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Short-Period perturbations of coorbital motion about an oblate primary

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ABSTRACT

There are many examples for co-orbital motion in the Solar system, including temporary co-orbital companions of the Earth, raising many interesting questions. The problem of co-orbital motion is formulated in the Hamiltonian form when the larger primary is an oblate body. Different forms of the disturbing function are outlined; the relevant form is developed in terms of Laplace's coefficients. The Hamiltonian of the problem is formed in Delaunay-like canonical elements. The ratio of the primaries'masses is considered as a small parameter of the first order while the leading oblateness term of the first primary is considered of second order. Finally, the short-period terms are eliminated from the Hamiltonian using the procedure based on Lie series and Lie transform, leaving the Hamiltonian as a function of only the secular and critical (resonant) terms.

Introduction

Although the term co-orbital was first used to describe the configuration of Janus and Epimetheus, satellites of Saturn, it is now used in the connection to any material that shares the same orbit with a larger perturber. The co-orbital motion problem can be modeled by using either a restricted three-body approach, in which one of the co-orbiting objects is assumed to have an infinitesimal mass or a more general treatment in which the co-orbital satellites are assumed to have small, but nonzero, masses. The latter approach was originally developed by Hill and is known as Hill's restricted three-body problem. Applications of Hill's problem are ubiquitous. It has been shown that Hill's problem is characterized by the same generality as that of the restricted three-body problem, in the sense that the mass ratio of the co-orbiting objects is arbitrary. The most known co-orbital objects are Jupiter's Trojan asteroids, sharing their orbits with Jupiter, while also librating around the triangular Lagrangian points L_4 or L_5 of the Sun Jupiter system. More than 2000 Trojan asteroids have already been discovered (Érdi et al., 2007). In the 1990s, two Trojan companions of Mars, 5261 Eureka and 1998 VF31, were discovered, librating about the L_5 Lagrange point, behind Mars in its orbit (Connors, et al., 2005).

There are presently 18 asteroids known with semi-major axes between 0.99 and 1.01 AU and thus potentially affected by resonant interaction with the Earth. Of these, 2002 AA29 and 2003 YN107 move on low-eccentricity orbits very similar to that of the Earth, with moderate inclinations of 4 and 11 degrees, respectively. These objects both have horseshoe orbits with respect to the Earth and are capable of being captured as quasi-satellites (which 2003 YN107 currently is). Two other objects are known to have horseshoe orbits deviating further from Earth's orbit. 54509 (2000 PH5) and short-arc object 2001 GO2 have eccentricities of about 0.2 and low inclinations. The remaining objects have eccentricity higher than 0.2 and may have high inclinations. At least one such object, 3753 Cruithne, nevertheless has a complex horseshoe-like orbit with respect to Earth (Connors and Innanen, 2004). Venus has one co-orbital object, 2002 VE68, which is also the first known QS. A detailed discussion of its orbit is given in Mikkola et al. (2004). It then will be a temporary L_5 Trojan of Venus for 700 years and then leave the co-orbital region. The object's eccentricity is large enough to bring it close to the Earth and frequent close approaches occur since the descending node stays close to the Earth's orbit.

Several authors have dealt with the co-orbital motion in the framework of the planar three body problem and it seams worth to sketch some of the most important works:

The analytical studies of the co-orbital motion have so far addressed the case of quasi-circular orbits. The short- and long-term evolution of tadpole orbits is presented in the book of Brown and Shook (1964) and in the work of Érdi (1981). In (1981a, b) Dermott and Murray gave a description of the co-orbital motion of 1980S1 and 1980S3 (Janus and Epimetheus) based on a combination of numerical integration and perturbation theory; they first studied the case where the mass of the third body was negligible, and generalized some of the results to include the case where the third body had sufficient mass to affect the other satellite (see also Murray and Dermott, 1999). Yoder et al. (1983) studied the long-term evolution of the co-orbiting satellites Janus and Epimetheus; however, because the orbit's precession due to the oblateness of Saturn dominates that due to the satellites interaction, the latter was not fully investigated. In later work, Salo and Yoder (1988) gave sufficiency conditions for the stability of a system of n co-orbital objects. Namouni (1999) performed an extensive study of co-orbital motion using Hill's three-body problem and orbital elements based generating solution. Broucke (1999) derived a Lagrangian formulation for the study of the 1:1 motion commensurability in the restricted three-body problem and computed families of stable horseshoe periodic orbits which approach the actual motion of 1980S1 and 1980S3. Christou (2000) has studied the behavior of probable Earth and Venus co-orbitals and found that 10563 Izdhubar, 3362 Khufu and 1994WF2 may become Earth co-orbitals in the future while 1989 VA may become a Venus co-orbital. Llibre and Ollé (2001) showed that the motion of Saturn co-orbital satellites is closely related to some periodic orbits of this restricted three-body

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problem. Morais and Morbidelli (2002) obtained the size and orbital distributions of Near-Earth Asteroids (NEAs) that are expected to be co-orbitals of the Earth in a steady-state scenario. They predicted 0.65 objects with absolute magnitudes H <18 and 16 with H < 22 and concluded that these objects are not easily observed as they are distributed over a large area in the sky and spend most of the time away from opposition where they may be too faint. Gurfil and Kasdin (2003) solved the Hamilton-Jacobi equation and treated the gravitational interaction of the co-orbiting objects as a perturbation using new canonical orbital elements, which they termed epicyclic orbital elements, as constants of the relative motion. Sicardy and Dubois (2003) showed the dynamics of a test particle co-orbital with a satellite of mass m_s which revolves around a planet of mass $M_0 >> m_s$ with a mean motion n_s and semi-major axis a_s. They studied the long-term evolution of the particle motion under slow variations of the mass of the primary, M_0, the mass of the satellite, m_s and the specific angular momentum of the satellite. Brasser et al. (2004) analyzed the orbital behavior of co-orbital NEOs and the Earth horseshoe object 2002 AA29. The objects are 2001 CK32, a 3753 Cruithne-like co-orbital of Venus, 2001 GO2 and 2003 YN107, two objects with motion similar to 2002 AA29. 2001 CK32 is on a compound orbit.

