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Approximate solution of fractional-order nonlinear sine-Gordon equation

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ABSTRACT

In this letter, the fractional modified decomposition method has been implemented for solving nonlinear sine-Gordon equation of fractional order. The fractional derivatives are described in the Caputo sense. In these schemes, the solution constructed in power series with easily computable components. The method is powerful tool for obtaining analytic and approximate solutions for different types of fractional differential equations.

Introduction

The nonlinear sine-Gordon equation is one of the basic equations of the modern nonlinear wave theory [1]. Firstly it appeared in the study of differential geometry of surfaces with constant Gaussian curvature. This equation also appeared in many scientific research areas such as the magnetic-flux propagation in large Josephson junctions, the motion of rigid pendulum attached to a stretched wire, solid state physics, nonlinear optics, and dislocations in metals, etc. [2,3]. Due to its wide applications and important mathematical properties, a great deal of potential has been devoted to studying the different solutions and physical phenomena related to this equation [4-9].

Preliminaries

The fractional order Klein-Gordon equation in one dimensional space has the form [4]

$$D_t^{\alpha} u - u_{xx} + \frac{dV(u)}{dt} = 0; \quad 1 < \alpha \le 2, \tag{2.1}$$

where $V = V(u)$ is a general nonlinear function of u. We will consider a particular case of equation (1), the so-called time-fractional sine-Gordon equation which has the form

$$D_t^{\alpha} u - u_{xx} + \sin(u) = 0; \quad 1 < \alpha \le 2, \; L_0 < x < L_1, \; t > t_0, \tag{2.2}$$

$$u(x,t_0) = f(x), \; u_t(x,t_0) = g(x); \qquad L_0 \le x \le L_1.$$

Differential equations of fractional order have been paid particular attention of many studies due to their frequent appearance in various applications in fluid mechanics, economic, viscoelasticity, biology, physics and engineering. This is because of the fact that the realistic modeling of a physical phenomenon does not depend only on the instant time, but also on the history of the previous time which can also be successfully achieved by using fractional calculus. In other words, previous values of the solution and the derivatives in fractional order differential equations are required to obtain a solution at a particular instance. The memory effect of the convolution in the fractional integral gives the equation increased expressive power. Also using fractional -order differential equations can help us to minimize the errors arising from the neglected parameters in modeling real life phenomena.

The basic goal of this work is devoted to introduce an analytical technique, namely fractional modified Laplace decomposition method (FMLDM) [10] to solve time-fractional sine-Gordon equation (2.1). The proposed method (FMLDM) is coupling of Adomian decomposition method (ADM) and the Laplace decomposition method (LDM). The main advantage of this proposed method is its capability of combining two powerful methods for obtaining rapid convergent series for fractional partial differential equations.

Fractional calculus

Fractional calculus is a generalization of classical differentiation and integration to arbitrary (non-integer) order. The use of fractional-orders differential and integral operators in mathematical models has become increasingly widespread in recent years [11]. Many mathematicians and applied researchers have tried to model real processes using the fractional calculus [12-15].

Definition 2.1. The fractional integral of order $\beta \in \mathbf{R}^+$ of the function $f(t)$, $t > 0$ is defined by

$$I^{\beta} f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds, \tag{3.1}$$

and the Caputo fractional derivative of order $\alpha \in (n-1, n]$ of the function $f(t), t > 0$ is defined by

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$$D^{\alpha} f(t) = I^{n-\alpha} D^{n} f(t),$$
$$D \equiv \frac{d}{dt}. \quad (3.2)$$

Definition 2.2. The Laplace transform $L[f(t)]$ of the Riemann-Liouville fractional integral is defined as [2]:

$$L[I^{\alpha} f(t)] = s^{-\alpha} F(s) \quad (3.3)$$

Definition 2.3. The Laplace transform $L[f(t)]$ of the Caputo fractional derivative is defined as [2]

$$L[D_t^{\alpha} u(x,t)] = s^{\alpha} L[u(x,t)] - \sum_{k=0}^{n-1} s^{\alpha-k-1} u^{(k)}(x,0), \quad n-1<\alpha\le n. \quad (3.4)$$

For the main properties of the fractional derivatives and integrals ([16-19].

