Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 81 (2015) 31681-31686

ISSN: 2229-712X

A Study on Analytic Solution of Burger's Equation arising in Longitudinal Dispersion Phenomenon in Groundwater Flow

Shreekant P Pathak and Twinkle Singh

Department of Applied Mathematics & Humanities, S. V. National Institute of Technology, Surat-395007, India.

ARTICLE INFO

Article history:

Received: 27 February 2015;

Received in revised form: 15 March 2015;

Accepted: 6 April 2015;

Keywords

Burger's equation,

Dispersion phenomenon,

Optimal homotopy analysis method

ABSTRACT

The present paper discusses an approximate solution of the Burger's equation arising in longitudinal dispersion phenomenon in groundwater flow. In the groundwater flow pure water displaced in longitudinal direction by salt water (or contaminated water) form a non linear partial differential equation which is known as of Burger`s equation. The longitudinal dispersion phenomenon may be rise in miscible or immiscible fluid flow through porous media. The problem of miscible displacement can be seen in the coastal areas, where fresh water beds are gradually moved by salt water (sea water). Longitudinal dispersion phenomenon plays an important role to control salinity of the soil in seashore region. We use the optimal homotopy analysis method based on the homotopy analysis method to solve the governing non-linear Burger`s equation and the numerical and graphical representations have been given.

Introduction

The major problem which is in any development and management of a water resources system is water quality [1, 2]. In fact, with the increased demand for water in most parts of the world, and with the intensification of water utilization, the quality problem becomes the limiting factor in the development of water resources in many parts of the world. Although in such regions, the quality of both surface and groundwater resources deteriorates as a result of pollution. Main reason of the pollution of groundwater in aquifers is their very slow velocity although; groundwater is more protected than surface water against pollution. The term "quality" usually refers either to energy in the form of heat or nuclear radiation or to materials contained in the water. Many materials dissolve in water. Due to new materials are coming into the market every day groundwater quality can be measured in terms of practically hundreds of parameters. The relevance of any of these materials depends on the use that is being considered. For example, salinity may be important if the water is intended for drinking, for irrigation, or for certain industries, but less important for recreation. Groundwater already contains a certain amount of dissolved matter, the term "pollutant" to denote dissolved matter carried with the water and accumulating in the aquifer. The Groundwater pollution is usually traced back from four sources, (1) Environmental (2) Domestic (3) Industrial (4) Agricultural. Aquifer is the geological formation that carries the water and allowed to pass through it. There are two types of aquifer based on the present or absent of water table known as unconfined or confined aquifer respectively. Coastal aquifers constitute an important source for water, especially in arid and semi-arid zones which border the sea. Many coastal areas are also heavily urbanized, a fact which makes the need for fresh water even more acute. In coastal aquifer hydraulic gradient exists toward the sea that serves as a recipient for the excess of their fresh water (replenishment minus pumpage). Owing to the presence of sea water in the aquifer formation under the sea bottom, a zone of contact is formed between the lighter fresh water (specific weight γ_f) flowing to the sea and the heavier underlying sea water (specific weight $\gamma_s > \gamma_f$). Typical cross sections with interfaces under natural conditions are shown in figure 1 gives a coastal phreatic aquifer with exploitation. The region of the soil that is unsaturated is known as the vadose zone (or unsaturated zone), and this is the region where the most interesting nonlinear hysteretic behaviour is observed [2, 3]. In the unsaturated zone, only part of the void space is occupied by water and the remainder being occupied by a gaseous and possibly by a non-aqueous phase liquid. In saturated ground water aquifers, all available pore spaces are filled with water. Above a capillary fringe some part of pore spaces also contains air. The mixing of pollutants with surrounding water flows generally includes underground porous layers, and the dispersion is a macroscopic phenomenon caused by a combination of molecular diffusion and hydrodynamic mixing occurring with laminar flow through porous medium. Analyzing the generic case of two fluids in contact flowing (simultaneous flow) through a porous medium it is observed that mixing is almost associated to the random walk of fluid (or tracer) particles through the disordered structure of the pore volume and thermal molecular agitation is dominant only at very low mean flow velocities. The steps of the random walk are much larger than those of thermal brownian motion so that the corresponding spreading scale and the width of the dispersion front are correspondingly increased. Of course, the minimum size of heterogeneities of the mixture obtained in this way is also larger; however, if the medium is adequately homogeneous, this size is of the order of the grain diameter so that molecular diffusion can generally complete the mixing. This mixing is known as hydrodynamic dispersion and it is referred to as mixing of miscible fluids. The flow of groundwater in coastal aquifers can be treated as an interface problem in which two fluids of different densities, fresh water and salt water, have a clear interface rather than a transition zone. This flow problem assumes that the fresh water flows over the salt water which is at rest. Saltwater intrusion, in which saline water displaces or mixes with fresh groundwater, is one of the major causes of groundwater contamination. Saltwater intrusion is usually caused when the hydrodynamic balance between the fresh water and the saline water is disturbed, such as when fresh groundwater is over pumped