The present work is concerned with the co-orbital motion with the Earth as a central body. The problem is treated analytically in the Hamiltonian frame-work and the equations of motion are integrated using a Lie transformation technique developed and applied by Deprit (1969) and Kamel (1969), modified to account to the co-orbital commensurability. The main features of the work can be summarized as

1- The disturbing function for the cases of co-orbital motion is represented in different forms with particular emphasis on the relevant form developed in terms of Laplace's coefficients then the problem is expressed in canonical form in terms of a set of Delaunay-like elements..

2- The ratio of the primary mass is considered as a small parameter of the first order while the leading oblateness term of the primary is considered of second order.

3- The short-periodic terms are eliminated using the usual Lie-Deprit-Kamel transform.

Hamiltonian of the Problem

The Hamiltonian of the restricted three-body problem can be written as (see for instant Nesvorný et al., 2002 or Brown and Shook, 1964)

$$\mathrm{H} = \mathrm{K} + \mathrm{R} + \mathrm{T} + \tilde{V} \tag{1}$$

where

$$\mathrm{K} = -\frac{\mu}{2a}$$

is the Keplerian part depending on the semi-major axis a of the infinitesimal body, R is the perturbation (disturbing function), a function of the positions of the infinitesimal body and the secondary mass. Assuming that m moves under the central attraction of M with m_s acting mainly as a disturbing body, the disturbing function can be written as

$$\mathrm{R} = \mu_s \left(\frac{1}{|\vec{r}_s - \vec{r}|} - \frac{\vec{r}.\vec{r}_s}{|\vec{r}_s|^3} \right) \tag{2}$$

with $\mu_s = \mathrm{G} m_s$ and $\mu = \mathrm{G}(M+m)$ where G is the universal constant of gravitation. And T is the canonical momentum conjugate to the time variable where the time appears due to the neglect of the effects of m, which necessitates the augmentation of the canonical variables with the time variable and its conjugate. Assuming the fixed orbit of the secondary mass to be circular, $\mathrm{T} = N_s \Lambda_s$, where N_s is the frequency of the mean longitude (λ_s) and Λ_s is the canonical momentum conjugate to λ_s. Finally $\tilde{V}$ is the perturbing potential.

Expansion of the Disturbing Function

The disturbing function R can be expanded in terms of standard orbital elements using the methods developed by Kaula (1961, 1962), in which the disturbing function for an inner secondary is expanded in an infinite series in the osculating elliptic elements referred to the equator of the primary. The expression for R in equation (2) can be written as

$$\begin{aligned}
\mathrm{R} = &\sum_{l=2}^{8} \frac{\mu_s}{a_s} \sum_{m=0}^{l} \alpha^l (-1)^{l-m} k_m \frac{(l-m)!}{(l+m)!} \sum_{p,p'=0}^{l} F_{lmp}(I) F_{lmp'}(I_s) \\
&\times \sum_{q,q'=-\infty}^{\infty} X_{l-2p+q}^{l,l-2p}(e) X_{l-2p'+q'}^{-(l+1),l-2p'}(e_s) \\
&\times \cos\left[(l-2p'+q')\lambda_s - (l-2p+q)\lambda - q'\varpi_s + q\varpi \right. \\
&\left. + (m-l+2p')\Omega_s - (m-l+2p)\Omega \right]
\end{aligned} \tag{3}$$

where $\alpha = a/a_s$, λ_s and λ are the mean longitudes, ϖ_s and ϖ are the longitudes of pericentre, and $k_0 = 0$ and $k_m = 2$ for $m \neq 0$.

The $F_{lmp}(I)$ are the inclination functions defined as

$$F_{lmp}(I) = \frac{i^{l-m}(l+m)!}{2^l p!(l-p)!}\sum_k (-1)^k \binom{2l-2p}{k}\binom{2p}{l-m-k} c^{3l-m-2p-2k} s^{m-l+2p+2k}$$

where $i = \sqrt{-1}$, $s = \sin(I/2)$, $c = \cos(I/2)$,

k is summed from $k = \max(0, l-m-2p)$ to $k = \min(l-m, 2l-2p)$.

The quantities $X_c^{a,b}(e)$ are Hansen coefficients defined by

$$X_c^{a,b}(e) = e^{|c-b|}\sum_{\sigma=0}^{\infty} X_{\sigma+\rho,\sigma+\beta}^{a,b}\, e^{2\sigma} \tag{4}$$

In this context $\rho = \max(0, c-b)$, $\beta = \max(0, b-c)$, and the $X_{c,d}^{a,b}$ are the Newcomb operators, which can be defined recursively by

$$X_{0,0}^{a,b} = 1$$

$$X_{1,0}^{a,b} = b - a/2$$

and, for $d = 0$,

$$4cX_{c,0}^{a,b} = 2(2b-a)X_{c-1,0}^{a,b+1} + (b-a)X_{c-2,0}^{a,b+2}$$

or, for $d \neq 0$,

$$4dX_{c,d}^{a,b} = -2(2b+a)X_{c,d-1}^{a,b-1} - (b+a)X_{c,d-2}^{a,b-2} - (c-5d+4+4b+a)X_{c-1,d-1}^{a,b} + 2(c-d+b)_{j\geq 2}(-1)^j \binom{3/2}{j} X_{c-j,d-j}^{a,b}$$

Also, $X_{c,d}^{a,b} = 0$ if $c < 0$ or $d < 0$. If $d > c$ then $X_{c,d}^{a,b} = X_{d,c}^{a,-b}$.

Ellis and Murray (2000) wrote the disturbing function (R) in equation (2) due to external perturber as

$$\mathrm{R} = \frac{\mu_s}{a_s}\left(\mathrm{R}_D + \alpha \mathrm{R}_E\right)$$

where $\alpha = a/a_s$. In this expression R_D is derived from the direct part of the disturbing function, R_E comes from the indirect part due to an external perturber. They gave explicit formulae for finite series associated with a specific argument expanded to

$$\phi = j_1\lambda_s + j_2\lambda + j_3\varpi_s + j_4\varpi + j_5\Omega_s + j_6\Omega$$

with $N_{\max}$ being the maximum order of the expansion. Ellis and Murray (2000) showed that the expression for R_D associated with ϕ is

