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Magic Square of Squares Proof

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ABSTRACT

The proof will demonstrate the non-existence of a 3x3 magic square of squares. I remember reading a Scientific America article and at the bottom of the article was the link to an article which described this problem. To me it seemed bizarre that a problem that had such a clear start point and was based on a concept so simple to understand could have no proof. I decided then that I would prove that such a square could never exist. I was inspired that the problem had been unresolved such it was first asked in 1984 and that it could be dated back to great mathematicians such as Leonhard Euler. Like many people interested in maths I am in awe of much of the the work that Leonhard Euler did so to be able to solve a problem whose roots can be traced back to him was exciting. People have been working with magic squares for centuries and yet nobody has presented a proof showing why a 3x3 magic square comprised entirely of square numbers cannot exist. I decided that night that I would provide such a proof. I was unaware how complex the problem was and how complex the tools I would need to solve the problem were but it was the start of the most wonderful journey that I wish I could relive. For hundreds of years people have been constructing magic squares. The definition of a magic square which I will refer to extensively throughout this proof states the following. The sum of all the elements in the rows, columns and diagonals must be equal. Each element must be unique in any square and must be a natural number. Therefore when I say something has been proven false through the defintion of a magic square this is the definition I am referrring to. Well reading an article in Scientific America I came across something that rather astounded me. The article claimed that no one had found an example of a 3x3 magic square that contains only magic square numbers. Furthermore no onw has proven that such a square cannot exist. This type of problem can be traced back all the way to Leonhard Euler who is the first person known to construct a 4x4 magic square of squares. I have always been fascinated in mathematical mysteries as the world is written in the mathematics to understand mathematics is to understand the world. Therefore by proving something in maths I am making the world a little more interesting. In the proof that follows I will show why a 3x3 magic square of squares can never exist. I will do this in 10 parts. The first part will be a general proof as to why the lowest element must be the middle or the corner for any magic square of squares to exist. I will then show that the lowest element can never occupy the middle or the corner of any square without violating the definition of what a magic square is.

Introduction

In this part of the proof the element x has the lowest value of all the elements in the square.

$X = x^2$
$A = (x + a)^2$
$B = (x + b)^2$
$C = (x + c)^2$
$D = (x + d)^2$
$E = (x + e)^2$
$F = (x + f)^2$
$G = (x + g)^2$
$H = (x + h)^2$

The magic square I will be working with in this proof looks as follows

$$\begin{matrix} G & X & F \\ C & D & E \\ B & H & A \end{matrix}$$

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If this is a magic square then the following equations must be true

$H + D + X = B + H + A$

$D + X = B + A$

$D = B + A - X$

Therefore we can write this magic square as follows

$$\begin{matrix} G & X & F \\ C & B + A - X & E \\ B & H & A \end{matrix}$$

As this is a magic square we can write the following equation

$G + C + B = B + A - X + A$

$C + B = 2A - X + B$

$C = 2A - X$

I can therefore re-write the magic square as follows

$$\begin{matrix} G & X & F \\ 2A - X & B + A - X & E \\ B & H & A \end{matrix}$$

Through the definition of a magic square of squares the following statement must result in a natural number

$\sqrt[2]{2A - X}$

$= \sqrt[2]{2x^2 + 4xa + 2a^2 - x^2}$

$= \sqrt[2]{x^2 + 4xa + 2a^2}$

I will now assume that the following statement is written in its most simplified form

$a = \frac{mx}{y}$

When I substitute this value into the above expression I get the following results

$\sqrt[2]{x^2 + \frac{4mx^2}{y} + \frac{2m^2x^2}{y^2}}$

$= \sqrt[2]{\frac{x^2y^2 + 4x^2my + 2m^2x^2}{y^2}}$

$= \frac{x}{y}\sqrt[2]{y^2 + 4my + 2m^2}$

$= \frac{x}{y}\sqrt[2]{(y + 2m)^2 - 2m}$

Therefore the sum of the differenes between

$(y + 2m)^2$

and

$(y + 2m)^2 - 2m^2$

Can be expressed as follows

$2(y + 2m - 1) + 1 + 2(y + 2m - 2) + 1 + ...2(y + 2m - n) + 1 = 2m^2$

The variabe n must be an even number. As the number of terms being summed on the left hand side must result in an even number as any number on the right hand side must be even.