Tele:

E-mail addresses: sppathak.svnit@gmail.com

in coastal aquifers. Saltwater intrusion can also occur when the natural barriers that separate fresh and saline water are destroyed, such as in construction of coastal drainage canals that enable tidal water to advance inland and percolate into a freshwater aquifer. In coastal aquifers, according to Ghyben-Herzberg [4, 5], fresh water exists below sea level for every meter of fresh water above sea level.

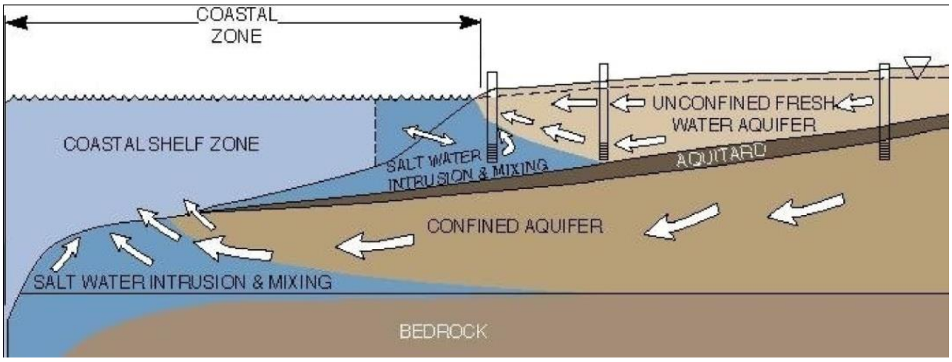

Figure 1. Typical cross section of a coastal aquifer

Mathematical model of the problem

In order to construct the mathematical model for the Burger's equation arising in longitudinal dispersion phenomena in groundwater flow, we have considered certain assumptions. We have considered that the dispersion zone is in one direction i.e. $x-$ direction only, so we avoids the transverse component of dispersion. Also we have assumed that the medium is homogeneous.
According to Darcy's law [1] the equation of continuity for the mixture, in the case of incompressible fluids are given as

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\left(\rho\bar{V}\right)=0, \tag{1}$$

where, ρ is the density for mixture and $\bar{V}$ is the seepage velocity vector.
The equation of diffusion for a fluid flow through a homogeneous porous medium, without increasing or decreasing the dispersing material is given by

$$\frac{\partial C}{\partial t}+\nabla\cdot\left(C\bar{V}\right)=\nabla\cdot\left(\rho\bar{D}\nabla\left(\frac{C}{\rho}\right)\right), \tag{2}$$

where, C is the concentration of the fluid f_s into the other host fluid f_f (i.e. C is the mass of fluid f_s per unit volume of the mixture) and $\bar{D}$ is the tensor coefficient of dispersion with nine components D_{ij} .In a laminar flow through a homogeneous porous medium at constant temperature ρ may be considered as constant, then

$$\nabla\cdot\bar{V}=0 \tag{3}$$

Therefore equation (2) can be rewritten as

$$\frac{\partial C}{\partial t}+\bar{V}\cdot\nabla C=\nabla\cdot\left(\bar{D}\nabla C\right), \tag{4}$$