$$\begin{aligned}\mathrm{R}_D = &\sum_{i=0}^{i_{\max}} \frac{(2i)!(-1)^i}{i!2^{2i+1}}\alpha^i \times \sum_{s=s_{\min}}^{i}\sum_{n=0}^{n_{\max}} \frac{(2s-4n+1)(s-n)!}{2^{2n}n!(2s-2n+1)!}\sum_{m=0}^{s-2n} k_m \\ &\times \frac{(s-2n-m)!}{(s-2n+m)!}(-1)^{s-2n-m} F_{s-2n,m,p}(I) F_{s-2n,m,p'}(I_s)\sum_{l=0}^{i-s}\frac{(-1)^s 2^{2s}}{(i-s-l)!} \\ &\times \sum_{l=0}^{l_{\max}}\frac{(-1)^l}{l!}\sum_{k=0}^{l}\binom{l}{k}(-1)^k \alpha^l \frac{d^l}{d\alpha^l} b_{i+\frac{1}{2}}^{(j)}(\alpha) X_{-j_2}^{i+k,-j_2-j_4}(e) X_{j_1}^{-(i+k+1),j_1+j_3}(e_s) \\ &\times \cos\left[j_1\lambda_s + j_2\lambda + j_3\varpi_s + j_4\varpi + j_5\Omega_s + j_6\Omega\right]\end{aligned} \tag{5}$$

where, as before, $k_0 = 1$ and $k_m = 2$ for $m \neq 0$. With the help of some useful relationships hold throughout the calculation; this relations can be found (see for instant Ellis and Murray, 2000 or Murray and Dermott, 1999).

For the indirect parts Ellis and Murray, in the same work, wrote them as

$$\mathrm{R}_E = -k_m \frac{(1-m)!}{(1+m)!} F_{1,m,p}(I) F_{1,m,p'}(I_s) X_{-j_2}^{-2,-j_2-j_4}(e) X_{j_1}^{1,j_1+j_3}(e_s)$$
$$\times\cos\left[j_1\lambda_s + j_2\lambda + j_3\varpi_s + j_4\varpi + j_5\Omega_s + j_6\Omega\right] \tag{6}$$

where each of the quantities p, p' and m must be integers and equal to 0 or 1.

Adapted Expansion

First the different forms of the perturbing function are given in the previous section. One is Kaulas expansion, based on Hansens series, having the advantage of being valid for arbitrary inclinations. However, this expansion is according to the powers of the ratio of the semi-major axes of the co-orbiting bodies, which being close to 1 rises question regarding the usefulness of this expansion for the co-orbital problem. This can also be said on the other expansion based on the Laplace series and Laplace coeffcients. So the disturbing function equation (2) can be written as

$$\begin{aligned} \mathrm{R} &= \frac{\mu_s}{r_s}\left(\frac{1}{\sqrt{1+(r/r_s)^2-2(r/r_s)\cos\phi}}-(r/r_s)\cos\phi\right) \\ &= \frac{\mu_s}{r_s}\left(\frac{1}{\sqrt{1+\alpha^2-2\alpha\cos\phi}}-\alpha\cos\phi\right) \end{aligned} \tag{7}$$

where $\alpha = r/r_s$, and we will suppose that $r_s > r$, ϕ is the angle between the position vectors $\vec{r}_s, \vec{r}$. The terms $\left(1+\alpha^2-2\alpha\cos\phi\right)^{-p}$ can be expanded as, Smart (1953),

$$\begin{aligned} \mathrm{D}^{-p} &= \left(1+\alpha^2-2\alpha\cos\phi\right)^{-p} \\ &= \frac{1}{2}\mathrm{B}_0^p + \sum_{n=1}^{\infty} \mathrm{B}_n^p \cos n\phi \end{aligned} \tag{8}$$

where the coefficients B_n^p are the Laplace coefficients given by

$$\frac{1}{2}\mathrm{B}_0^p = 1 + p^2\alpha^2 + \frac{(p.p+1)^2}{(2!)^2}\alpha^4 + \frac{(p.p+1.p+2)^2}{(3!)^2}\alpha^6 + ...$$

$$\frac{1}{2}\mathrm{B}_n^p = \frac{p.p+1..........p+n-1}{n!}\alpha^n \mathrm{F}\left(p, p+n, n+1; \alpha^2\right) \qquad (n \ge 1)$$

and $\mathrm{B}_n^p = \mathrm{B}_{-n}^p$. Since $\alpha < 1$, the series of $\frac{1}{2}\mathrm{B}_0^p$, and $\frac{1}{2}\mathrm{B}_n^p$ are clearly convergent. B_0^p can be now written simply as

$$\frac{1}{2}\mathrm{B}_0^p = \sum_{k=0}^{\infty} \frac{(p)_k^2}{(k!)^2}\alpha^{2k} = \sum_{k=0}^{\infty} \mathrm{L}_{0,k,p}\alpha^{2k}$$

with $(p)_k = p(p+1)(p+2).........(p+k-1)$, $(p)_0 = 1$, and $\mathrm{L}_{0,k,p} = (p)_k^2/(k!)^2$.

The hypergeometric series can be expressed as

$$\mathrm{F}\left(a,b,c;x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\frac{x^n}{n!}$$

hence, $\frac{1}{2}\mathrm{B}_n^p$ can be more simply written as

$$\frac{1}{2}\mathrm{B}_n^p = \frac{(p)_n}{n!}\alpha^n \sum_{k=0}^{\infty} \frac{(p)_k (p+n)_k}{(n+1)_k}\frac{\alpha^{2k}}{k!} = \sum_{k=0}^{\infty} \mathrm{L}_{n,k,p}\alpha^{2k+n}$$

where $\mathrm{L}_{n,k,p} = \frac{(p)_n}{n!}\frac{(p)_k (p+n)_k}{k!(n+1)_k}$. So that

$$\mathrm{D}^{-p} = \sum_{k=0}^{\infty} \mathrm{L}_{0,k,p}\alpha^{2k} + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \mathrm{L}_{n,k,p}\alpha^{2k+n}\cos n\phi$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathrm{L}_{n,k,p}\alpha^{2k+n}\cos n\phi$$

for $p = \dfrac{1}{2}$

$$\mathrm{D}^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathrm{L}_{n,k,\frac{1}{2}}\alpha^{2k+n}\cos n\phi$$
$$= \sum_{k=0}^{\infty} \mathrm{L}_{0k\frac{1}{2}}\alpha^{2k} + \sum_{k=0}^{\infty} \mathrm{L}_{1,k,\frac{1}{2}}\alpha^{2k+1}\cos\phi + \sum_{n=2}^{\infty}\sum_{k=0}^{\infty} \mathrm{L}_{n,k,\frac{1}{2}}\alpha^{2k+n}\cos n\phi$$

