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Proof of Beal's Conjecture

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ABSTRACT

This paper is devoted to obtain a proof of Beal's conjecture. In this paper we have given proof of Beal's conjecture for the following two cases.

Case 1:- If (x, y, z) = (mn, m, mn+1), then same prime divides A, B and C ;$\forall \boldsymbol{A, B, C, m, n \epsilon N \& m > 2}$.

Case 2:- If (x, y, z) = (m, m, m+1), then same prime divides A, B and C,; $\forall \boldsymbol{A, B, C, m \epsilon N, m > 2}$.

Introduction

Beal's conjecture is a conjecture in number theory: If A, B, C, x, y, and z are positive integers with x, y, z > 2, and $A^x + B^y = C^z$ then A, B, and C have a common prime factor .

In 1993 Andrew Beal discovered this conjecture while investigating generalizations of Fermat's last theorem. Later Robert Tijdeman and Don Zagier invented the same conjecture. While "Beal conjecture" is the more commonly accepted reference, it has also been referred to as the "Tijdeman–Zagier conjecture" in one published article. Beal initially offered a prize of US $5,000 in 1997, gradually raising this amount up to US $1,000,000.

Relation to other conjectures

Fermat Last Theorem established that $A^n + B^n = C^n$ has no solutions for n > 2 and for positive integers A, B, and C. If any solutions had existed to Fermat's Last Theorem, then by dividing out every common factor, there would also exist solutions with A, B, and C coprime. Hence, Fermat's Last Theorem can be seen as a special case of the Beal conjecture restricted to x = y = z. The abc conjecture would imply that there are at most finitely many counterexamples to Beal's conjecture.

Partial results proved by other

In the cases below where 2 is an exponent, multiples of 2 are also proven, since a power can be squared.

- The case x = y = z is Fermat Last Theorem, proven to have no solutions by Andrew wiles in 1994.
- The case gcd (x,y,z) > 2 is implied by Fermat's Last Theorem.
- The case (x, y, z) = (2, 4, 4) was proven to have no solutions by Pierre de Fermat in the 1600s. The case y = z = 4 has been proven for all x.
- The case (x, y, z) = (2, 3, 7) and all its permutations were proven to have only four solutions, none of them involving an even power greater than 2, by Bjorn Poonen, Edward F. Schaefer, and Michael Stoll in 2005.
- The case (x, y, z) = (2, 3, 8) and all its permutations are known to have only three solutions, none of them involving an even power greater than 2.
- The case (x, y, z) = (2, 3, 9) and all its permutations are known to have only two solutions, neither of them involving an even power greater than 2.
- The case (x, y, z) = (2, 3, 10) was proved by David Brown in 2009.
- The case (x, y, z) = (2, 3, 15) was proved by Samir Siksek and Michael Stoll in 2013.
- The case (x, y, z) = (2, 4, n) was proved for n ≥ 4 by Michael Bennet, Jordan Ellenberg, and Nathan Ng in 2009.
- The case (x, y, z) = (n, n, 2) has been proven for n any integer other than 3 or a 2 power.
- The case (x, y, z) = (n, n, 3) has been proven.
- The case (x, y, z) = (3, 3, n) has been proven for n equal to 4, 5, or 17 ≤ n ≤ 10000.
- The cases (5, 5, 7), (5, 5, 19) and (7, 7, 5) were proved by Sander R. Dahmen and Samir Siksek in 2013.
- The case A = 1 is implied by Catalan's conjecture, proven in 2002 by Preda Mihăilescu.
- Faltings' theorem implies that for every specific choice of exponents (x,y,z), there are at most finitely many solutions.
- Peter Norvig, Director of Research at Google, reported having conducted a series of numerical searches for counterexamples to Beal's conjecture. Among his results, he excluded all possible solutions having each of x, y, z ≤ 7 and each of A, B, C ≤ 250,000, as well as possible solutions having each of x, y, z ≤ 100 and each of A, B, C ≤ 10,000.

Note: Every positive integer greater than 1 is always written as product of power of prime.

Note: Every positive integer n greater than 1 is always written as $n = 1 + k^m$; $\forall k, m \geq 1$.

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Main Result

Theorem 2.1: If A = C = y', B= (k y'^{n}), where y' = (1+k^{m}) is any natural number and (x, y, z) = (m n, m, m n + 1), then Beal's conjecture hold. $\forall\, n, m, k \in N$ and $m > 2$.

Proof: Let A = C = y', B= (k y'^{n}), where y' = (1+k^{m}) and (x, y, z) = (m n, m, m n + 1), then

Consider $A^x + B^y = A^{mn} + B^m$

$$= y'^{mn} + (k^m y'^{mn})$$
$$= y'^{mn}(1 + k^m)$$
$$= y'^{mn}\, y'$$
$$= y'^{mn+1}$$

i.e. $A^x + B^y = C^z$. $\forall\, n, m, k \in N$ and $m > 2$.

