



Strategy of finding the value function for financial problem described by non-deterministic system

Entisar Alrasheed¹ and Ismaiel Elsanosi²

¹University of Bahri- College of Applied and Industrial Sciences, Sudan.

²University of Alneelen, Sudan.

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ABSTRACT

The purpose of this paper is to find the value function for financial problem whose dynamic is described by non-deterministic differential equation.

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Introduction

The theory of stopping time and stopping time problems are concerned with the problems of choosing a time to take a particular action, in order to maximize an expected reward or minimize an expected cost. The nondeterministic system with jump component has great importance and many applications in different areas of science and in real life, for example in modeling asset prices in finance. The general problem of this kind of processes with jumps is considered in the literature, for example see [1], [2]. This type of problems also can be formulated with delay see [3], [4].

Consider $(\Omega, \mathcal{F}, \{f_t\}_{t \geq 0}, P)$ to be a complete filtered probability space satisfying the usual conditions or usual hypotheses, on which we will define the following concepts.

Random variable: An n -dim random variable X is a measurable function from the set of all possible outcomes Ω to R^n that is $X : \Omega \rightarrow R^n$, $\forall \omega \in \Omega$, $X(\omega) \in R$

Random Process: Denotes by X_t , is mathematical model defined by the map $X : \Omega \times T \rightarrow R^n$ such that $\forall t \in T$, $\omega \rightarrow X(\omega, t)$ is measurable.

Sample path: for fixed ω , the sample path of the random process X_t is the map $t \rightarrow X(t, \omega)$. A process X_t is said to be continuous time random process if its sample path is continuous function otherwise X_t is called discontinuous random process.

Stopping time: The random variable $\tau : \Omega \rightarrow I$ is called a stopping time if $\tau \leq t \in \mathcal{F}_t$ for all t in I . In other word random process T is called a stopping time if you can tell when it happens.

Stopping time is often defined by a stopping rule or a mechanism for deciding whether to continue or to stop a process on the basis of the present position and past events, and which will almost always lead to a decision to stop at some finite time. Stopping times usually occur in decision theory, see [5].

Lévy process: An adopted random process $X_t = \{X_t : t \geq 0\}$ is said to be a Lévy process if it satisfies the following properties:

1. $X_0 = 0$ almost surely
2. Independence of increments: For any $0 \leq s < t$, $X_t - X_s$ is independent of $\mathcal{F}_s$ [increment independent of the past].
3. Stationary increments: For any $s < t$, $X_t - X_s$ is equal in distribution to X_{t-s} .
4. Continuity in probability: For any $\varepsilon > 0$ and $t \geq 0$ it holds that $\lim_{h \rightarrow 0} P(|X_{t+h} - X_t| > \varepsilon) = 0$

If X_t is a Lévy process then one may construct a version of X_t such that $t \rightarrow X_t$ is almost surely right continuous with left limits (cadlag).

Radom differental equation : (or stochastic differential equation), is differential equation in which one or more of the terms is random process, resulting in a solution which is itself a random process , see [6].

Itô diffusion: Let $X_t : \Omega \times T \rightarrow R^n$ defined on a probability space (Ω, f, P) the process X_t is Ito diffusion if it satisfying a random differential equation of the form:

$$d X_t = \alpha (X_t) dt + \beta (X_t) dB_t$$

where B_t is an m -dimensional Brownian motion and $\alpha : R^n \rightarrow R^n$ and $\beta : R^n \rightarrow R^{n \times m}$ are the drift and diffusion fields respectively. Itô diffusion named after the Japanese mathematician Kiyoshi Itô.

Infinitesimal Generator: The infinitesimal generator $A = A_x$ of X_t is the operator which is defined to act on suitable functions $f : R^n \rightarrow R$ by:

$$A f(x) = \lim_{t \downarrow 0} \frac{E^x [f(X_t)] - f(x)}{t}, \quad x \in R^n$$

where the limit exist.

The set function $f : R^n \rightarrow R$ such that the limit exist at x denoted by $D_A(x)$ while D_A denoted to the set of functions which their limits exist for all $x \in R^n$, the infinitesimal generator of random process is a partial differential operator that encodes a great of information about the process .