Then the disturbing function can be written as, (Rahoma, 2009),

$$\mathrm{R} = \frac{\mu_s}{r_s}\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathrm{D}_{n,k,\frac{1}{2}}\alpha^{2k+n}\cos n\phi\right) \qquad (9)$$

where

$$\mathrm{D}_{n,k,\frac{1}{2}} = \begin{cases} \mathrm{L}_{0,k,\frac{1}{2}} & n = 0, k \geq 0 \\ \mathrm{L}_{1,k,\frac{1}{2}} & n = 1, k > 0 \\ \mathrm{L}_{1,0,\frac{1}{2}} - 1 & n = 1, k = 0 \\ \mathrm{L}_{n,k,\frac{1}{2}} & n \geq 2, k \geq 0 \end{cases}$$

The next step is to express $\cos\phi$ in terms of the orbital elements. It is clear from Fig. (1) that the base vectors along $\vec{r}$ and $\vec{r}_s$ may be defined

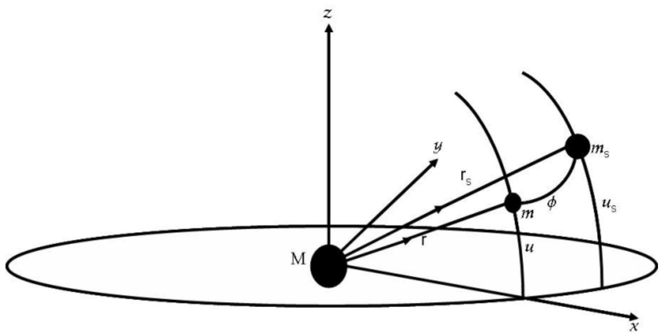

Fig. (1): *Relation between the angle ϕ and the orbital elements of any two objects with mass m and m_s*

$$\vec{r} = r\begin{pmatrix} \cos u \\ \cos I \sin u \\ \sin I \sin u \end{pmatrix}, \qquad \vec{r}_s = r_s\begin{pmatrix} \cos\Delta\Omega\cos u_s - \cos I_s \sin\Delta\Omega\sin u_s \\ \sin\Delta\Omega\cos u_s + \cos I_s\cos\Delta\Omega\sin u_s \\ \sin I_s \sin u_s \end{pmatrix}$$

where r is the modulus of vector $\vec{r}$, from which

$$\cos\phi = \left(\vec{r}.\vec{r}_s\right)/\left(r r_s\right) = \cos u\left(\cos\Delta\Omega\cos u_s - \cos I_s \sin\Delta\Omega\sin u_s\right)$$
$$+\cos I \sin u\left(\sin\Delta\Omega\cos u_s + \cos I_s\cos\Delta\Omega\sin u_s\right)$$
$$+\sin I\sin u\sin I_s\sin u_s$$

where $u = f + \omega,$ usually known as the argument of latitude, and $\Delta\Omega = \Omega_s - \Omega,$ is the difference of the nodal longitudes.

In the subsequent developments the adopted reference frame is an equatorial system with the positive x-axis towards the node of the orbit of m, z-axis towards the north pole of the equator of the primary, and the y-axis completing a right handed system. The last equation can be rewritten in compact form as

$$\cos\phi = \sum_{i,j=-1}^{1} A_{i,j}\cos\left(f+\omega+i\left(f_s+\omega_s\right)+j\left(\Omega_s-\Omega\right)\right) \tag{10}$$

where the non-vanishing coefficients are

$$A_{-1,-1} = \frac{1}{4}\left(1+\cos I+\cos I_s+\cos I\cos I_s\right)$$

$$A_{-1,0} = \frac{1}{2}\sin I\sin I_s$$

$$A_{-1,1} = \frac{1}{4}\left(1-\cos I-\cos I_s+\cos I\cos I_s\right)$$

$$A_{1,-1} = \frac{1}{4}\left(1+\cos I-\cos I_s-\cos I\cos I_s\right)$$

$$A_{1,0} = -\frac{1}{2}\sin I\sin I_s$$

$$A_{1,1} = \frac{1}{4}\left(1-\cos I+\cos I_s-\cos I\cos I_s\right)$$

Taking the primary (e.g. the Earth) to be an oblate spheroid, a contribution to the perturbing potential come from the oblateness, this can be expressed as

$$\begin{aligned}\tilde{V} = -\frac{1}{r}\Bigg[&\frac{J_2}{2}\frac{R_\oplus^2}{r^2}\left(1-3\sin^2\phi'\right)+\frac{J_3}{2}\frac{R_\oplus^3}{r^3}\left(3\sin\phi'-5\sin^3\phi'\right)\\ &+\frac{J_4}{8}\frac{R_\oplus^4}{r^4}\left(3-30\sin^2\phi'+35\sin^4\phi'\right)+\ldots\Bigg]\end{aligned} \tag{11}$$

where ϕ' is the geocentric latitude and $R_\oplus, J_2, J_3, J_4$ are constants representing the equatorial radius of the Earth and the three leading zonal harmonics of the geopotential. And μ is taken as unity. Equation (1) can now be written as

$$\begin{aligned}\mathrm{H} = &-\frac{1}{2a}+N_s\Lambda_s+\frac{\mu_s}{r_s}\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathrm{D}_{n,k,\frac{1}{2}}\alpha^{2k+n}\cos n\phi\right)\\ &-\frac{J_2R_\oplus^2}{4r^3}\left[\left(1-3C^2\right)-3S^2\cos\left(2\omega+2f\right)\right]\end{aligned} \tag{12}$$

where the series of the perturbing potential is truncated beyond J_2. With $\sin\phi' = S\sin(\omega+f)$ where f, ω are the of true anomaly and argument of periapsis of the infinitesimal body and I is the inclination, with $S=\sin I$, $C=\cos I$. The appropriate canonical variables (Delaunay like variables) for the 1:1 commensurability are (Nesvorný et al., 2002)

$$\begin{aligned} l_1 &= \lambda-\lambda_s & L_1 &= L\\ l_2 &= -\varpi & L_2 &= L-G\\ l_3 &= -\Omega & L_3 &= G-H\\ l_4 &= \lambda_s & L_4 &= \Lambda_s+L \end{aligned}$$

where $L=\sqrt{a},\ G=L\sqrt{1-e^2}$ and $H=G\cos I$ are the Delaunay momenta, and $a,\ e,\ I,\ l,\ \varpi,\ \Omega$ are usual orbital elements. The mean and perihelion longitudes are related to the Delaunay angles (mean anomaly l, pericentric argument ω and nodal longitude Ω) by: $\lambda = l+\varpi$ and $\varpi=\omega+\Omega$ (regarding the mutual independence of each set). In these variables, the resonant angle l_1 and the conjugate momentum L_1 will describe the co-orbital resonant motion, $l_2,\ L_2$ and $l_3,\ L_3$ are related to planar and spatial secular evolutions, and $\lambda_s,\ \Lambda_s$ is the degree of freedom associated to short periodic effects.

