac{x^3}{3!}+B_5\frac{x^4}{4!}+B_6\frac{x^5}{5!}+\int_0^x \left[e^{-s}y^2{}_n(s)\left\{-\frac{s^5}{5!}+\frac{xs^4}{4!}\right]-\frac{x^2s^3}{12}-\frac{x^4s}{4!}+\frac{x^5}{5!}\right\}ds$$

Using the boundary conditions, we have

$B_1= B_3 = B_5 = 1$,

$$B_2=\frac{1}{120\left(\frac{307e}{3}-\frac{472}{3}\right)}-\frac{1}{120}\int_0^1 \left[e^{-s}y^2{}_n(s)\left\{5s^4-\frac{20}{3}s^3-s^2+\frac{8}{3}s\right]\right\}ds,$$

$$B_5=\frac{1}{6}(5e-8)-1/6\int_0^1 e^{-s}\,y^2{}_n(s)\{-s^3+3s^2-2s\}ds$$

$$B_6=(e-1)+\int_0^1 e^{-s}\,y^2{}_n(s)\{s-1\}ds$$

Consequently we have the following approximations

$y_0(x) = 1+1.00697922x+0.5x^2+ 0.1553169206x^3+ 0.04166667x^4 + 0.0143190152x^5$

$y_1(x)=1+0.99998538x+0.5x^2+0.1666907042x^3+ 0.04166667x^4 +0.008321486957x^5 + 0.0013888884x^6 + 0.0020118223x^7 + 0.00002480400282x^8 + 2.379\times 10^{-6}x^9 -2.746\times 10^{-7}x^{10} +6.149\times 10^{-8}x^{11} -1.88\times 10^{-9}x^{12} -3.27\times 10^{-10}x^{13} -2.053\times 10^{-11}x^{14} + 3.421\times 10^{-11}x^{15} + o(x^{16})$

$y_2(x) =1+x+0.5x^2+0.1666666166x^3+ 0.0416667x^4 +0.0083358007x^5 + 0.001388884x^6 + 1.984097\times 10^{-4}x^7 + 2.4804\times 10^{-5}x^8 +2.7552\times 10^{-6}x^9 +2.753\times 10^{-7}x^{10} + 2.492\times 10^{-8}x^{11} +2.0876\times 10^{-9}x^{12} +1.6518\times 10^{-10}x^{13} +1.1\times 10^{-11}x^{14} - 5.456\times 10^{-13}x^{15} + o(x^{16})$........

Now applying DTM, The differential transform of (11) is:

$$Y(r+5)=\frac{1}{\prod_{i=1}^{6}(r+i)}\sum_{l=0}^{r}\sum_{m=0}^{l}\frac{(-1)^m}{m!}Y(l-m)Y(r-l) \qquad (13)$$

Differential transformation of the boundary conditions (13) is:

$y(0) =1, y(2) = 0.5,\ Y(2)=\frac{1}{4!}$

$$\sum_{r=0}^{N}Y(r)=\sum_{r=0}^{N}r(r-1)Y(r)=\sum_{r=0}^{N}\prod_{k=0}^{3}(r-k)Y(r)=e$$

Using (12) (13) and the inverse differential transform, the following series solution of the problem, up to higher terms obtained.

$$y(x) = 1+ x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + 0.24802\times10^{-4}x^8+ 2.75573\times10^{-6}x^9+$$

$$2.75573\times10^{-7}x^{10}+2.50523\times10^{-8}x^{11}+ 2.08768\times10^{-9}x^{12}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

Table Error Estimation 1.2

X	Exact Solution	DTM Solution	VOPM Solution	Error Estimate of DTM Solution	Error Estimate of VOPM Solution
0	1.000000000	1.000000000	1.000000000000000	-5.4×10^{-9}	0.0
0.1	1.105170918	1.105170918	1.105170918075648	-1.0×10^{-8}	-3.4×10^{-14}
0.2	1.221402758	1.221402758	1.221402758160170	-1.4×10^{-8}	-2.6×10^{-13}
0.3	1.349858808	1.349858808	1.349858807576003	-1.7×10^{-8}	-8.8×10^{-13}
0.4	1.491824698	1.491824698	1.491824697641270	-1.8×10^{-8}	-2.0×10^{-12}
0.5	1.648721271	1.648721271	1.648721270700128	-1.8×10^{-8}	-3.8×10^{-12}
0.6	1.822118800	1.822118800	1.822118800390509	-1.5×10^{-8}	-6.3×10^{-12}
0.7	2.013752707	2.013752707	2.013752707470477	-9.3×10^{-8}	-9.3×10^{-12}
0.8	2.225540928	2.225540928	2.225540928492468	-1.1×10^{-8}	-1.2×10^{-11}
0.9	2.459603111	2.459603111	2.459603111156950	-6.0×10^{-9}	-1.6×10^{-11}
1.0	2.718281828	2.718281828	2.718281828459045	-1.1×10^{-13}	-1.8×10^{-11}

Conclusion

In this paper, we have introduced a new technique to solve higher order boundary value problems by the Differential transform method and variation of parameter method .The numerical results in the Tables [1.1-1.2], show that the Differential transform method provides highly accurate numerical results as compared to variation of parameter method. It can be concluded that is Differential transform method a highly efficient method for solving higher order boundary value problems arising in various fields of engineering and science.

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