Clearly y' divides A, B, C. Hence if prime r divides y', then same prime r divide A, B and C.

Corollary 2.2: If A = C = y', B= (k y'), where y' = (1+k^{m}) is any natural number and (x, y, z) = (m, m, m + 1), then Beal's conjecture hold. $\forall\, m, k \in N$ and $m > 2$, $n = 1$.

Proof: Let A = C = y', B= (k y'), where y' = (1+k^{m}) and (x, y, z) = (m, m, m + 1), then

Consider $A^x + B^y = A^m + B^m$

$$= y'^{m} + (k^m y'^{m})$$
$$= y'^{m}(1 + k^m)$$
$$= y'^{(m+1)}$$

i.e. $A^x + B^y = C^z$; $\forall\, m, k \in N$ and $m > 2, n = 1$.

Clearly y' divide A, B, C. Hence if prime r divides y', then same prime r divide A, B and C.

Corollary 2.3: If A = C = y', B= (2y'), where y' = $(1 + (2)^{2^{2^m}})$ and (x, y, z) = $(2^{2^m}, 2^{2^m}, 2^{2^m} + 1)$, then Beal's conjecture hold. $\forall\, m \in N$.

Proof: Let, A = C = y', B= (2y'), where y = $(1 + (2)^{2^{2^m}})$ and (x, y, z) = $(2^{2^m}, 2^{2^m}, 2^{2^m} + 1)$, then

Consider, $A^x + B^y = A^{2^{2^m}} + B^{2^{2^m}}$

$$= y'^{\,2^{2^m}} + (2y')^{2^{2^m}}$$
$$= y'^{\,2^{2^m}}\,(1 + (2)^{2^{2^m}})$$
$$= y'^{\,2^{2^m}+1}$$
$$= C^{2^{2^m}+1}$$
$$= C^z \qquad \forall\, m \in N\,.$$

*C*learly y' divides A, B, C. Hence if prime r divides y', then same prime r divides A, B and C.

Thus Beal's Conjecture is hold in this case.

Corollary 2.4: If A = C = y', B= (2y'), where y' = $(1 + (2)^{2^m})$ and (x, y, z) = $(2^m, 2^m, 2^m + 1)$, then Beal's conjecture hold. $\forall\, m \in N \;\&\; m > 2$.

Proof: Let, A = C = y', B= (2y'), where y' = $(1 + (2)^{2^m})$ and (x, y, z) = $(2^m, 2^m, 2^m + 1)$, then

Consider

$$A^x + B^y = A^{2^{2^m}} + B^{2^{2^m}}$$
$$= y'^{\,2^m} + (2y')^{2^m}$$
$$= y'^{\,2^m}\,(1 + (2)^{\;m})$$
$$= y'^{\,2^m+1}$$
$$= C^{2^m+1}$$
$$= C^z \qquad \forall\, m \in N \;\&\; m > 2\,.$$

Clearly y' divides A, B, and C. Hence if prime r divides y', then same prime r divides A, B and C. Thus Beal's Conjecture is hold in this case.

Note: An natural number y' = $(1 + (2)^{2^m})$ is called as Fermat Prime number.

Corollary 2.5: If A = C = y', B= (p y'), where y' = $(1 + (p)^{p^{p^m}})$ and (x, y, z) = $(p^{p^m}, p^{p^m}, p^{p^m} + 1)$, then Beal's conjecture hold, Where p is odd prime.

Proof: Let A = C = y', B= (p y'), where y' = $(1 + (p)^{p^{p^m}})$, and (x, y, z) = $(p^{p^m}, p^{p^m}, p^{p^m} + 1)$, then

Consider

$$A^x + B^y = A^{p^{p^m}} + B^{p^{p^m}}$$
$$= y'^{\,p^{p^m}} + (p\,y')^{p^{p^m}}$$
$$= y'^{\,p^{p^m}}\,(1 + (p)^{p^{p^m}})$$
$$= y'^{\,p^{p^m}+1}$$
$$= C^{p^{p^m}+1}$$
$$= C^z \qquad \forall\, m \in N\,.$$

Clearly y' divide A, B and C. Hence if prime r divides y', then same prime r divides A, B and C. Thus Beal's Conjecture is hold in this case.

Corollary 2.6: If A = C = y', B= (p y'), where y' = $(1 + (p)^{p^m})$ and (x, y, z) = (x, y, z) = $(p^m, p^m, p^m + 1)$, then Beal's conjecture hold, Where p is odd prime.