Ordering

The aim of this section is to expand the Hamiltonian H in the form $\sum_{t=0}^{2}\frac{\varepsilon^t}{t!}\mathrm{H}_t$ taking the Moon as the secondary body. The integrable part of the Hamiltonian, H_0, depends only on L_1, and L_4 so (in the nonresonant case) l_1, and l_4 are the only fast variables. Then we can write the zero order, using the Delaunay like variables, as

$$\mathrm{H}_0 = -\frac{1}{2L_1^2} + N_s\left(L_4 - L_1\right) \tag{13}$$

With the Moon as the secondary body, $\mu_s \cong 0.0123$ can be considered of order ε. Then H_1 is given by

$$\mathrm{H}_1 = \frac{1}{r_s}\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathrm{D}_{n,k,\frac{1}{2}}\alpha^{2k+n}\cos n\phi\right) \tag{14}$$

which can be rewritten in general form as

$$\mathrm{H}_1 = \frac{1}{r_s}\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{[n/2]}\mathrm{D}_{n,k,\frac{1}{2}}\alpha^{2k+n}T_t^{(n)}\left(\cos\phi\right)^{n-2t}\right)$$

$$= \frac{1}{r_s}\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i,j=-1}^{1}\sum_{t=0}^{[n/2]}\mathrm{D}_{n,k,\frac{1}{2}}\alpha^{2k+n}T_t^{(n)}\left(A_{i,j}\cos\left(f+\omega+i\left(f_s+\omega_s\right)+j\Delta\Omega\right)\right)^{n-2t}\right)$$

or

$$\mathrm{H}_1 = \frac{1}{r_s}\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i,j=-1}^{1}\sum_{t=0}^{[n/2]}\sum_{\sigma=0}^{[(n-2t)/2]}\mathrm{D}_{n,k,\frac{1}{2}}T_t^{(n)}2^{1-n+2t}A_{i,j}\left(1-\frac{1}{2}\delta_{(n-2t)/2,\sigma}\right)\right.$$
$$\left.\times C_\sigma^{(n-2t)}\alpha^{2k+n}\cos\left[\left(n-2t-2\sigma\right)\left(f+\omega+i\left(f_s+\omega_s\right)+j\left(\Omega_s-\Omega\right)\right)\right]\right) \tag{15}$$

where

$$T_t^{(n)} = \left(-1\right)^t\left[\frac{n}{2}\right]\frac{\left(n-t-1\right)!}{\left(t\right)!\left(n-2t\right)!}2^{n-2t}, \qquad n \ge 1$$

$$T_t^{(0)} = 1$$

$$C_k^Z = \frac{Z!}{k!\left(Z-k\right)!}$$

where $[n/2]$ is the greatest integer less than or equal to $n/2,$ see Ahmed, (1994). Truncating beyond α^4, or $2k+n=4.$ Equation (14) reduces to

$$\mathrm{H}_1 = \frac{1}{a_s}\sum_{n,t=0}^{4}\sum_{\rho=-4}^{4}\mathrm{A}_{t,\sigma,n}\left(\frac{a}{a_s}\right)^n\left(\left(\frac{a_s}{r_s}\right)^{n+1}\left(\frac{r}{a}\right)^n\cos\left(t\left(f+\omega\right)+\sigma\left(f_s+\omega_s\right)+\rho\left(\Omega_s-\Omega\right)\right)\right) \tag{16}$$

where the non-vanishing coefficients are

$$\mathrm{A}_{0,\sigma,0} = \mathrm{D}_{0,k,\frac{1}{2}}$$

$$\mathrm{A}_{0,\sigma,2} = \mathrm{D}_{2,k,\frac{1}{2}}\left(A_{\sigma,\rho}^2 - 1\right)$$

$$\mathrm{A}_{0,\sigma,4} = \mathrm{D}_{4,k,\frac{1}{2}}\left(3A_{\sigma,\rho}^4 - 4A_{\sigma,\rho}^2 + 1\right)$$

$$\mathrm{A}_{1,\sigma,1} = \mathrm{D}_{1,k,\frac{1}{2}}A_{\sigma,\rho}$$

$$\mathrm{A}_{1,\sigma,3} = 3\mathrm{D}_{3,k,\frac{1}{2}}\left(A_{\sigma,\rho}^2 - 1\right)A_{\sigma,\rho}$$

$$\mathrm{A}_{2,\sigma,2} = \mathrm{D}_{2,k,\frac{1}{2}}A_{\sigma,\rho}^2$$

$$\mathrm{A}_{2,\sigma,4} = 4\mathrm{D}_{4,k,\frac{1}{2}}\left(A_{\sigma,\rho}^2 - 1\right)A_{\sigma,\rho}^2$$

$$\mathrm{A}_{3,\sigma,3} = \mathrm{D}_{3,k,\frac{1}{2}}A_{\sigma,\rho}^3$$

$$\mathrm{A}_{4,\sigma,4} = \mathrm{D}_{4,k,\frac{1}{2}}A_{\sigma,\rho}^4$$

Using the relations of Hansen coefficients as defined by

$$\left(\frac{a}{r}\right)^n \cos\left(mf + t\omega\right) = \sum_{k=-\infty}^{\infty} X_k^{-n,m} \cos\left(kl + t\omega\right)$$

Then H_1 can be rewritten as

$$\mathrm{H}_1 = \sum_{i_3=-4}^{8} \sum_{|i_1|,|i_2|,|i_4|,|i_5|,|i_6|=0}^{\infty} \mathrm{X}_{i_1,i_2,i_3,i_4} \cos\left[i_1 l_1 + i_4 l_4 + i_2 l_2 + i_3 l_3 + i_5 \Omega_s + i_6 \omega_s\right] \quad (17)$$

where the coefficients $\mathrm{X}_{q,k,t,\sigma}$ are functions of (a,e,I) of the infinitesimal body and the secondary body, and they are given by

$$\mathrm{X}_{q,k,t,\sigma} = \frac{1}{a_s}\sum_{n=0}^{4} \mathrm{A}_{t\sigma n}\left(\frac{a}{a_s}\right)^n X_k^{n,t}(e) X_q^{-(n+1),\sigma}(e_s)$$