Here, we assume that the seepage velocity $\bar{V}$ is along the $x-$ axis, then $\bar{V}=u(x,t)$ and none zero component will be $D_{11}\cong D_L=\gamma$ (coefficient of longitudinal dispersion) and other components will be zero by Polubarinova [4].

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}=D_L\frac{\partial^2 C}{\partial x^2}, \tag{5}$$

where, u is the component of flow velocity V along the $x-$ axis. It is depending on distance also it is time dependent along the $x-$ axis in the non-negative direction. By Mehta [6-9] it has been observed that seepage flow velocity u is related with concentration of the dispersing material as $u=\frac{C\left(x,t\right)}{C_0};x>0,$

where, the concentration of the contaminated water at $x=0$ is very high and it is constant $C_0\approx 1$. Hence,

$$\frac{\partial C}{\partial t}+C\frac{\partial C}{\partial x}=\gamma\frac{\partial^2 C}{\partial x^2}, \tag{6}$$

subject to appropriate initial and boundary conditions are taken as

$$\begin{aligned} C\left(x,0\right)&=e^{-x} \quad ;x>0 \\ C\left(0,t\right)&=1 \qquad ;t>0 \end{aligned} \tag{7}$$

Solution of mathematical model by OHAM

According to OHAM first we construct the zeroth order deformation equation as

$$(1-q)\mathcal{L}\left[\phi(x,t;q)-C_0(x,t)\right]=c_0 q \mathrm{N}\left[\phi(x,t;q)\right] \tag{8}$$

Where, $q\in[0,1]$ is the embedding parameter, $c_0\neq 0$ is an auxiliary parameter also know as convergence control parameter. We choose $\mathcal{L}=\frac{\partial}{\partial t}$ is an auxiliary linear operator [12] and $C_0(x,t)=(1+x\cdot t)e^{-x}$ is an initial guess of $C(x,t)$ [13]. According to OHAM expanding $\phi(x,\ t;q)$ in Maclaurin series with respect to q, then the corresponding *m*th order deformation equation is given by

$$\mathcal{L}\left[C_m(x,t)-\chi_m C_{m-1}(x,t)\right]=c_0\delta_m\left[C_{m-1}(x,t)\right], \tag{9}$$

where

$$\delta_m\left[C_{m-1}(x,t)\right]=\left(C_{m-1}\right)_t-\left(\sum_{r=0}^{m-1}C_rC_{m-1-r}\right)_x-\left(C_{m-1}\right)_{xx}$$

Taking inverse operator

$$C_m(x,t)=\chi_m C_{m-1}(x,t)+c_0\mathcal{L}^{-1}\delta_m\left[C_{m-1}(x,t)\right], \tag{10}$$

where

$$\chi_m=\begin{cases}0, & m\leq 1\\ 1, & m>1\end{cases}$$

To, find the optimal value of convergence c_0 applying Yabushita's approach [14] by means of finding the minimum square residual error as

$$E_m(c_0)=\frac{1}{(M+1)(N+1)}\sum_{i=0}^{M}\sum_{j=0}^{N}\left\{\mathrm{N}\left[\sum_{n=0}^{m}C_n\left(\frac{i}{M},\frac{j}{N}\right)\right]^2\right\} \tag{11}$$

The optimal value of convergence control parameter $c_0=-0.0024146449553088577$ with minimum square residual error $E_6=2.3854E-01$ at six order approximation, which can be notice by figure 2.