Proof: Let A = C = y', B= (p y'), C= y', where y' = = $(1 + (p)^{p^m})$ and (x, y, z) = $(p^m, p^m, p^m + 1)$, then Consider

$$\begin{aligned} A^x + B^y &= A^{p^m} + B^{p^m} \\ &= y'^{p^m} + (p\,y')^{p^m} \\ &= y'^{p^m}\ (1 + (p)^{p^m}) \\ &= y'^{p^m+1} \\ &= C^{p^m+1} \\ &= C^z \qquad \forall\, m \in N \,\&\, m > 2\,. \end{aligned}$$

Clearly y' divide A, B and C. Hence if prime r divides y', then same prime r divides A, B and C. Thus Beal's Conjecture is hold in this case.

Corollary 2.7: If A = C = y', B= (p y'), where y' = $(1 + p^p)$ and (x, y, z) = $(p, p, p + 1)$, then Beal's conjecture hold , Where p is odd prime.

Proof: Let A = C = y', B= (p y'), where y' = $(1 + p^p)$ and (x, y, z) = $(p, p, p + 1)$, then Consider

$$\begin{aligned} A^x + B^y &= A^p + B^p \\ &= y'^p + (p\,y')^p \\ &= y'^p\ (1 + p^p) \\ &= y'^{p+1} \\ &= C^{p+1} \\ &= C^z \end{aligned}$$

Clearly y' divide A, B and C. Hence if prime r divides y', then same prime r divides A, B and C.
Thus Beal's Conjecture is hold in this case.

Corollary 2.8: If A = C = y', B= (q y'), where y' = $(1 + (q)^{p^{p^m}})$ and (x, y, z) = $(p^{p^m}, p^{p^m}, p^{p^m} + 1)$, then Beal's conjecture hold. $\forall\, q, m, p \in N \,\&\, p \text{ is prime}$.

Proof: Let A = C = y', B= (q y'), where y' = $(1 + (q)^{p^{p^m}})$, and (x, y, z) = $(p^{p^m}, p^{p^m}, p^{p^m} + 1)$, then, Consider

$$\begin{aligned} A^x + B^y &= A^{p^{p^m}} + B^{p^{p^m}} \\ &= y'^{p^{p^m}} + (q\,y')^{p^{p^m}} \\ &= y'^{p^{p^m}}\ (1 + (q)^{p^{p^m}}) \\ &= y'^{p^{p^m}+1} \\ &= C^{p^{p^m}+1} \\ &= C^z \qquad \forall\, q, m, p \in N \,\&\, p \text{ is prime}. \end{aligned}$$

Clearly y' divide A, B and C. Hence if prime r divides y', then same prime r divides A, B and C. Thus Beal's Conjecture is hold in this case.

Corollary 2.9: If A = C = y', B= (q y'), where y' = $(1 + (q)^{p^m})$ and (x, y, z) = $(p^m, p^m, p^m + 1)$, then Beal's conjecture hold. . $\forall\, q, m, p \in N \,\&\, m > 2 \,\&\, p \text{ is prime}$.

Proof: Let A = C = y', B= (q y'), where y' = $(1 + (q)^{p^m})$ and (x, y, z) = $(p^m, p^m, p^m + 1)$, then Consider

$$\begin{aligned} A^x + B^y &= A^{p^m} + B^{p^m} \\ &= y'^{p^m} + (q\,y')^{p^m} \\ &= y'^{p^m}\ (1 + q^{p^m}) \\ &= y'^{p^m+1} \\ &= C^{p^m+1} \\ &= C^z \quad \forall\, q, m, p \in N \,\&\, m > 2 \,\&\, p \text{ is prime}. \end{aligned}$$

Clearly y' divide A, B and C. Hence if prime r divides y', then same prime r divides A, B and C.
Thus Beal's Conjecture is hold in this case.

Corollary 2.10: If A = C = y', B= (q y'), where y' = $(1 + q^p)$ and (x, y, z) = $(p, p, p + 1)$, then Beal's conjecture hold. Where p is odd prime.

Proof: Let A = C = y', B= (q y'), where y' = $(1 + q^p)$ and (x, y, z) = $(p, p, p + 1)$, then Consider

$$\begin{aligned} A^x + B^y &= A^p + B^p \\ &= y'^p + (q\,y')^p \\ &= y'^p\ (1 + q^p) \\ &= y'^{(p+1)} \\ &= C^{p+1} \\ &= C^z \end{aligned}$$

Clearly y' divide A, B and C. Hence if prime r divides y', then same prime r divides A, B and C. Thus Beal's Conjecture is hold in this case.

Corollary 2.11: If A = C = y', B= (k y'^n), where y' = $(1+k^m)$ and (x, y, z) = (m n, m, m n + 1) and if k is odd natural number then 2 divides A, B and C. $\forall\, n, m, k \in N \text{ and } m > 2$.

Proof: Let k is odd natural number then y’ = $(1+k^m)$ is even number, hence 2 divides y’. Let A = C = y’, B= $(k\ y'^n)$, where y’ = $(1+k^m)$ and (x, y, z) = (m n, m, m n + 1), then by Theorem 2.1, y’ divides A, B and C. Hence 2 divides A, B and C.