Also, e, and e_s are the orbital eccentricities of the infinitesimal body and the secondary body, respectively. or If we write $J_2 = \mathrm{A}\varepsilon^2$ where A is a constant. Then H_2 is given by

$$\mathrm{H}_2 = -\frac{\mathrm{A}R_\oplus^2}{2r^3}\left[\left(1-3C^2\right) - 3S^2\cos\left(2\omega + 2f\right)\right] \quad (18)$$

which can be rewritten as

$$\mathrm{H}_2 = -\frac{\mathrm{A}R_\oplus^2}{2a^3}\sum_{k=-\infty}^{\infty}\left[\left(1-3C^2\right)X_k^{-3,0}\cos\left(k\left(l_1+l_4+l_2\right)\right)\right.$$
$$\left. -3S^2 X_k^{-3,2}\cos\left(k\left(l_1+l_4\right)+\left(k-2\right)l_2+2l_3\right)\right] \quad (19)$$

This order arises directly from the effect of the oblateness of the primary which is a direct effect performed by the primary on the infinitesimal.

Perturbation Technique

This section discusses the technique that will be employed to find an approximate analytical solution of a system of differential equations. Typically this method is applied to problems where the exact solution is either impossible or impractical to find by other means. Such systems are often nonlinear and/or nonautonomous in nature. Although the concept of the Lie transform dates back to more than a century ago, this concept has been introduced into perturbation methods by Hori (1966) through his pioneer work. Hori constructed an algorithm using the Lie series to determine the transformed Hamiltonian from the old one. Deprit (1969) constructed another algorithm to generate the new Hamiltonian recursively using the Lie transform allowing for the generator to depend explicitly on a small parameter.Let ε be the small parameter of the problem, and let the considered system of differential equations be

$$\dot{u} = \mathrm{H}_U \qquad \dot{U} = -\mathrm{H}_u \quad (20)$$

where (u,U,t) is the n (angle, action)-vector of adopted canonical variables, in addition to time variable. The Hamiltonian H is assumed expandable as

$$\mathrm{H} = \mathrm{H}_0 + \sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\mathrm{H}_n \quad (21)$$

and the system with $\mathrm{H} = \mathrm{H}_0$ is assumed integrable with $\mathrm{H}_0 = \mathrm{H}_0(U_1)$, what is required is to construct two (or more) canonical transformations $(u,U,t;\varepsilon) \rightarrow (u',U',t)$ analytic in ε at $\varepsilon = 0$. The transformed Hamiltonian and the corresponding generators will be assumed expandable as

$$\mathrm{H}^*\left(-,u_2',u_3',U',t;\varepsilon\right) = \mathrm{H}_0^*\left(U_1'\right) + \sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!}\mathrm{H}_n^*\left(-,u_2',u_3',U',t\right) \quad (22)$$

$$W\left(u',U',t;\varepsilon\right) = \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}W_{n+1}\left(u',U',t\right) \quad (23)$$

The basic identities are: Kamel (1969)

$$\mathrm{H}_0^* = \mathrm{H}_0 \quad (24)$$

$$\mathrm{H}_n^* = \tilde{\mathrm{H}}_n - \frac{DW_n}{Dt}, \qquad n \geq 1 \quad (25)$$

where

$$\tilde{\mathrm{H}}_n = \mathrm{H}_n + \sum_{1\le j\le n-1} \left[C_{j-1}^{n-1}\left(\mathrm{H}_{n-j};W_j\right) + C_j^{n-1} G_j \mathrm{H}_{n-j}^{*} \right] \tag{26}$$

Let u_1 be the fast variable in H_n. We choose H_n^* to be the average of $\tilde{\mathrm{H}}_n$ over $u_1^{'}$, i.e.

$$\mathrm{H}_n^* = \left\langle \tilde{\mathrm{H}}_n \right\rangle_{u_1^{'}} \tag{27}$$

and

$$\begin{aligned} P_n &= \tilde{\mathrm{H}}_n - \mathrm{H}_n^* = \frac{DW_n}{Dt} \\ &= \frac{\partial W_n}{\partial t} + \left(W_n;\mathrm{H}_0\right) \\ &= \frac{\partial W_n}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial W_n}{\partial u_1^{'}}\frac{\partial \mathrm{H}_0}{\partial U_1^{'}} \end{aligned} \tag{28}$$

so that

$$W_n = \left[\frac{\partial \mathrm{H}_0}{\partial U_1^{'}} + \frac{\partial u_1^{'}}{\partial t}\right]^{-1} P_n du_1^{'} \tag{29}$$

Elimination of the Short-Period Terms

Before proceeding to effect the short period transformation it is worthy to note that

1- The angle l_4 is the fast angle that is to be eliminated first.

2- the angle l_1 is a critical (resonant) angle so that its rate depends on the degree of the resonance considered. Assuming a case of shallow resonance we may consider l_1 as neither a fast nor a slow variable. Thus we may assume terms containing derivatives of trigonometric functions of l_1 as of periods of order $\sqrt{\varepsilon}$.

3- l_2 and l_3 are slow variables.

Zero Order

From equation (13) we can write

$$\mathrm{H}_0 = -\frac{1}{2L_1^2} + N_s\left(L_4 - L_1\right)$$

Hence

$$\mathrm{H}_0^* = \mathrm{H}_0 \,|_{L_1 = L_1^{'}, L_4 = L_4^{'}}$$

where N is the frequency of the mean longitude (λ), then

$$\mathrm{H}_0^* = \mathrm{H}_0 \,|_{L_1 = L_1^{'}, L_4 = L_4^{'}} = -\frac{1}{2L_1^2} + N_s\left(L_4 - L_1\right) \tag{30}$$

where the primes will be ignored for the sake of simplicity of writing till the elements of the transformation be obtained.