The modified fractional Laplace decomposition method (MFLDM)

The basic idea of the fractional modified Laplace decomposition for the fractional partial differential equation can be clarified as follow [10]. Consider the following general nonlinear fractional partial differential equation

$$D_t^{\alpha} u(x,t) + R[u(x,t)] + N[u(x,t)] = g(x,t), \quad t>0, \; x\in R, \; n-1<\alpha\le n, \quad (4.1)$$

where $R[u], N[u]$ indicate the linear and nonlinear terms and $g(x,t)$ are continuous functions. The methodology consists of applying Laplace transform first on both sides of Eq. (4.1), we get

$$L[u(x,t)] = s^{-\alpha}\left(\sum_{k=0}^{n-1} s^{\alpha-k-1} u^{(k)}(x,0) - L\big(R[u(x,t)] + N[u(x,t)] - g(x,t)\big)\right) \quad (4.2)$$

Operating the inverse Laplace transform on both sides in Eq. (4.2), we get

$$u(x,t) = G(x,t) - L^{-1}\left(s^{-\alpha} L\big(R[u(x,t)] + N[u(x,t)]\big)\right), \quad (4.3)$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions.

The Laplace transform decomposition admits a solution in the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \quad (4.4)$$

The nonlinear term $N[u(x,t)]$ decomposed as

$$N[u(x,t)] = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n), \quad (4.5)$$

where A_n are Adomian polynomials of $u_0, u_1, \ldots, u_n$ and it can be calculated by the following formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left[\sum_{k=0}^{\infty} \lambda^k u_k\right]\Bigg|_{\lambda=0} \qquad n\ge 0$$

Substituting Eqs. (8) and (9) in Eq. (7), we get

$$\sum_{n=0}^{\infty} u_n(x,t) = G(x,t) - L^{-1}\left[s^{-\alpha} L\left(R\left(\sum_{n=0}^{\infty} u_n(x,t)\right) + \sum_{n=0}^{\infty} A_n\right)\right] \quad (4.6)$$

The modified Laplace decomposition method suggests that if the zeroth component $u_0(x,t) = G(x,t)$ and the function $G(x,t)$ can be dividing into two parts such as $G_0(x,t)$ and $G_1(x,t)$, then one can formulate the recursive algorithm $u_0(x,t)$ and general term in a form of the modified recursive scheme as follows

$$u_0(x,t) = G_0(x,t), \quad (4.7)$$
$$u_1(x,t) = G_1(x,t) - L^{-1}\left(s^{-\alpha} L\big(R[u_0] + A_0\big)\right), \quad (4.8)$$
$$u_{n+1}(x,t) = -L^{-1}\left(s^{-\alpha} L\big(R[u_n] + A_n\big)\right), \quad n\ge 1. \quad (4.9)$$

Applications

Example 1. Consider the following time-fractional sine-Gordon equation

$$D_t^{\alpha} u - u_{xx} + \sin(u) = 0; \; 1 < \alpha \le 2, \; -\infty < x < \infty, \; t > 0, \tag{5.1}$$

Subject to initial condition

$$u(x,0) = 0, \; u_t(x,0) = 4\operatorname{sech} x,$$

the exact solution, for the special case $\alpha \to 1$, is given by

$$u(x,t) = 4\tan^{-1}[t \operatorname{sech} x]. \tag{5.2}$$

By using Eqs. (4.7)-(4.9) we could be able to calculate some of the terms of the decomposition series (4.4) as

$$u_0(x,t) = 0,$$

$$u_1(x,t) = 4t \operatorname{sech} x,$$

$$u_2(x,t) = -\frac{8\operatorname{sech}^3 x}{\Gamma(\alpha+2)} t^{\alpha+1},$$

$$u_3(x,t) = -\frac{32(\cosh 2x - 2)\operatorname{sech}^5 x}{\Gamma(2\alpha+2)} t^{2\alpha+1},$$

and so on. Substituting $u_0, u_1, u_2, u_3, \ldots$ into Eq. (4.4) gives the solution $u(x,t)$ in the series form:

$$u(x,t) = 4t\operatorname{sech} x - \frac{8\operatorname{sech}^3 x}{\Gamma(\alpha+2)} t^{\alpha+1} - \frac{32(\cosh 2x - 2)\operatorname{sech}^5 x}{\Gamma(2\alpha+2)} t^{2\alpha+1} - \ldots \tag{5.3}$$