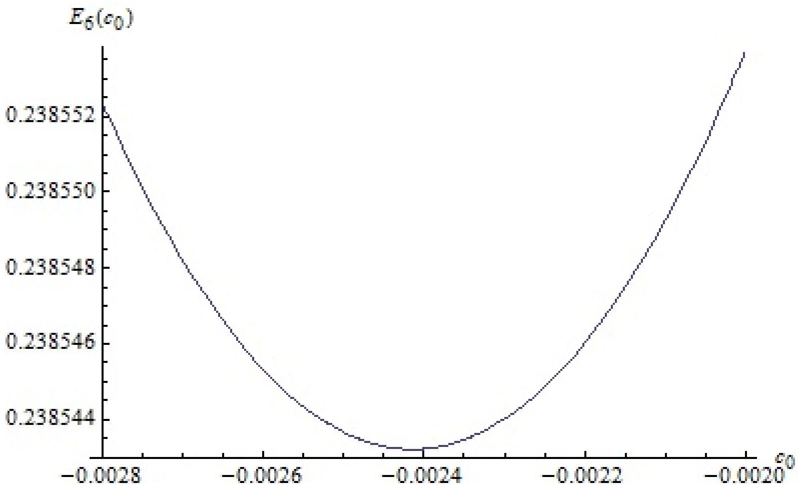

Figure 2. Square residual error at 6th-order approximation

Substituting the optimal value of convergence control parameter $c_0=-0.0024146449553088577$ and solve equation (10) for different values of m we have following approximations

$$C_1=-0.0024(-e^{-2x}t-e^{-x}t+\frac{1}{2}e^{-2x}t^2+e^{-x}t^2+e^{-x}tx$$
$$-e^{-2x}t^2x-\frac{1}{2}e^{-x}t^2x+\frac{1}{3}e^{-2x}t^3x-\frac{1}{3}e^{-2x}t^3x^2)$$

$$C_2 = -0.0024(-e^{-2x}t - e^{-x}t + \frac{1}{2}e^{-2x}t^2 + e^{-x}t^2 + e^{-x}tx - e^{-2x}t^2x$$
$$-\frac{1}{2}e^{-x}t^2x + \frac{1}{3}e^{-2x}t^3x - \frac{1}{3}e^{-2x}t^3x^2) - 0.0024(0.0024e^{-2x}t$$
$$+0.0024e^{-x}t - 0.0036e^{-3x}t^2 - 0.001e^{-2x}t^2 - 0.006e^{-x}t^2$$
$$+0.003e^{-3x}t^3 + 0.0082e^{-2x}t^3 + 0.0016e^{-x}t^3 - 0.0005e^{-3x}t^4$$
$$-0.002e^{-2x}t^4 - 0.0024e^{-x}tx + 0.005e^{-2x}t^2x + 0.0024e^{-x}t^2x$$
$$-0.005e^{-3x}t^3x - 0.008e^{-2x}t^3x - 0.0004e^{-x}t^3x + 0.003e^{-3x}t^4x$$
$$+0.004e^{-2x}t^4x - 0.0003e^{-3x}t^5x + 0.0024e^{-2x}t^3x^2 - 0.0024e^{-3x}t^4x^2$$
$$-0.0014e^{-2x}t^4x^2 + 0.0001e^{-3x}t^5x^2 - 0.0005e^{-3x}t^5x^3$$

Using initial guess and all sixth order approximation we get the approximate solution of Burger`s equation arising in costal aquifer in groundwater flow.