7.2 First Order

Recalling equation (16) we have

$$\mathrm{H}_1 = \sum_{i_3=-4}^{8} \sum_{|i_1|,|i_2|,|i_4|,|i_5|,|i_6|=0}^{\infty} \mathrm{X}_{i_1,i_2,i_3,i_4} \cos\left[i_1 l_1 + i_4 l_4 + i_2 l_2 + i_3 l_3 + i_5 \Omega_s + i_6 \omega_s\right]$$

setting

$$\tilde{\mathrm{H}}_1 = \mathrm{H}_1$$

Choosing

$$\mathrm{H}_1^* = \left\langle \tilde{\mathrm{H}}_1 \right\rangle_{l_4} = \sum_{i_3=-4}^{8} \sum_{|i_1|,|i_2|,|i_5|,|i_6|=0}^{\infty} \mathrm{X}_{i_1,i_2,i_3,0} \cos\left[i_1 l_1 + i_2 l_2 + i_3 l_3 + i_5 \Omega_s + i_6 \omega_s\right] \tag{31}$$

then the generating function W_1 can be written as

$$W_1 = \sum_{i_3=-4}^{8} \sum_{|i_1|,|i_2|,|i_4|,|i_5|,|i_6|=0\, i_4\neq 0}^{\infty} \left(i_4 \frac{\partial \mathrm{H}_0}{\partial L_4}\right)^{-1} \mathrm{X}_{i_1,i_2,i_3,i_4} \sin\left[i_1 l_1 + i_4 l_4 + i_2 l_2 + i_3 l_3 + i_5 \Omega_s + i_6 \omega_s\right]$$
$$= \sum_{i_3=-4}^{8} \sum_{|i_1|,|i_2|,|i_4|,|i_5|,|i_6|=0\, i_4\neq 0}^{\infty} \frac{1}{i_4 N_s} \mathrm{X}_{i_1,i_2,i_3,i_4} \sin\left[i_1 l_1 + i_4 l_4 + i_2 l_2 + i_3 l_3 + i_5 \Omega_s + i_6 \omega_s\right] \tag{32}$$

Second order

Firstly, we calculate $\tilde{\mathrm{H}}_2$ as

$$\tilde{\mathrm{H}}_2 = \mathrm{H}_2 + \left(\left(\mathrm{H}_1 + \mathrm{H}_1^*\right); W_1\right)$$

Now $\left(\mathrm{H}_1 + \mathrm{H}_1^*\right)$ can be written as

$$\left(\mathrm{H}_1 + \mathrm{H}_1^*\right) = \sum_{i_3=-4}^{8} \sum_{|i_1|,|i_2|,|i_4|,|i_5|,|i_6|=0}^{\infty} \mathrm{Y}_{i_1,i_2,i_3,i_4} \cos\left[i_1 l_1 + i_4 l_4 + i_2 l_2 + i_3 l_3 + i_5 \Omega_s + i_6 \omega_s\right] \tag{33}$$

where

$$\mathrm{Y}_{i_1,i_2,i_3,i_4} = \begin{cases} 2\mathrm{X}_{i_1,i_2,i_3,0} & |i_4| = 0 \\ \mathrm{X}_{i_1,i_2,i_3,i_4} & |i_4| \neq 0 \end{cases}$$

$$\tilde{\mathrm{H}}_{21} = \sum_{z_3=-12}^{16} \sum_{|z_1|,|z_2|,|i_4|,|i_4'|,|z_5|,|z_6|=0\, i_4'\neq 0}^{\infty} \frac{1}{i_4' N_s}$$
$$\times\Big[\mathrm{Z}_{z_1,z_2,z_3,i_4,i_4',z_5,z_6} \cos\left[z_1 l_1 + \left(i_4 - i_4'\right) l_4 + z_2 l_2 + z_3 l_3 + z_5 \Omega_s + z_6 \omega_s\right]$$
$$+\mathrm{W}_{z_1,z_2,z_3,i_4,i_4',z_5,z_6} \cos\left[z_1 l_1 + \left(i_4 + i_4'\right) l_4 + z_2 l_2 + z_3 l_3 + z_5 \Omega_s + z_6 \omega_s\right]\Big] \tag{34}$$

where the coefficients can be compiled as

$$\mathrm{Z}_{z_1,z_2,z_3,i_4,i_4',z_5,z_6} = \frac{1}{2}\left\{-\left(i_1 \frac{\partial \mathrm{X}_{i_1',i_2',i_3',i_4'}}{\partial L_1} + i_2 \frac{\partial \mathrm{X}_{i_1',i_2',i_3',i_4'}}{\partial L_2} + i_3 \frac{\partial \mathrm{X}_{i_1',i_2',i_3',i_4'}}{\partial L_3}\right)\mathrm{Y}_{i_1,i_2,i_3,i_4}\right.$$
$$\left.-\left(i_1' \frac{\partial \mathrm{Y}_{i_1',i_2',i_3',i_4'}}{\partial L_1} + i_2' \frac{\partial \mathrm{Y}_{i_1',i_2',i_3',i_4'}}{\partial L_2} + i_3' \frac{\partial \mathrm{Y}_{i_1',i_2',i_3',i_4'}}{\partial L_3}\right)\mathrm{X}_{i_1,i_2,i_3,i_4}\right\}$$

$$\mathrm{W}_{z_1,z_2,z_3,i_4,i_4',z_5,z_6} = \frac{1}{2}\left\{\left(i_1 \frac{\partial \mathrm{X}_{i_1',i_2',i_3',i_4'}}{\partial L_1} + i_2 \frac{\partial \mathrm{X}_{i_1',i_2',i_3',i_4'}}{\partial L_2} + i_3 \frac{\partial \mathrm{X}_{i_1',i_2',i_3',i_4'}}{\partial L_3}\right)\mathrm{Y}_{i_1,i_2,i_3,i_4}\right.$$
$$\left.-\left(i_1' \frac{\partial \mathrm{Y}_{i_1',i_2',i_3',i_4'}}{\partial L_1} + i_2' \frac{\partial \mathrm{Y}_{i_1',i_2',i_3',i_4'}}{\partial L_2} + i_3' \frac{\partial \mathrm{Y}_{i_1',i_2',i_3',i_4'}}{\partial L_3}\right)\mathrm{X}_{i_1,i_2,i_3,i_4}\right\}$$

with $z_n = i_n - i_n', i_n, i_n' \in [-\infty, \infty],\ n = 1, 2, ..., 6\ \neq 3,$ and $z_3 = i_3 - i_3', i_3, i_3' \in [-4, 8].$