Corollary 2.12: If A = C = y’, B= $(k\ y'^n)$, where y’ = $(1+k^m)$ and (x, y, z) = (m n, m, m n + 1) and if k is even natural number then odd prime divides A, B and C. $\forall\ n, m, k \epsilon N\ and\ m > 2.$

Proof: Let k is even natural number then y’ = $(1+k^m)$ is odd number, hence odd prime r divides y’. Let A = C = y’, B= $(k\ y'^n)$, where y’ = $(1+k^m)$ and (x, y, z) = (m n, m, m n + 1), then by Theorem 2.1, y’ divides A, B and C. Hence odd prime r divides A, B and C.

Corollary 2.13: If A = C = y’, B= (k y’), where y’ = $(1+k^m)$ and (x, y, z) = (m, m, m + 1) and if k is odd natural number then 2 divides A, B and C. $\forall\ m, k \epsilon N\ and\ m > 2.$

Proof: Let k is odd natural number then y’ = $(1+k^m)$ is even number, hence 2 divides y’. Let A = C = y’, B= (k y’), where y’ = $(1+k^m)$ and (x, y, z) = (m, m, m + 1), then by Theorem 2.1, y’ divides A, B and C. Hence 2 divides A, B and C.

Corollary 2.14: If A = C = y’, B= (k y’), where y’ = $(1+k^m)$ and (x, y, z) = (m, m, m + 1) and if k is even natural number then odd prime divides A, B and C. $\forall\ m, k \epsilon N\ and\ m > 2.$

Proof: Let k is even natural number then y’ = $(1+k^m)$ is odd number, hence odd prime r divides y’. Let A = C = y’, B= (k y’), where y’ = $(1+k^m)$ and (x, y, z) = (m, m, m + 1) then by Theorem 2.1, y’ divides A, B and C. Hence odd prime r divides A, B and C.

Examples

Example 3.1: Take k=2, m=2, n=3 then we obtain A= $(1+2^2)$ = 5, B= $(2.A^3)$ = 10, C = 5

Substitute all these values in result $A^{mn} + B^m = c^{mn+1}$, we obtain.

$$(5)^{2.3} + (2^2.5^6) = 5^7$$

Clearly, 5 divide A, B & C.

Hence Beal’s conjecture is hold in this case.

Example 3.2: If A = $(1 + 3^2)$ = 10, B=(3.10^n) & C = A, where k = 3, m = 2

Substitute all these values in result $A^{mn} + B^m = c^{mn+1}$, we obtain.

$$A^{mn} + B^m = 10^{2n} + (3.10^n)^2$$
$$= 10^{2n}(1 + 3^2)$$
$$= 10^{2n+1}$$

Hence, $10^{2n} + (3.10^n)^2 = 10^{2n+1}$; $\forall n \geq 1$

Clearly 10 divide A, B & C and hence primes 2 & 5 divides A, B & C. Hence Beal’s conjecture is hold in this case also.

Discussion and Conclusion

This paper is devoted to obtain a proof of Beal’s conjecture. In this paper we have given proof of Beal’s conjecture for the following two cases.

Case 1:- If (x, y, z) = (mn, m, mn+1), then same prime divides A, B and C; $\forall\ A, B, C, m, n \epsilon N$ & $m > 2$.

Case 2:- If (x, y, z) = (m, m, m+1), then same prime divides A, B and C; $\forall\ A, B, C, m \epsilon N$, & $m > 2$.

References

[1] Titu Andreescu. Dorin Andrica, Zuming Feng.,104 Number Theory Prob- lems, Universities Press(India)Private Limited.(Indian Edition), (2010).

[2] Jean-Marie De Koninck, Armeel Mercier, Problems in Classical Number Theory, American Mathematical Society, (Indian Edition), (2010).

[3] Debnath L.and Thomas J. Berkeley Problems in Mathematics, Springer International Edition, Third Edition.(1976),559-593.

[4] Tom M. Apostol, Introduction to Analytic Number Theory, Narosa Publication House, Springer International Student Edition, New York Inc.(1998).

[5] Devid M. Burton, Elementary Number Theory, Tata McGraw-Hill edition, Sixth edition, (2011).

[6] G. H. Hardy and E. M. Wright, An Introduction to The Theory Numbers, Oxford University Press, (Sixth edition), (2008).

[7] Ivan Nivan, Herbert S. Zuckerman, Hagh L. Montgomery, An Introduction to The Theory Numbers, (Fifth Edition), (2010).

[8] Gareth A. Tones, and J. Mary Jones, Elementary Number Theory, Springer International edition, (2006).

[9] Watson E.J., Aspect of Combinatorics and Combinatorial number theory, Narosa publishing House (2003).