We can simplify this expression to say the following

$2ny + 4mn + n^2 = 2m^2$

This illustrates why n must be even. The sum of the elements on the right hand side of the equation always results in an even number so n must be even so that the sum of the elements on the left hand side are also even. Therefore I can say the following

$n = 2p$

Therefore I can re-write thie equation as follows

$4py + 8mp = 4p^2 = 2m^2$

$2py + 4mp + 2p^2 = m^2$

There m must be even as the elements on the left hand side sum to an even number and therefore the element on the right handside must be equal.

Therefore I can say the following

$m = 2k$

I can now re-write the equation as follows

$2py + 4mp + 2p^2 = 4k^2$

$py + 4kp + p^2 = 2k^2$

$k^2 + 4kp + 2p^2 = 3k^2 - py + p^2$

Therefore I notice that this is the same as the equation that described 2A – X. I can therefore say the following

$x^2 + 4xa + 2a^2 = 3x^2 - ab + a^2$
$4xa - a^2 = 2x^2 - ab$
$4xa - a^2 + ab = 2x^2$
$a(4x - a + b) = 2x^2$
$a = \frac{2x^2}{4x - a + b}$
As
$a = \frac{mx}{y}$

$\frac{mx}{y} = 2x^2 \div \left(4x - \frac{mx}{y} + b\right)$
$\frac{mx}{y} = 2x^2 \div \left(\frac{4xy - mx + by}{y}\right)$
$\frac{mx}{y} = \frac{2x^2 y}{4xy - mx + by}$
$\frac{4mx^2y - m^2x^2 + mxby}{y} = 2x^2y$
$\frac{4mx^2 - m^2x^2 + mxby}{2xy^2} = x$
I will substitute this value into the equation
$\frac{mx}{y}$
This gives me the following
$\frac{m(4mx^2 - m^2x^2 + mxby)}{2xy^3}$
But both the numerator and denomitor are even and therefore this fraction can be simplified. Therefore the assumption that
$\frac{mx}{y}$
is fully simplified was false. Therefore a cannot be rational. Therefore any value for "a" will violate the definition of a magic square and therefore I can say the following. If I have the following magic square

n_1 n_2 n_3
n_4 n_5 n_6
n_7 n_8 n_9

The smallest value of any magic square of squares may never occupy position n_2. But any magic square will exhibit properties of rotational symmetry therefore the smallest value of any magic square may never occupy positions n_2, n_4, n_6 or n_8. This completes Part 1 of the proof.

Part 2

For the remainder of the proof I will now be using the following definitions and relations for all the possible positions of any element in the magic square.
$X < A < B < C < D < E < F < G < H$
I do not need to specify that any of these elements need to be square as the rest of the proof will apply to all magic squares.
I will now prove that X can never occupy position n_5.
Any magic square whose smallest element occupies the position n_5 will look as follows.

n_1 n_2 n_3
n_4 X n_6
n_7 n_8 n_9

The remaining elements that have to be placed in this magic square are: A, B, C, D, E, F, G and H. I will now show that this can never be a magic square by showing that none of the remaining elements can occupy position n_1.
I will start by placing A in position n_1. When I do this the magic square looks as follows.

A n_2 n_3
n_4 X n_6
n_7 n_8 n_9

I will prove that such a square can never exist by showing that H can never occupy any of the remaining positions. I will start by proving that H can never occupy position n_2, n_3, n_4 or n_7. I will show this by demonstrating that any row or column will always sum to a larger value than the sum of the diagonal containing X and A. I will prove this by demonstrating why it is true when H occupies position n_2. When I place H in position n_2. The magic square looks as follows

A H n_3
n_4 X n_6
n_7 n_8 n_9

The remaining elements that have to be placed in this square are: B, C, D, E, F and G. I will show that the sum of the elements in the row containing H and A will always be larger than the diagonal containing X and A by placing the largest remaining element in position n_9 and the smallest remaining element in position n_3. When I do this the magic square looks as follows

$A \quad H \quad B$

$n_4 \quad X \quad n_6$

$n_7 \quad n_8 \quad G$

If this is a magic square then the following equation must be true

$A + H + B = A + X + G$

$H + B = X + G$

But this cannot be true because of the following

$G < H$

$X < B$

Therefore

$H + B > X + G$

$A + H + B > A + X + G$

Therefore this cannot be a magic square as the sum of the elements in the row or column that contains H and A will always be larger than the sum of the elements in the diagonal which contains A and X.