One can show that any compactly-supported C^2 (twice differentiable with continuous second derivative) function f lies in D_A and that

$$A f(x) = \sum_i b_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j} (\sigma(x) \sigma(x)^T)_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}$$

or, in terms of the gradient and scalar and Frobenius inner products,

$$A f(x) = b(x) \cdot \nabla_x f(x) + \frac{1}{2} (\sigma(x) \sigma(x)^T) : \nabla_x \nabla_x f(x)$$

Itô processes: Let B_t be 1-dimensional Brownian motion on (Ω, f, P) . An Ito process (also called random integral) is random processes X_t on (Ω, f, P) adopted to f_t of the form:

$$d X_t = u(s, w) dt + v(s, w) dB_t$$

Or in integral form:

$$X_t = X_0 + \int_0^t u(s, w) ds + \int_0^t v(s, w) dB_t$$

Theorem(Itô formula): Ito's formula is the fundamental theorem of stochastic calculus, just as one speaks of the fundamental theorem of ordinary integral/differential calculus.

Let X_t be an Ito processes given by $d X_t = u(t) dt + v(t) dB_t$, let $g(x, t) \in C^2([0, \infty) \times R)$, then $Y_t = g(t, X_t)$ is again Ito processes and

$$d Y_t = \frac{\partial g(t, X_t)}{\partial t} dt + \frac{\partial g(t, X_t)}{\partial x} d X_t + \frac{1}{2} \frac{\partial^2 g(t, X_t)}{\partial x^2} (d X_t)^2$$

For the proof see [7].

Assume that on the complete probability space (Ω, f, P) , the dynamics of the asset price $X(t)$ follows the differential equation with jump of the form:

$$d X(t) = \mu X(t) dt + \sigma X(t) dB(t) + \int_0^\infty h(X_t, y) \hat{N}(dt, dy) \quad (1)$$

With initial condition

$$X(0) = x > 0 \quad (2)$$

Where $\mu(X), \sigma(X)$ and $h(X)$ are real valued functions and $\{B\}_t$ is 1-dimensional Brownian motion with respect to $\{f_t\}$ and $\hat{N}(dt, dy)$ is the Compensated Poisson random measure given by :

$$\hat{N}(dt, dy) = N(dt, dy) - dt \pi(dy)$$

Where $\pi(dy)$ is the Levy measure associated to N and for integrability reason assume that:

$$\int_{-1}^{\infty} (1 \wedge y^2) \pi(dy) < \infty \quad (3)$$

$$\int_{-1}^{\infty} y \pi(dy) < \infty \quad (4)$$

we also assume that the size of a jump is greater than -1 , so that $X(t)$

remain non- negative for all $t \geq 0$ a.s .

Suppose that $g : R^2 \rightarrow R$ is continuous function such that:

$$g(X_t) = e^{-\rho t} (X_t - c) \quad (5)$$

Statement of the Problem

Our considered problem is to find the value function φ and stopping time τ such that the following supremum is attained:

$$\Phi(s, x) = \sup_{\tau} E^{(s, x)} [g(X_{\tau}) \cdot \chi_{[\tau < \infty]}] = E^{(s, x)} [e^{-\rho \tau^*} (X_{\tau^*} - c) \cdot \chi_{[\tau^* < \infty]}] \quad (6)$$

The supremum is being taken over all stopping time τ where τ^* is the optimal stopping time. The price X_t is modeled by the geometric Levy process, $c > 0$ is the transaction coast and $\rho > 0$ is the discounting factor.

If we can find a function φ then we have found a solution $\varphi = \Phi$ and the stopping time τ .

Verification theorem

Consider that $Q = R^+ \times R$

Suppose we can find appositive function φ such that $\varphi(s, x) \in C^{1,2}$ and satisfy the following conditions:

$$a) \quad \varphi(s, x) \geq g(s, x) \text{ in } Q$$

$$b) \quad \forall (s, x) \in Q \Rightarrow L\varphi(s, x) = \frac{\partial \varphi}{\partial s} + \mu(x) \frac{\partial \varphi}{\partial x} + \frac{1}{2} \sigma^2(x) \frac{\partial^2 \varphi}{\partial x^2} + \int_{-1}^{\infty} \left[\varphi(s, x + h(x, y)) - \varphi(s, x) - h(x, y) \frac{\partial \varphi(s, x)}{\partial x} \right] \pi(dy) \leq 0.$$

$$c) \quad \text{Assume that for } T_R = T \wedge R \wedge \inf \{t > 0, X_t > R\} \text{ and } R > 0,$$