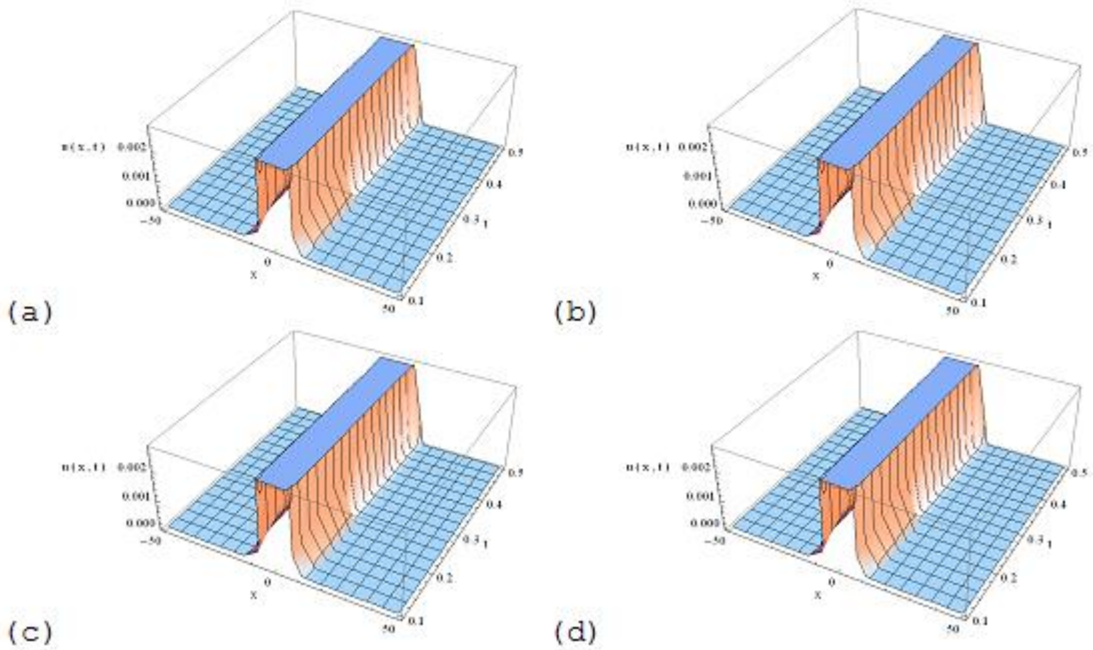

Fig 1. The approximate solution of Eq. (5.1) shown in the figure (a) in comparison with the exact solution Eq. (5.2) shown in the figure (b) when $\alpha = 2$, in (c) when $\alpha = 1.99$ and in (d) when $\alpha = 1.95$

Example 2. Consider the following time-fractional sine-Gordon equation

$$D_t^{\alpha} u - u_{xx} + \sin(u) = 0, \; 1 < \alpha \le 2, \; -\infty < x < \infty, \; t > 0 \tag{5.4}$$

subject to initial condition

$$u(x,0) = \pi + \varepsilon \cos \mu x, \; u_t(x,0) = 0.$$

We could be able to calculate some of the terms of the decomposition series (4.4) by using Eqs. (4.7)- (4.9) as

$$u_0(x,t) = \pi + \varepsilon \cos(\mu x),$$

$$u_1(x,t) = \frac{\varepsilon(1-\mu^2)\cos(\mu x)}{\Gamma(\alpha+1)} t^{\alpha},$$

$$u_2(x,t) = \frac{\varepsilon(1-\mu^2)^2 \cos(\mu x)}{\Gamma(2\alpha+1)} t^{2\alpha},$$

$$u_3 = \frac{\varepsilon(1-\mu^2)^3 \cos(\mu x)}{\Gamma(3\alpha+1)} t^{3\alpha} - \frac{\varepsilon^3 \cos^3(\mu x)}{6\Gamma(\alpha+1)} t^{\alpha},$$