The magic square I am now working with looks as follows

$A \quad n_2 \quad n_3$

$n_4 \quad X \quad n_6$

$n_7 \quad n_8 \quad n_9$

I will now prove that H cannot occupy positions n_4, n_6 or n_8. I will prove this by showing that when H occupise any of these postions the sum of the elements in the row or column that contains H I will prove this by showing that this is the case when H occupies position n_6. When H occupies position n_6 the magic square looks as follows.

$A \quad n_2 \quad n_3$

$n_4 \quad X \quad H$

$n_7 \quad n_8 \quad n_9$

The remaining elements that have to be placed in this magic square are: B, C, D, E, F and G. I will show that the sum of the elements in the row that contains H will always be larger than the sum of the elements in the diagonal that contains A and X. I will prove this by showing that this is true when I place the lowest remaining element in position n_4 and the highest remaining element in position n_9.

When I do this the magic square looks as follows

$A \quad n_2 \quad n_3$

$B \quad X \quad H$

$n_7 \quad n_8 \quad G$

As this is a magic square the following equation must be true

$A + X + G = B + X + H$

$A + G = B + H$

But this cannot be true because of the following

$A < B$

$G < H$

Therefore

$A + G < B + H$

$A + X + G < B + X + H$

Therefore this cannot be a magic square as the sum of the elements in the complete diagonal is lower than the sum of the elements in the complete row. Therefore the sum of the elements in the row or column that contains H will always be larger than the sum of the elements in the diagonal that contains X and A when H occupies positions n_4, n_6 or n_8. Therefore H cannot occupy poisitions n_4, n_6 or n_8.

The magoc square I am now working with looks as follows

$A \quad n_2 \quad n_3$

$n_4 \quad X \quad n_6$

$n_7 \quad n_8 \quad n_9$

I will now show that H can never occupy position n_9. When H occupies position n_9 the magic square looks as follows

$A \quad n_2 \quad n_3$

$n_4 \quad X \quad n_6$

$n_7 \quad n_8 \quad H$

The remaining elements that have to be placed in this magic square are B, C, D, E, F and G. I will show that none of these remaining elements can be placed in positions n_3 and n_6. I will prove that this is true by showing that when I place the two lowest remaining elements in positions n_3 and n_6 the sum of the elements in this row will always be larger than the sum of the elements in the complete diagonal.

When I place the two lowest remaining elements in these positions the magoc square looks as follows

$A \quad n_2 \quad B$
$n_4 \quad X \quad C$
$n_7 \quad n_8 \quad H$

As this is a magic square the following equation must be true

$A + X + H = B + C + H$
$A + X = B + C$

Therefore this cannot be true because of the following

$A < B$
$X < C$

Therefore

$A + X < B + C$
$A + X + H < B + C + H$

Therefore this cannot be a magic square as the sum of the elements in the complete diagonal will always be smaller than the sum of the elements in the column that contains H. Therefore this cannot be a magic square when h occupies position n_9.

There when X occupies position n_5 and A occupies position n_1 there can never be a largest element in the magic square. Therefore for this to be a magic square all the elements would have to be equal in value. This violates the definition of a magic square. Through the rotational symmetry of any magic square we can also now say that when X occupies position n_5 then A can never occupy position n_1, n_3, n_7 and n_9.

I will now prove that when X occupies position n_5 that A can never occupy position n_2.

If this were to happen then the magic square would look as follows

$n_1 \quad A \quad n_3$
$n_4 \quad X \quad n_6$
$n_7 \quad n_8 \quad n_9$

I will now show that this can never be a magic square. I will prove this by showing that none of the remaining elements can occupy positions n_7 and n_9. For this to be a magic square A+X would have to equal the sum of the two elements placed in these positions. The remaining elements that have to be placed in this magic square are: B, C, D, E, F, G and H. I will show that the sum of any two of these elements is larger than X and A by placing the two smallest remaining elements in these positions.