$$C(x,t) = \begin{cases} (1+x\cdot t)e^{-x} - 0.0024(-e^{-2x}t - e^{-x}t + \frac{1}{2}e^{-2x}t^2 + e^{-x}t^2 \\ +e^{-x}tx - e^{-2x}t^2x - \frac{1}{2}e^{-x}t^2x + \frac{1}{3}e^{-2x}t^3x - \frac{1}{3}e^{-2x}t^3x^2) \\ -0.0024(-e^{-2x}t - e^{-x}t + \frac{1}{2}e^{-2x}t^2 + e^{-x}t^2 + e^{-x}tx - e^{-2x}t^2x \\ -\frac{1}{2}e^{-x}t^2x + \frac{1}{3}e^{-2x}t^3x - \frac{1}{3}e^{-2x}t^3x^2) - 0.0024(0.0024e^{-2x}t \\ +0.0024e^{-x}t - 0.0036e^{-3x}t^2 - 0.001e^{-2x}t^2 - 0.006e^{-x}t^2 \\ +0.003e^{-3x}t^3 + 0.0082e^{-2x}t^3 + 0.0016e^{-x}t^3 - 0.0005e^{-3x}t^4 \\ -0.002e^{-2x}t^4 - 0.0024e^{-x}tx + 0.005e^{-2x}t^2x + 0.0024e^{-x}t^2x \\ -0.005e^{-3x}t^3x - 0.008e^{-2x}t^3x - 0.0004e^{-x}t^3x + 0.003e^{-3x}t^4x \\ +0.004e^{-2x}t^4x - 0.0003e^{-3x}t^5x + 0.0024e^{-2x}t^3x^2 - 0.0024e^{-3x}t^4x^2 \\ -0.0014e^{-2x}t^4x^2 + 0.0001e^{-3x}t^5x^2 - 0.0005e^{-3x}t^5x^3 + \ldots \end{cases} \quad (12)$$

Numerical and graphical representation

The following table 1 is shows the numerical values of solution of Burger's equation up to six approximations of OHAM using MATHEMATICA coding.

Table 1. Numerical value for the concentration $C(x,t)$ of contaminated water for $\gamma = 1$

X	T=0	T=0.1	T=0.2	T=0.3	T=0.4	T=0.5	T=0.6	T=0.7	T=0.8	T=0.9	T=1.0
0.1	0.9048	0.9161	0.9270	0.9375	0.9477	0.9576	0.9671	0.9763	0.9851	0.9935	1.0016
0.2	0.8187	0.8369	0.8548	0.8724	0.8897	0.9068	0.9235	0.9400	0.9562	0.9721	0.9877
0.3	0.7408	0.7645	0.7879	0.8112	0.8342	0.8569	0.8795	0.9018	0.9239	0.9458	0.9674
0.4	0.6703	0.6983	0.7261	0.7537	0.7811	0.8084	0.8355	0.8624	0.8891	0.9156	0.9419
0.5	0.6065	0.6378	0.6689	0.6998	0.7307	0.7613	0.7919	0.8223	0.8525	0.8826	0.9126
0.6	0.5488	0.5824	0.6160	0.6494	0.6827	0.7159	0.7490	0.7820	0.8149	0.8476	0.8802
0.7	0.4966	0.5319	0.5671	0.6022	0.6373	0.6722	0.7071	0.7419	0.7766	0.8113	0.8458
0.8	0.4493	0.4857	0.5219	0.5582	0.5943	0.6304	0.6664	0.7024	0.7383	0.7741	0.8099
0.9	0.4066	0.4434	0.4803	0.5170	0.5538	0.5905	0.6271	0.6637	0.7002	0.7367	0.7732
1	0.3679	0.4048	0.4418	0.4787	0.5156	0.5524	0.5892	0.6260	0.6627	0.6994	0.7361