And so on, so the solution $u(x,t)$ in the series form:

$$u(x,t) = \pi + \varepsilon \cos(\mu x) + \left(\frac{\varepsilon(1-\mu^2)\cos(\mu x)}{\Gamma(\alpha+1)} - \frac{\varepsilon^3 \cos^3(\mu x)}{6\Gamma(\alpha+1)} \right) t^{\alpha} + \frac{\varepsilon(1-\mu^2)^2 \cos(\mu x)}{\Gamma(2\alpha+1)} t^{2\alpha}$$

$$+ \frac{\varepsilon(1-\mu^2)^3 \cos(\mu x)}{\Gamma(3\alpha+1)} t^{3\alpha} + \dots \qquad (5.5)$$

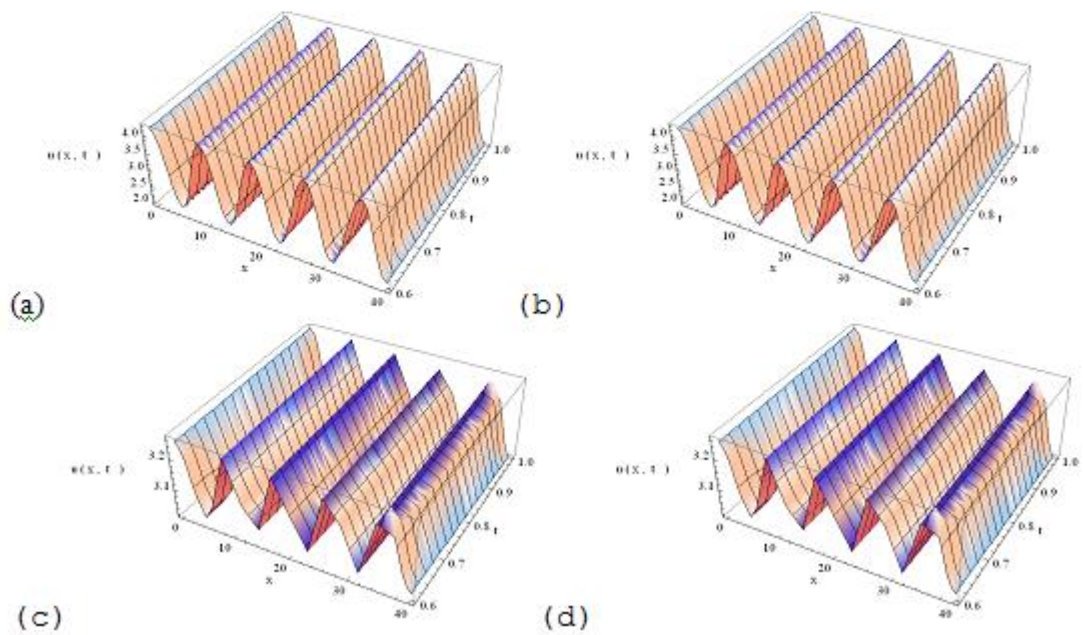

Fig 2. The approximate solution of Eq. (5.4) shown in the figure (a) when $\alpha = 2, \varepsilon = 1$, in (b) when $\alpha = 1.95, \varepsilon = 1$, in (c) when $\alpha = 2, \varepsilon = 0.1$ and in (d) when $\alpha = 1.95, \varepsilon = 0.1$.

Conclusion

In this letter, the fractional modified Laplace decomposition method has been successfully applied to obtain the numerical solutions of the time fractional nonlinear sine-Gordon equation. The reliability of this method and reduction in computations give this method a wider applicability. From Figs. 1-2, we deduce the behavior of the approximate solutions is the same behavior of the exact solution at some different values of α. The modified fractional Laplace decomposition method was clearly very efficient and powerful technique in finding the approximate solutions of the proposed equations.

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