When I do this the magic square looks as follows

$n_1 \quad A \quad n_3$
$n_4 \quad X \quad n_6$
$B \quad n_8 \quad C$

If this is a magic square the following equation must be true

$A + X = B + C$

But this cannot be true because of the following

$A < B$
$X < C$

Therefore

$A + X < B + C$

Therefore this can never be a magic square as the sum of the elements in the row n_7, n_8 and n_9 will always be larger than the sum of the elements that contain X and A. Therefore this can never be a magic square as it violates the definition of a magic square.

Therefore when X occupies position n_5 A cannot occupy position n_2. But a square exhibits rotational symmetry so we can say that when X occupies position n_5 then A can never occupy position n_2, n_4, n_6 or n_8.

Therefore through all that has been shown in this part of the proof we can say that when X occupies position n_5 A cannot occupy positions n_1, n_2, n_3, n_4, n_6, n_7, n_8 or n_9. Therefore any magic square where the lowest element occupies position n_5 cannot contain a second smallest element. Therefore any square would need to have all values that were equal. This would violate the definition of a magic square. Therefore X can never occupy position n_5.

Part 3

In this part of the proof I will show that A can never occupy position n_5. I will prove this by showing that when A occupies position n_5 any magic square of squares will not have a smallest element.

When A occupies position n_5 and X occupies position n_1 the magic square looks as follows

$X \quad n_2 \quad n_3$
$n_4 \quad A \quad n_6$
$n_7 \quad n_8 \quad n_9$

The remaining elements that have to be placed in this magic square are: B, C, D, E, F, G and H. I will show none of these elements can occupy position n_9. I will first prove that B, C, D, E, F and G cannot occupy position n_9. I will prove this by showing that whenever of these elements are placed in position n_9 the sum of the elements in the diagonal will always be less than the row, column or diagonal that contains H and A. I will prove this by showing it is true when G occupies position n_9 and H occupies position n_2.

When H occupies position n_2 and G ocupies position n_9 the magic square looks as follows

$X \quad H \quad n_3$
$n_4 \quad A \quad n_6$
$n_7 \quad n_8 \quad G$

The remaining elements that have to be placed in this magic square are: B, C, D, E and F. I will now place the lowest remaining element in position n_8 to show that any row, column or diagonal that contains H and A will sum to a higher value than the diagonal that contains A and X.

When I place the lowest remaining element in position n_8 the magic square looks as follows.

$X \quad H \quad n_3$

$n_4 \quad A \quad n_6$

$n_7 \quad B \quad G$

If this is a magic square the following equation must be true

$X + A + G = H + A + B$

$X + G = H + B$

But this cannot be true because of the following

$X < B$

$G < H$

Therefore

$X + G < H + B$

$X + A + G < H + X + B$

Therefore this cannot be a magic square as the sum of the elements in the row that contains H and A is greater than the sum of the elements in the diagonal that contain X and A. Therefore B, C, D, E, F and G cannot occupy position n_9 as the sum of the elements in this diagonal will always be less than the sum of the elements in the row, column or diagonal that contains H and A.

Therefore H must occupy position n_9. This creates the following magic square

$X \quad n_2 \quad n_3$

$n_4 \quad A \quad n_6$

$n_7 \quad n_8 \quad H$

The remaining elements that have to be placed in this magic square are: B, C, D, E, F and G. I will show that none of these elements can be placed in positions n_3 and n_6. I will show this by showing that the sum of the elements in the column containing H will always be larger than the sum of the elements in the complete diagonal. I will prove this is true by showing that it is true when I place the two lowest remaining elements in positions n_3 and n_6.

When I do this the magic square looks as follows

$X \quad n_2 \quad B$

$n_4 \quad A \quad C$

$n_7 \quad n_8 \quad H$

As this is a magic square the following equation must be true

$X + A + H = B + C + H$

$X + A = B + C$

But this cannot be true because of the following

$X < B$

$A < B$

Therefore

$X + A < B + C$

$X + A + H < B + C + H$

Therefore this cannot be a magic square as the sum of the elements and at least one column will never be equal this violates the definition of a magic square. Therefore a magic square can never be created when A occupies position n_5 and X occupies position n_1.

