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Transitivity and Parameters Related to Domination

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ABSTRACT

This paper is about vertex transitive graphs and their domination related parameters. In particular we establish that all the vertices of a vertex transitive graph have the property that removal of any vertex from the graph either decreases the value of the parameter or does not change the value of the parameter.

Introduction

The vertex transitive graphs are highly symmetric as they have many automorphisms. They have been studied from algebraic view point so far. It may be interesting to study them from a different angle and perspective. The aim of this paper is to consider these graphs for domination related parameters. In particular we observe the effect of removing a vertex from the graph on these parameters. In fact we show that the values of these parameters do not change or decreases when any vertex is removed.

Preliminaries

Definition -1: Vertex Covering Set [4]

Let G be a graph. A set $S \subseteq V(G)$ is said to be a vertex covering set of the graph G if every edge has at least one end point in S.

Definition 2: Minimal Vertex Covering Set [4]

If S is a vertex covering set such that no proper subset of S is a vertex covering set then S is called minimal vertex covering set.

Definition 3: Minimum Vertex Covering Set [4]

A vertex covering set with minimum cardinality is called minimum vertex covering set. It is also called α_0 set.

Note that every minimum vertex covering set is a minimal vertex covering set.

Definition 4: Vertex Covering Number [4]

The vertex covering number of the graph G is the cardinality of any minimum vertex covering set of the graph G. It is denoted by $\alpha_0(G)$ or simply α_0.

Definition 5: Vertex Transitive Graph [4]

Let G be a graph then G is said to be vertex transitive if for every $u,v \in V(G)$ there is an automorphism $f:V(G) \to V(G)$ such that $f(u) = v$.

Definition 6: Minimum Dominating set [4]

A Dominating set with minimum cardinality is called a minimum dominating set or a γ set.

Definition 7: Domination number [4]

The cardinality of a minimum dominating set of G is called the domination number of G and it is denoted as $\gamma(G)$.

The following notations can be found in [1]

- $V^0_{cr} = \{v \in V(G): \alpha_0(G-v) = \alpha_0(G)\}$
- $V^+_{cr} = \{v \in V(G): \alpha_0(G-v) > \alpha_0(G)\}$
- $V^-_{cr} = \{v \in V(G): \alpha_0(G-v) < \alpha_0(G)\}$

Vertex covering and vertex transitivity

Lemma 8: [2] If $v \in V(G)$ then

(1) $\alpha_0(G-v) \leq \alpha_0(G)$.

(2) If $\alpha_0(G-v) < \alpha_0(G)$ then $\alpha_0(G-v) = \alpha_0(G) - 1$.

Theorem 9: [2] Let G be a graph and $v \in V(G)$ then $v \in V^-_{cr}$ if and only if there is a α_0 set S_1 such that $v \in S_1$.

Corollary 10: [2] Let G be a graph and $v \in V(G)$ then $v \in V^0_{cr}$ if and only if v does not belongs to any minimum vertex covering set of the graph G.

Corollary 11 :[2] Suppose $S_1, S_2, \ldots S_k$ are all α_0 sets of the graph G and $v \in V^0_{cr}$ then N(v) is subset of $S_1 \cap S_2 \cap \ldots \cap S_k$.