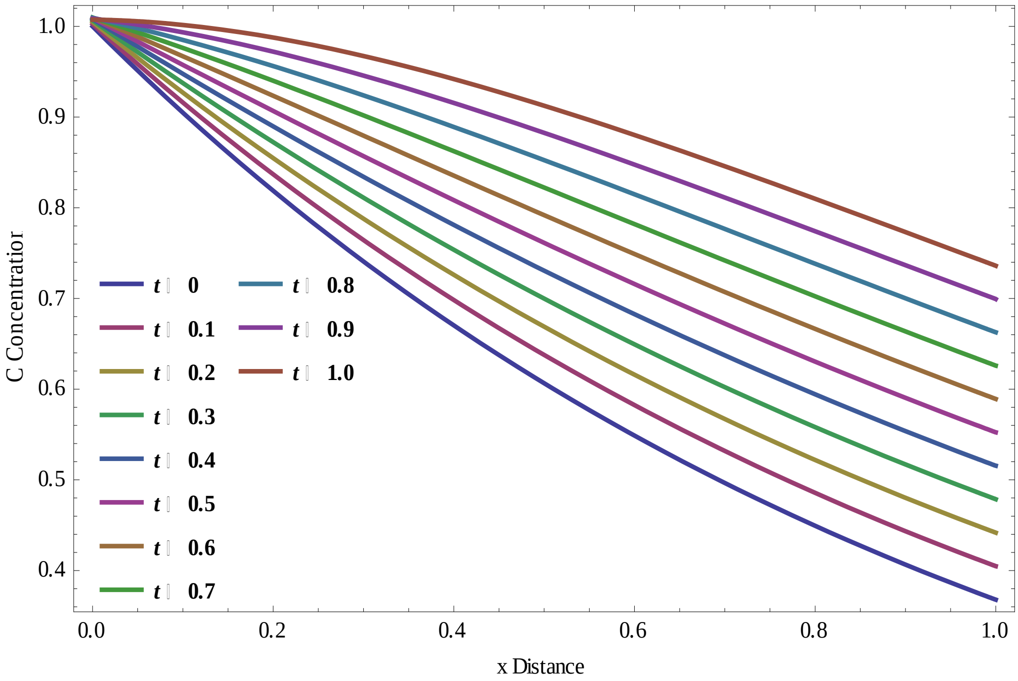

Figure 3. Concentration of contaminated water $C(x,t)$ at different distance x with fix time level $t = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$

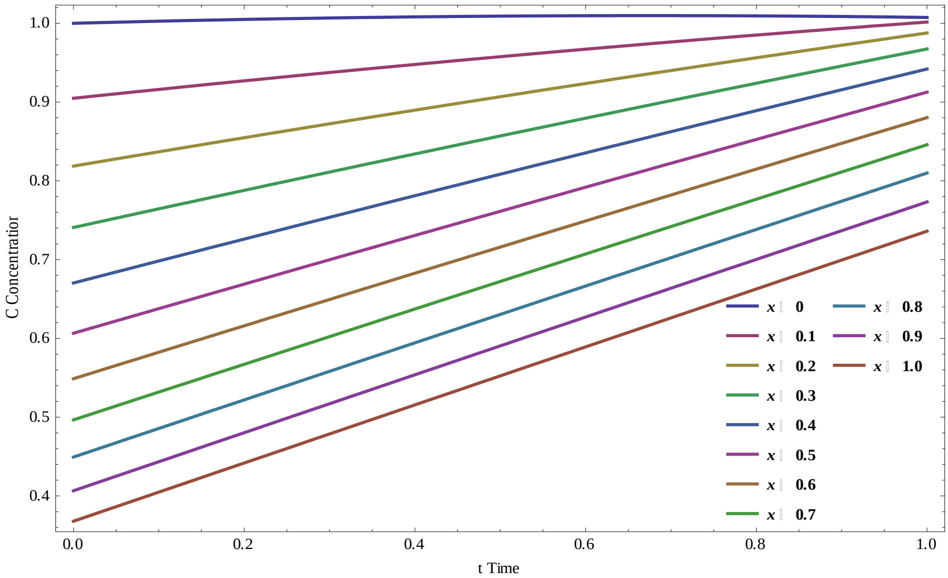

Figure 4. concentration of contaminated water $C(x,t)$ at different time intervals t with fix distance $x = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$

The figure 3 shows the approximate solution of mathematical model of Burger's equation arising in longitudinal dispersion phenomena in groundwater flow. From that one can predict the value of concentration at any time and any distance. Figure 4 shows that graph of concentration vs time at fix different distances.