Through the rotational symmetry of a square we can now say the following.

When A occupies position n_5 X cannot occupy position n_1, n_3, n_7 or n_9. Through the work in Part 1 of the proof I have proven that when A occupies position n_5 that X cannot occupy position n_2. When the rotational symmetry of a square is applied to this statement we can say that when A occupies position n_5 X cannot occupy positions n_2, n_4, n_6 or n_8.

When these statements are combined we can conclude that when A occupies position n_5 any magic square of squares will not have a least element. This means that they will have to be at least one occasion where an element will have to be repeated. Therefore a magic square of squares cannot be created when A occupies position n_5.

Part 4

I will now prove that when either B, C, D, E, F or G occupy position n_5 and X occupies position n_1 that H cannot occupy positions n_2, n_3, n_4, n_6, n_7 or n_8.

When X occupies position n_1 the magic square looks as follows

$X \quad n_2 \quad n_3$

$n_4 \quad n_5 \quad n_6$

$n_7 \quad n_8 \quad n_9$

I will show that neither B, C, D, E, F or G can occupy position n_5 using the following fact about magic squares

$X + n_9 = n_2 + n_8 = n_3 + n_7 = n_4 + n_6$

This equation is derived from the fact that in any magic square the sum of all the elements in all the rows and columns must be equal. We know that each of these equals the sum the elements in a row, column or diagonal when n_5 is added to these equations. But this is added to each equation. Therefore the above equation must be true.

I will show that the sum of X and any other element placed in postion n_9 will always be less than the sum of H and another element that is across from H through the middle.

In other words if, in the above equation I replace n_2, n_8, n_3, n_7, n_4 or n_6 the sum of the elements in that row, column or diagonal must be more than the sum of the elements in the diagonal which contains X. I will do this by placing the highest element other than H in position n_9, H in position n_2 and the lowest remaining element in position n_8.
When this is done the magic square looks as follows

$X \quad H \quad n_3$
$n_4 \quad n_5 \quad n_6$
$n_7 \quad B \quad G$

For this to be a magic square the following must be true
$X + G = H + B$
But this cannot be true because of the following
$X < B$
$G < H$
Therefore
$X + G < H + B$

Therefore this cannot be a magic square as it violates the definition of a magic square. This must also be true if H occupies any other position other than n_9 and n_5 well X occupies position n_1 as shown through the equation derived at the start of this part of the proof.
Therefore as this is true when X occupies position n_1 we can apply the rotational symmetry of a square which means that this proof must also be true when X occupies position n_1, n_3, n_7 or n_9.
Therefore for every subsequent part of this proof H will occupy position n_9 until the last part of the proof, part 11, where it will occupy position n_5. Therefore when stipulating other squares I will not state that H occupies position n_9 as the reason for this was shown in this part of the proof.

Part 5

I will now prove that when B occupies position n_5 that X cannot occupy position n_1
When I place B in position n_5, X in position n_1 and H in position n_9 I create the following magic square

$X \quad n_2 \quad n_3$
$n_4 \quad B \quad n_6$
$n_7 \quad n_8 \quad H$

The remaining elements that have to be placed in this magic square are: A, C, D, E, F adnd G. I will show that none of these remaining elements can occupy positions n_3 and n_6. I will show that this is the case by showing that when the two lowest remaining elements occupy these positions the sum of the elements in column H will be larger than the sum of the elements in the complete diagonal.
When I place the two lowest remaining elements in positions n_3 and n_6 I create the following magic square.