For the Hamiltonian

$$\mathrm{H}_{21}^* = \left\langle \tilde{\mathrm{H}}_{21} \right\rangle_{l_4} = \frac{1}{2\pi}\int_0^{2\pi} \left(\left(\mathrm{H}_1 + \mathrm{H}_1^*\right); W_1\right) dl_4$$

which by (23) becomes

$$\mathrm{H}_{21}^* = \sum_{z_3=-12}^{16} \sum_{|z_1|,|z_2|,|i_4|,|i_4'|,|z_5|,|z_6|=0\, i_4'\neq 0}^{\infty} \frac{1}{i_4' N_s}$$
$$\times \mathrm{Q}_{z_1,z_2,z_3,i_4,i_4',z_5,z_6} \cos\left[z_1 l_1 + z_2 l_2 + z_3 l_3 + z_5 \Omega_s + z_6 \omega_s\right] \tag{35}$$

where

$$\mathrm{Q}_{z_1,z_2,z_3,i_4,i_4',z_5,z_6} = \begin{cases} \mathrm{Z}_{z_1,z_2,z_3,i_4,i_4',z_5,z_6} & i_4 = i_4' \\ \mathrm{W}_{z_1,z_2,z_3,i_4,i_4',z_5,z_6} & i_4 = -i_4' \end{cases}$$

then the generating function W_{21} can be written as

$$W_{21} = \sum_{z_3=-12}^{16} \sum_{|z_1|,|z_2|,|i_4|,|i_4'|,|z_5|,|z_6|=0\,|i_4'|\neq 0}^{\infty} \frac{1}{i_4' N_s^2}$$
$$\times\left[\left(\frac{1}{i_4-i_4'}\right)\mathrm{Z}_{z_1,z_2,z_3,i_4,i_4',z_5,z_6}\cos\left[z_1l_1+\left(i_4-i_4'\right)l_4+z_2l_2+z_3l_3+z_5\Omega_s+z_6\omega_s\right]\right.$$
$$\left.+\left(\frac{1}{i_4+i_4'}\right)\mathrm{W}_{z_1,z_2,z_3,i_4,i_4',z_5,z_6}\cos\left[z_1l_1+\left(i_4+i_4'\right)l_4+z_2l_2+z_3l_3+z_5\Omega_s+z_6\omega_s\right]\right] \tag{36}$$

Now

$$\mathrm{H}_{22}^* = \left\langle \mathrm{H}_2 \right\rangle_{l_4} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{H}_2 dl_4 = -\frac{\mathrm{A}R_\oplus^2}{2a^3}\left[\left(1-3C^2\right)X_0^{-3,0} - 3S^2X_0^{-3,2}\cos\left(2l_3-2l_2\right)\right] \tag{37}$$

then the generating function W_{22} can be written as

$$W_{22} = -\frac{\mathrm{A}R_\oplus^2}{a^3}\sum_{|k|=1}^{\infty}\frac{1}{kN_s}\left[\left(1-3C^2\right)X_k^{-3,0}\cos\left(k\left(l_1+l_4+l_2\right)\right)\right.$$
$$\left.-3S^2X_k^{-3,2}\cos\left(k\left(l_1+l_4\right)+\left(k-2\right)l_2+2l_3\right)\right]27 \tag{38}$$

Finally we can write

$$\mathrm{H}_2^* = \mathrm{H}_{21}^* + \mathrm{H}_{22}^*$$
$$= \sum_{z_3=-12}^{16}\sum_{|z_1|,|z_2|,|i_4|,|i_4'|,|z_5|,|z_6|=0\,i_4'\neq 0}^{\infty}\frac{1}{i_4'N_s}\mathrm{Q}_{z_1,z_2,z_3,i_4,i_4',z_5,z_6}\cos\left[z_1l_1+z_2l_2+z_3l_3+z_5\Omega_s+z_6\omega_s\right]$$
$$-\frac{\mathrm{A}R_\oplus^2}{2a^3}\left[\left(1-3C^2\right)X_0^{-3,0}-3S^2X_0^{-3,2}\cos\left(2l_3-2l_2\right)\right] \tag{39}$$

and $W_2 = W_{21} + W_{22}$, which can be obtained as

$$W_2 = \sum_{z_3=-12}^{16}\sum_{|z_1|,|z_2|,|i_4|,|i_4'|,|z_5|,|z_6|=0\,|i_4'|\neq 0}^{\infty}\frac{1}{i_4'N_s^2}$$
$$\left[\left(\frac{1}{i_4-i_4'}\right)\mathrm{Z}_{z_1,z_2,z_3,i_4,i_4',z_5,z_6}\cos\left[z_1l_1+\left(i_4-i_4'\right)l_4+z_2l_2+z_3l_3+z_5\Omega_s+z_6\omega_s\right]\right.$$
$$\left.+\left(\frac{1}{i_4+i_4'}\right)\mathrm{W}_{z_1,z_2,z_3,i_4,i_4',z_5,z_6}\cos\left[z_1l_1+\left(i_4+i_4'\right)l_4+z_2l_2+z_3l_3+z_5\Omega_s+z_6\omega_s\right]\right]$$
$$-\frac{\mathrm{A}R_\oplus^2}{a^3}\sum_{|k|=1}^{\infty}\frac{1}{kN_s}\left[\left(1-3C^2\right)X_k^{-3,0}\cos\left(k\left(l_1+l_4+l_2\right)\right)\right.$$
$$\left.-3S^2X_k^{-3,2}\cos\left(k\left(l_1+l_4\right)+\left(k-2\right)l_2+2l_3\right)\right] \tag{40}$$

Conclusion

As previously stated in the introduction the present work stressed on developing the Hamiltonian framework for the analysis of coorbital motion in terms of a set of Delaunay-like elements. Therefore we can conclude that the approach presented above is very useful for formulating the co-orbital motion. It renders an analytic insight to co-orbital motion problem. The disturbing function for the cases of co-orbital motion is represented in different forms with particular emphasis on the relevant form developed in terms of Laplace coefficients. The short periodic terms are eliminated using the usual Lie-Deprit-Kamel transform. The method presented for the solution of the problem will be applied in subsequent works for some special cases. Since the present general form of the

Hamiltonian mix it very difficult form to the problem to be treated using elliptic function and integrals which consequently complicate much analyzing to phase space portrait.

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