$\lim_{R \rightarrow \infty} E^{(s,x)}[\varphi(s+T_R), X(T_R)] = 0$, then:

$$\varphi(s, x) \geq \Phi(s, x) \quad \forall (s, x) \in Q \quad (7)$$

Define the continuation region D as:

$$D = \{(s, x) \in Q : \varphi(x) > g(x)\} \quad (8)$$

d) The family $\varphi(s, \tau), \tau \leq \tau_D$ is uniformly integrable w.r.t Q

e) Suppose $L\varphi(s, x) = 0$ for all $(s, x) \in D$, and let $\tau_D = \inf \{t > 0, X_t \notin D\} < \infty$

then:

$$\varphi(s, x) = \Phi(s, x), \quad \forall (s, x) \in Q \quad (9.)$$

Proof

Verify Ito formula for semi martingale (see [8]), we have

$$\begin{aligned} \varphi(s+T_R, X(T_R)) - \varphi(s, X(0)) &= \int_0^{T_R} \left(\frac{\partial \varphi}{\partial s} + \mu(x) \frac{\partial \varphi}{\partial x} + \frac{1}{2} \sigma^2(x) \frac{\partial^2 \varphi}{\partial x^2} + \right. \\ &\quad \left. \int_{-1}^{\infty} \left[\varphi(s, x+h(x, y)) - \varphi(s, x) - h(x, y) \frac{\partial \varphi(s, x)}{\partial x} \right] \pi(dy) \right) dt \\ &\quad + \int_0^{T_R} \sigma(x) \frac{\partial \varphi}{\partial x} dB(t) + \int_0^{T_R} \int_{-1}^{\infty} [\varphi(s, x+h(x, y)) - \varphi(s, x)] \hat{N}(dt, dy) \\ &\quad + \sum_{0 < t_j < T_R} \left\{ \varphi(t_j, x_{t_j}) - \varphi(t_j, x_{t_j-}) - \frac{\partial \varphi}{\partial x}(s+t_j, x_{t_j}) \Delta x_{t_j} \right\} \end{aligned} \quad (10)$$

the sum is taken over all jumping times $t_j \in [0, T_R]$ and $\Delta x_{t_j} = x_{t_j} - x_{t_j-}$.

When we apply the Dynkin's formula to (10) (see[2]), we get

$$\begin{aligned} E^{(s,x)}[\varphi(s+T_R, X(T_R))] &= \varphi(s, x) + E^{(s,x)} \left[\int_0^{T_R} L\varphi(s, x) dt - g(s, x) \right] \\ &\quad + E^{(s,x)} \left[\sum_{0 < t_j < T_R} \left\{ \varphi(t_j, X_{t_j}) - \varphi(t_j, X_{t_j-}) - \frac{\partial \varphi}{\partial x}(s+t_j, X_{t_j}) \Delta X_{t_j} \right\} \right] \end{aligned} \quad (11)$$

from condition (b) we have

$$E^{(s,x)}[\varphi(s+T_R, X(T_R))] \leq \varphi(s, x) - E^{(s,x)}[g(s, x) dt] +$$

$$E^{(s,x)} \left[\sum_{0 < t_j < T_R} \left\{ \varphi(t_j, X_{t_j}) - \varphi(t_j, X_{t_j-}) - \frac{\partial \varphi}{\partial x}(s+t_j, \hat{X}_{t_j-}) \Delta x_{t_j} \right\} \right] \quad (12)$$

By using mean value property on the terms under the summation above

we can see that:

$$\Delta \varphi(t_j, X_{t_j}) = \frac{\partial \varphi}{\partial x}(s+t_j, \hat{X}_{t_j}) \Delta X_{t_j} \quad (13)$$

For some point $\hat{X}_{t_j}$ joining the two points X_{t_j} and X_{t_j-} from (13) and

condition (d) we get the following result

$$\varphi(s, x) \geq E^{(s,x)}[g(s, x) + \varphi(s+T_R, X(T_R))] \quad (14)$$

Now we use the limiting argument as $R \rightarrow \infty$ to get

$$\varphi(s, x) \geq E^{(s,x)}[g(X_t)] + \lim_{R \rightarrow \infty} E^{(s,x)}[\varphi(s+T_R, X(T_R))] \quad (15)$$

Using (iii) we get

$$\varphi(s, x) \geq E^{(s,x)}[g(X_t)] \geq \Phi(s, x) \quad (16)$$

This proof the requirement in (7).

b- For the second part of the proof, conceder D as in (iv), and when we

apply conditions (iii) and (v) in (11) we get

$$\varphi(s, x) = E^{(s,x)}[g(s, x)] \quad (17)$$

This result together with (i) prove the requirements (9)

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