$X \quad n_2 \quad A$
$n_4 \quad B \quad C$
$n_7 \quad n_8 \quad H$

If this is a magic square the following equation must be true
$X + B + H = A + C + H$
$X + B = A + C$
But this cannot be true because of the following
$X < A$
$B < C$
Therefore
$X + B < A + C$
$X + B + H < A + C + H$

Therefore this cannot be a magic square as the sum of the elements and in a diagonal and the sum of the elements in a column are not equal. This violates the definition of a magic square.
Therefore when B occupies position n_5 and X occupies position n_1 a magic square cannot be created. Through the rotational symmetry of a square we can expand this statement to say that when B occupies position n_5 in any mangic square X cannot occupy positions n_1, n_3, n_7 or n_9. From part one we know that in any magic square of squares X cannot occupy poisition n_2. This can be expanded using the rotational symmetry of a square to say that a magic square of squares cannot be created when X occupies position n_2, n_4, n_6 or n_8.
When these two statements are combined we can say the following. That when B occupies position n_5 in a 3x3 magic square of squares X cannot occupy any of the other remaining positions. Therefore there cannot be a least element in any magic square of squares when B occupies position n_5. This means that some elements will have to be repeated which violates the definition of a magic square of squares.

Part 6

In this part of the proof I will show that when C occupies position n_5 that X cannot occupy position n_1 without violating the definition of a magic square.
When I place C in position n_5, X in position n_1 and H in position n_9 the magic square looks as follows

$X \quad n_2 \quad n_3$
$n_4 \quad C \quad n_6$
$n_7 \quad n_8 \quad H$

The remaining elements that have to be placed in this magic square are: A, B, D, E, F and G. I will prove that D, E, F and G can only occupy positions n_2 and n_4. Therefore it is impossible to place 4 elements into only two positions. This will show that it will be impossible to create any magic square based on the form above without violating the definition of the magic square.

I will show that that this must be true by placing D, the lowest of the elements I wish to prove can only occupy positions n_2 and n_4, and show that any row or column that has both D and H will sum to a larger value than the complete diagonal. I will prove this by proving that it is true when D occupies position n_3.

When D occupies position n_3 the magic square looks as follows

$X \quad n_2 \quad D$

$n_4 \quad C \quad n_6$

$n_7 \quad n_8 \quad H$

I will now place the lowest of the remaining elements in position n_6 to show that if any other remaining element is placed in this position the sum of the elements in this column will always be larger than the sum of the elements in the complete diagonal.

When I place the lowest remaining element in position n_6 the magic square looks as follows.

$X \quad n_2 \quad D$

$n_4 \quad C \quad A$

$n_7 \quad n_8 \quad H$

If this is a magic square then the following equation must be true

$X + C + H = D + A + H$

$X + C = D + A$

But this cannot be true because of the following

$X < A$

$C < D$

Therefore

$X + C < D + A$

$X + C + H < D + A + H$

Therefore this can never be a magic square when any element that is equal in size or larger than element D shares the same row or column as element H. As when this does occur there will be a violation of the definition of what a magic square is.

But it is impossible to construct such a square as four elements will have to occupy two positions which is not possible. Therefore this cannot be a magic square when C occupies position n_5 and X occupies position n_1. A square exhibits rotational symmetry. We can therefore use this proof and expand the statement to say that when C occupies position n_1, n_3, n_7 or n_9.

Part 1 says that when X occupies position n_2 a magic square of squares cannot be created. We can expand that statement to say that no magic square of squares can be constructed when X occupies position n_2, n_4, n_6 or n_8.

Combining these two revelations we can conclude that when C occupies position n_5 that X cannot occupy any of the other positions. Therefore there can not be a lowest element in the magic square. This means that all the elements must be equal in size which violates the definition of a magic square. Therefore when C occupies position n_5 no magic square of squares can be created.

Part 7

In this part of the proof I will prove that when D occupies position n_5 that X cannot occupy position n_1 without violating the definition of a magic square. When D occupies position n_5, X occupies position n_1 and H occupies position n_9 the magic square looks as follows

$X \quad n_2 \quad n_3$

$n_4 \quad D \quad n_6$

$n_7 \quad n_8 \quad H$

The remaining elements that have to be placed in this magic square are: A, B, C, E, F and G. I will show that A, B, and C have to occupy positions n_6 and n_8. This is of course impossible as three different elements cannot occupy two positions. This will prove that the above square can never be used to create a magic square. I will show this by proving that when A, B, or C share a row or column with X the sum of the elements in this row or column will be less than the sum of the elements in the complete diagonal. I will prove this for A, B and C by proving that it is true for C and must therefore be true for any element that has a smaller value than C.