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Corollary -12:[2] Let G be a graph and $v \in V^0_{cr}$ such that v is not an isolated vertex in G then $S_1 \cap S_2 \cap \ldots \cap S_k$ is non empty.
Theorem -13: [1] if G is vertex transitive graph with at least one edge then every vertex $v \in V^-_{cr}$.
Proof: Let S be a minimum vertex covering set in G and v be a vertex of the graph G. If $v \in S$ then $v \in V^-_{cr}$ by Theorem -9.
If $v \notin S$ then, let $u \in S$. Let f be an automorphism of the graph G such that f(u) = v (because G is vertex transitive graph). Now consider the set f(S) which is minimum vertex covering set of the graph G and it contains f(u) = v that is $v \in f(S)$. Thus, f(S) is a minimum vertex covering set of G such that $v \in f(S)$. So, again by Theorem - 9 $v \in V^-_{cr}$. Thus, every vertex of the graph G belongs to V^-_{cr}.
Theorem 14: [1] If G is a graph without isolated vertices and if S_1 and S_2 are disjoint vertex covering set of the graph G then,
(1) G is a bipartite graph.
(2)S_1 and S_2 are minimal vertex covering set of the graph G.
Proof: (1) Let e = uv be an edge of graph G then either $u \in S_1$ and $v \in S_2$ or $u \in S_2$ and $v \in S_1$. Thus, every edge joins a vertex of S_1 to a vertex of S_2 .
Moreover if x is any vertex of graph G and if e is an edge whose one end vertex is x then $x \in S_1$ or $x \in S_2$. Thus every vertex of the graph G belongs to either S_1 or S_2. Thus, G is a bipartite graph.
(2) Now let v be any vertex of S_1 and if e is an edge whose end vertex is v then S_1-{v} does not contain the end vertex v of the edge e. Thus, S_1-{v} is not a vertex covering set of the graph G. Hence S_1 is a minimal vertex covering set of the graph G. Similarly S_2 is a minimal vertex covering set of the graph G.
Corollary 15: [1] If G is a graph without isolated vertices and if G has an odd number of vertices then any two minimum vertex covering set have non empty intersection.
Proof: Suppose S_1 and S_2 are disjoint minimum vertex covering set of the graph G. Then by Theorem-14 graph G is a bipartite. Hence $|V(G)| = |S_1| + |S_2|$ .Since $|S_1| = |S_2|$, |V(G)| is an even number which is not true. Thus, $S_1 \cap S_2 \neq ¢$.
Theorem 16:[1] If G is a vertex transitive graph which is not null graph then
(1) There are at least two distinct minimum vertex covering sets in the graph G.
(2) The intersection of all minimum vertex covering sets of G is empty set.
Proof: (1) Since G is not a null graph there is a proper vertex covering set of the graph G, therefore there is a proper subset S of V (G) which is a minimum vertex covering set. Now let $y \in S$ and $x \notin S$. Since G is a vertex transitive graph. So, there is an automorphism f: V (G) → V (G) such that f(y) = x then f(s) is a minimum vertex covering set containing f(y) = x. Note that $S \neq f(S)$ because $x \in f(S)$ but $x \notin S$. Thus, S and f(S) are two distinct minimum vertex covering sets of G.
(2) Let $S_1,S_2,S_3,\ldots.S_k$ be all the minimum vertex covering sets of the graph G and we assume that $S_i \neq S_j$, if $i \neq j$.
Now suppose $S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_k \neq \phi$ Let $y \in S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_k$. Note that this intersection is a proper subset of S_i for every i.
Let $x \in S_1$ such that $x \notin S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_k$. Now since G is vertex transitive then there is an automorphism f such that f(y) = x. Now the set $\{S_1, S_2, S_3,\ldots S_k\} = \{f(S_1), f(S_2) f(S_3), \ldots f(S_k)\}$.
Now $f(y) \in f(S_1) \cap f(S_2) \cap f(S_3) \cap, \ldots \cap f(S_k)$, $x \in S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_k$ but
$x \notin S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_k$. This is a contradiction. Hence $S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_k = \phi$
Theorem 17: Suppose G is a vertex transitive graph which is not a null graph
(1)If G has exactly two minimum vertex covering sets then they are disjoint, the graph is bipartite graph and the $\alpha_0(G) = n/2$.
(2)If G is a bipartite graph and if G has n (even) vertices then G has exactly two disjoint minimum vertex covering sets and $\alpha_0(G) = n/2$.
Proof: (1) Suppose S_1 and S_2 are the only minimum vertex covering sets of the graph G then S_1 and S_2 are disjoint. (by last Theorem-16).
Now since every edge has one end point in S_1 and the other end point in S_2 then the graph is bipartite. Also by (Theorem-14) n must be even. Also note that S_1 or S_2 does not contain any isolated vertex.(In fact all vertices are of the graph G have the same degree. Since the graph is vertex transitive and therefore regular.) Thus, $\{S_1,S_2\}$ is partition of V(G) and since $|S_1| = |S_2|$. So, $\alpha_0(G) = n/2$.
(2) Let m = n/2. We will prove that $\beta_0(G) = m$. First we note that if $\{V_1,V_2\}$ is a bipartition of the graph G then $|V_1| = |V_2| = m$.This is because the graph G is vertex transitive and hence k-regular for some $k \geq 1$. The number of edges incident with vertices of V_1 is equal to $k|V_1|$ and the same edges are incident with vertices of V_2 and the number of such edges is equal to $k|V_2|$. Hence $k|V_1| = k|V_2|$.So, $|V_1| = |V_2| = m = n/2$.The set of vertices of V_1 is an independent set. Hence $\beta_0(G) \geq m$. Now we prove that any set with m+1 vertices cannot be an (maximum) independent set.
Let S be any set with m+1 vertices. S has at least one vertex fromV_1 and at least one vertex from V_2. Let t be the number of vertices in S which are in V_1 then m+1-t vertices of S are in V_2.
Suppose S is an independent set. Consider the edges which are incident with those vertices of S which are in V_1. The number of such edges equals to kt.The other end vertices of these edges (kt) will be different from m+1-t vertices of S which are in V_2. The number such vertices is equal to m-(m+1-t) = t-1 vertices. The Total number of edges which are incident with these t-1 vertices equal to k(t-1).This contradicts to above statement that those kt edges have end vertices in these t-1 vertices .Thus S cannot be an independent set .Hence the maximum size of an independent set is m .Thus $\beta_0(G) = m$.