Conclusion

In this study, optimal homotopy analysis method is use to find the analytic solution of the Burger's equation arising in longitudinal dispersion phenomenon in groundwater flow, and the optimal value of convergence control parameter $c_0 = -0.0024146449553088577$ is successfully obtained by minimizing the square residual error at sixth order approximation. The numerical values of the solution is sited in table 1 and also graphically demonstrated in order to consistence of the physical situation. From figure 3 we conclude that the concentration $C(x,t)$ of contaminated water decreases as the distance x increases for the given time $t > 0$. Here the initial concentration of contaminated water at $x = 0$ is highest and it decreases as distance x increases for given time $t > 0$, which is physically fact that at the source the concentration of contaminated water $C(x,t)$ is always highest and it is decreasing and dispersing from the source. It is also concluded from the figure 4 of the concentration $C(x,t)$ of contaminated water vs time t for fix distance x, the concentration of contaminated water is increasing for time t for fix distances x. Hence, it is fact that at the initial source the dispersion of contaminated water is not fast; therefore the concentration of contaminated water is slightly increasing with time t, for fixed distance x.

References

1. Bear, J., *Hydraulics of groundwater*. Courier Corporation, 2012
2. Marino, M. A., *Flow against dispersion in no adsorbing porous media.* Journal of Hydrology, 37(1), pp 149-158, 1978
3. Hunt B., *Dispersion sources in uniform groundwater flow*, J. Hydraulic Div., 104(1), pp 75-85, 197
4. Polubarinova-Kochina, P. Ya., *Theory of Ground water Movement,* Princeton Uni. Press, 1962
5. Scheidegger A. E., *General theory of dispersion in porous media,* J. Geophysics, Res., 66 (10), pp 3273-3278, 1961
6. Joshi, M. S., Desai, N. B., & Mehta, M. N., *Solution of the burger's equation for longitudinal dispersion phenomena occurring in miscible phase flow through porous media,* Journal of Engineering and Technological Sciences, 44(1), pp 61-76, 2012
7. Meher, R., Mehta, M. N., & Meher, S. K., *Adomian decomposition method for dispersion phenomena arising in longitudinal dispersion of miscible fluid flow through porous media.* Journal of advances in theoretical and applied mechanics, 3(5), pp 211-220, 2010
8. Mehta M.N., Patel T., *A solution of Burger's equation type one dimensional ground water recharge by spreading in porous media*, Journal of Indian Acad. Math. 28, pp 25-32, 2006
9. Borana, R., Pradhan, V., & Mehta, M., *Numerical Solution of Burger's equation arising in Longitudinal Dispersion Phenomena in Fluid Flow through Porous Media by Crank-Nicolson Scheme*, 5th International Conference on Porous Media and Their Applications in Science, Engineering and Industry, Eds, ECI Symposium Series, 2014.
10. Liao, S., *Homotopy analysis method in non-linear differential equations,* Heidelberg, New York, Higher Education Press, Beijing, Springer, 2012
11. Pathak S., & Singh T., *Optimal homotopy analysis methods for solving the linear and nonlinear fokker-planck equations*, British Journal of Mathematics & Computer Science, SCIENCEDOMAIN international, 7(3), pp 209-217, 2015
12. Baxter M, Van Gorder RA, Vajravelu K., *On the choice of auxiliary linear operator in the optimal homotopy analysis of the Cahn-Hilliard initial value problem.* Numerical Algorithms, 66(2), pp 269-298, 2014
13. Patel, K., Mehta, M. N., & Singh, T. R., *A solution of one-dimensional dispersion phenomenon by homotopy analysis method*, International Journal of Modern Engineering Research (IJMER), Vol. 3, Issue. 6, pp 3626-3631, 2013
14. Yabushita K, Yamashita M, Tsuboi K., *An analytic solution of projectile motion with the quadratic resistance law using the homotopy analysis method.* J Phys, pp 8403- 8416, 2007