When I place C in position n_2 the magic square looks as follows

$X \quad C \quad n_3$

$n_4 \quad D \quad n_6$

$n_7 \quad n_8 \quad H$

I will now place the largest remaining element in position n_3 to show that regardless which remaining element is placed in position n_3 the sum of the elements in that row will always be less than the sum of the elements in the complete diagonal.

When I place the largest remaining element in position n_3 the magic square looks as follows

$X \quad C \quad G$

$n_4 \quad D \quad n_6$

$n_7 \quad n_8 \quad H$

If this is a magic square then the following equation must be true

$X + C + G = X + D + H$

$C + G = D + H$

But this cannot be true because

$C < D$

$G < H$

Therefore
$C + G < D + H$
$X + C + G < X + D + H$

Therefore this can never be a magic square for the following reason. I cannot place A, B or C in positions n_2, n_3, n_4, or n_7 without violating the definition of a magic square. I must therefore place these three elements in positions n_6 or n_8. But this is impossible. Therefore this can never be a magic square when D occupies position n_5 and X occupies position n_1. I can now use the rotational symmetry of a squareto expand this statement to say that when D occupies position n_5 X cannot occupy position n_1, n_3, n_7 or n_9. Using part one of the proof we know that a magic square of squares cannot be created when X occupies position n_2. This can also be expanded to say that when X ocuppies position n_2, n_4, n_6 or n_8 a magic square of squares cannot be created.

When these two statements are combined we can say that when D occupies position n_5 in a magic square of squares X cannot occupy any of the other positions. Therefore there can be no elements with the lowest value. This forces all elements to have the same value which is a violation of the magic square. Therefore this proves that when D occupies position n_5 no magic square of squares can be created.

Part 8

In this part of the proof I will show that when E, F or G occupies poisition n_5 X cannot occupy position n_1 without violating the definition of a magic square of squares.

When X occupies position n_1 and H occupies position n_9 the magic square looks as follows.

$$\begin{matrix} X & n_2 & n_3 \\ n_4 & n_5 & n_6 \\ n_7 & n_8 & H \end{matrix}$$

I will now show that regardless of whether E, F or G occupy position n_5 that this cannot possibly be a magic square. I will show this by proving that the sum of the elements in the row or column that contain X will always be less that the diagonal that contains H and X when A, B, C or D occupy position n_2, n_3, n_4, or n_7. I will prove this by showing that this is true when D occupies any one of these positions. When D occupies position n_2 the magic square looks as follows

$$\begin{matrix} X & D & n_3 \\ n_4 & n_5 & n_6 \\ n_7 & n_8 & H \end{matrix}$$

The remaining elements that could occupy position n_5 are E, F and G. All of these elements are larger than D. So for the row that contains D to be able to sum to a value equal to the sum of the elements in the diagonal which contains X and H the element that is placed in position n_3 must be larger than H.

But the largest element that could occupy position n_3 is G this value is smaller than H. Therefore the sum of the elements in that row that contain D and X must be smaller than the sum of the elements in the diagonal that contain X and H. Therefore A, B, C and D cannot occupy positions n_2, n_3, n_4 and n_7. Therefore four elements must be placed in two squares. This is clearly impossible.

Therefore this cannot be a magic square when X occupies position n_1 and E, F, or G occupy position n_5. We also know that a square exhibits rotational symmetry so this proof also proves the following. When E, F, or G occupy position n_5 X cannot occupy positions n_1, n_3, n_7 or n_9. From part one of the proof we know that X cannot occupy position n_2. Due to the rotational symmetry of a square this can be expanded to say that X cannot occupy positions n_2, n_4, n_6 or n_8.

When these two statements are combined we realise that we have proved the following. When E, F or G occupy position n_5 in a magic square of squares the smallest element of the square cannot occupy any of the other positions. Therefore this cannot be a magic square of squares when E, F or G occupy position n_5.