Domination Number and vertex transitive graphs

Lemma 18: If G is a vertex transitive graph with at least one edge then G has at least two minimum dominating sets.
Proof:- Let S be a minimum dominating set of G. Let $x \in S$ and $y \notin S$. Since G is a vertex transitive graph there is an automorphism f such that f(x) = y. Now consider the set f(S), since f is an automorphism , f(S) is a minimum dominating set. Also S and f(S) are distinct sets because $y \in f(S)$ but $y \notin S$.

Remark 19: The vertex transitivity of the graph is necessary in the above lemma. For example the Path Graph P_3 with three vertices has only one minimum dominating set.

Theorem 20: Let G be a vertex transitive graph with at least one edge, then the intersection of all minimum dominating set is empty.

Proof: Let $\{S_1, S_2, S_3 \ldots S_k\}$ be the collection of all minimum dominating sets of G. Suppose $y \in \bigcap_{i=1}^{k} S_i$. Let $x \in S_1$ such that $x \notin \bigcap_{i=1}^{k} S_i$. Since G is vertex transitive there is an automorphism f such that f(y) = x. Now $\{f(S_1), f(S_2), \ldots f(S_k)\}$ is also the collection of all minimum dominating set of G. Since $y \in \bigcap_{i=1}^{k} S_i$, $f(y) \in \bigcap_{i=1}^{k} f(Si) = \bigcap_{i=1}^{k} S_i$.Thus, $x \in \bigcap_{i=1}^{k} S_i$, which is a contradiction. Hence the theorem.

Now for any graph G, $V^+ = \{v \in V(G): \gamma(G) < \gamma(G-v)\}$.

$V^- = \{v \in V(G): \gamma(G) > \gamma(G-v)\}$.

$V^0 = \{v \in V(G): \gamma(G) = \gamma(G-v)\}$.

Theorem 21: [3] A vertex $v \in V^+$ if and only if

(a) v is not an isolate vertex.

(b) v is in every γ set of G.

(c) No subset S of V(G) – N[v] with cardinality $\gamma(G)$ dominates G – {v}.

Corollary 22: If G is a vertex transitive graph then V^+ is the empty set.

Proof: If G has no edges then the result is obvious. Suppose G has at least one edge then the intersection of all minimum dominating sets is empty and hence condition (2)of theTheorem-21 is not satisfied by any vertex v of the graph G.

Definition 23: [3] Private Neighborhood of v with respect to a set.

Let S be a subset of V (G) and $v \in S$. Then the private neighborhood of v with respect to S = Pn[v,S] = $\{w \in V(G): N[w] \cap S = \{v\}.\}$

Theorem 24: [3]A vertex $v \in V^-$ if and only if there is a minimum dominating set S containing v such that Pn[v,S] = {v}.

Theorem 25: If G is a vertex transitive graph with at least one edge and $v \in V(G)$ such that $v \in V^-$ then $V^- = V(G)$.

Proof:- There is a minimum dominating set S containing v such that $P_n[v, S] = \{v\}$. Let w be any vertex of G. There is an automorphism f such that f(v) = w then w belongs to minimum dominating set f(S) and $P_n[w, f(S)] = \{w\}$. Thus, $w \in V^-$.

Corollary 26: If G is a vertex transitive graph and $v \in V(G)$ such that $v \in V^0$ then $V^0 = V(G)$.

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