Part 9

In this part of the proof I will show that X cannot occupy position n_1 when H occupies position n_5. When X occupies position n_1 and H occupies position n_5 the magic square looks as follows.

$$\begin{matrix} X & n_2 & n_3 \\ n_4 & H & n_6 \\ n_7 & n_8 & n_9 \end{matrix}$$

The remaining elements that have to be placed in this magic square are: A, B, C, D, E, F and G. I will now prove that element G cannot be placed in position n_2, n_3, n_4, n_6, n_7 or n_8. I will prove this by showing that any row, column or diagonal that contains the elements G, H and any element larger than X cannot exist in this smagic square without violating the definition of a magic square. I will prove that this is true when G occupies position n_2 to show that it is true for all the other positions.

When G occupies position n_2 the magic square looks as follows

$$\begin{matrix} X & G & n_3 \\ n_4 & H & n_6 \\ n_7 & n_8 & n_9 \end{matrix}$$

To show that such a square cannot exist I will place the lowest remaining element in position n_8 and the largest remaining element in position n_9. I will do this to show that any row, column or diagonal that contains G, H and any element other than X will always sum to a larger value than the sum of the elements in the diagonal that contain H and X. When I place the smallest remaining element in position n_8 and the largest remaining element in position n_9 I create the following magic square

$$\begin{matrix} X & G & n_3 \\ n_4 & H & n_6 \\ n_7 & A & F \end{matrix}$$

If this is a magic square then the following equation must be true

$X + H + F = G + H + A$
$X + F = G + A$
But this cannot be true because of the following
$X < A$
$F < G$
Therefore
$X + F < G + A$
$X + H + F < G + H + A$

Therefore this cannot be a magic square as the sum of the elements in at least one column and one row are not equal. This violates the definition of a magic square. This also proves that when G, H and another element other than X are in a row, column or diagonal the sum of the elements in this row, column or diagonal will always sum to a larger value than the sum of the elements in the diagonal that contain H and X. Therefore This can only ever be a magic square if G occupies position n_9.

When G now occupies position n_9 the magic square looks as follows

$X \quad n_2 \quad n_3$
$n_4 \quad H \quad n_6$
$n_7 \quad n_8 \quad G$

The remaining elements that have to be placed in this magic square are: A, B, C, D, E and F. I will show that none of these elements can occupy position n_2 and n_3. I will prove this by showing that when the two largest remainibg elements occupy these positions the sum of the elemnts in the row containing X will always be smaller than the sum of the elements in the complete diagonal.

When the two largest remainibg elements occupy positions n_2 and n_3 the magic square looks as follows

$X \quad F \quad E$
$n_4 \quad H \quad n_6$
$n_7 \quad n_8 \quad G$

If this is a magic square then the following equation must be true
$X + F + E = X + H + G$
$H + G = F + E$
But this cannot be true because of the following
$F < H$
$E < G$
Therefore
$F + E < H + G$
$X + F + E < X + H + G$

Therefore this cannot be a magic square as the sum of the elements in at least one row and one diagonal can never be equal. This violates the definition of a magic square.

As a result this cannot be a magic square when H occupies position n_5 and X occupies position n_1. Due to the fact that when this happens element G cannot exist in this magic square. Due to the rotational symmetries of a magic square this statement can be expanded to say the following

When H occupies position n_5 X cannot occupy positions n_1, n_3, n_7 or n_9. We also now from part one of the proof that X cannot occupy position n_2 in a magic square of squares. Due to the rotational symmetries of a sqaure this statement can be expanded to say that in a magic square of squares X cannot occupy positions n_2, n_4, n_6 or n_8.

When these two statements are combined we can conclude that when H occupies position n_5 in a magic square of square X cannot occupy any of the remaining positions. There therefore cannot be a smallest element in this square. Therefore some elements must be repeated which violates the definition of a magic square. Therefore when H occupies position n_5 a magic square of squares cannot be created.

Conclusion

If we take a look at an empty magic square of squares below.

$n_1 \quad n_2 \quad n_3$
$n_4 \quad n_5 \quad n_6$
$n_7 \quad n_8 \quad n_9$

The remaining elements that have to be placed in this magic square of squares are: X, A, B, C, D, E, F, G and H. From part 2-9 of the proof it becomes apparent that none of these remaining elements can occupy position n_5. This means that there will always be an empty position in this magic square. Therefore this can never be a magic square of squares.