Available online at www.elixirpublishers.com (Elixir International Journal)

Discrete Mathematics

ISSN: 2229-712X

Elixir Dis. Math. 83 (2015) 32778-32781

One Point Determining and Two Point Distinguishing Graphs

Elachini V. Lal[1], Palleeluveedu R. Sujith[2] and Karakkattu S. Parvathy[3]

[1]Department of Higher Secondary, Govt. HSS Kuttippuram, Kerala, India – 679571.

[2]Department of Higher Secondary, Paruthur HSS Pallippuram, Kerala, India – 679335.

[3]Department of Mathematics, St. Mary's College, Thrissur, Kerala, India – 680020.

ARTICLE INFO

Article history:

Received: 27 December 2014;

Received in revised form:

25 May 2015;

Accepted: 30 May 2015;

Keywords

One point determining graphs,

Two point distinguishing graphs.

ABSTRACT

A point determining graph is defined to be a graph in which distinct non adjacent points have distinct neighborhoods. If in addition any two distinct points have distinct closed neighborhoods, it is called point distinguishing graph. A graph **G** is said to be one point determining, if for any two distinct vertices $\mathbf{v_1}$ and $\mathbf{v_2}$ $\mathbf{N(v_1)}$and $\mathbf{N(v_2)}$ have at most one vertex in common. A graph **G** is said to be two point distinguishing if for any two distinct vertices $\mathbf{v_1}$ and $\mathbf{v_2}$, the closed neighborhood $\mathbf{N[v_1]}$ and $\mathbf{N[v_2]}$, have at most two vertices in common. Here we focus on some properties of one point determining and two point distinguish- ing graphs.

Introduction

By a graph $G = (V, E)$, we mean a finite, undirected, connected graph with no loop or multiple edges. For basic graph theoretic terminology, we refer to **[1, 2, 3]**. The point determining graphs were introduced by D.P.Summner **[4]** and point distinguishing graphs were defined by R.C Entringer and L.D Gassman **[5]**. For, a vertex 'a' $\in V\ G$), the open neighborhood of a, denoted by $N(a)$, is the set of all vertices in G that are adjacent to 'a'. A graph G is said to be point determining if for any two distinct non-adjacent vertices 'a' and 'b' of G, $N(a) \neq N(b)$. The closed neighborhood of a vertex 'p', denoted by $N[p]$ is defined to be $N(p) \cup \{p\}$. A graph G is point distinguishing if $N[p] \neq N[q]$ whenever $p \neq q$.

One point determining graphs

Definition 1: A graph G is said to be *one point determining*, if for any two distinct vertices v_1 and v_2 , $N(v_1)$and $N(v_2)$have at most one vertex in common.

Example 1: Consider the path graph of order 3 shown in Figure 1.

$N(v_1) = \{v_2\} = N(v_3)$

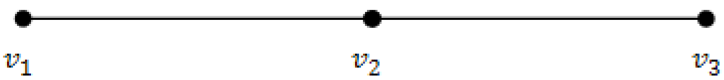

Figure 1. Example of a one point determining graph

Theorem 1. *A graph G is one-point determining if and only if G is C_4 free.*

Proof: If G contains a 4 – cycle, $v_1 v_2 v_3\ v_4 v_1$ then $v_1, v_3 \in N(v_2)$and $N(v_4)$. Hence the condition is sufficient.

If G is not one-point determining, then there exists v_1 and v_2 such that $N(v_1)$and $N(v_2)$contains at least two vertices in common, say, u and v. Then $v_1$is adjacent to both u and v. Also, v_2 is adjacent to both u and v. Thus $v_1 u\ v_2 v\ v_1$form a 4 – cycle. Hence the condition is necessary.

Theorem 2. *Let G be a one-point determining. G is point determining if and only if the following hold: If $u, v \in V(G), d(u) = d(v) = 1$, then $N(u) \neq N(v)$.*

Proof: Let G be a one-point determining. Then, for any two distinct vertices u and v, $N(u)$ and $N(v)$ have at most one vertex in common. If G is point determining, then for any two distinct vertices u and v, $N(u) \neq N(v)$. So the condition is necessary.

Conversely, let us assume that, $d(u) = d(v) = 1$,with $N(u) \neq N(v)$. Thus pendent vertices have distinct neighborhoods in G. We have to prove that G is point determining If $d(u) \neq d(v)$, then $N(u) \neq N(v)$.But, if $d(u) = d(v)$ with $d(u) > 1$ and $d(v) > 1$, then $N(u) \neq N(v)$.Otherwise, G is not one-point determining, which is a contradiction. Hence the condition is sufficient.

Theorem 3. *Let G be a bipartite cubic planar graph. Then G is not one-point determining.*

Proof: Let G be a bipartite cubic planar graph. Since G is cubic, every vertex is of degree three and hence there exists vertices u_1 and u_2 such that $N(u_1)$ and $N(u_2)$ have three vertices of which two of them is common, say v_1 and v_2. Then u_1 is adjacent to both v_1 and v_2. Also u_2 is adjacent to both v_1 and v_2. Thus $u_1 v_1 u_2\ v_2 u_1$ form a 4 – cycle. Hence, by Theorem1, G is not one-point determining.

Theorem 4: *If G is one-point determining, then $G + uv$ is also one-point determining if and only if there doesn't exist an edge $u'v'$ such that $u' \in N(u)$ and $v' \in N(v)$.*

Tele:

E-mail addresses: lamathematics@gmail.com

Proof: Given $\boldsymbol{G}$ is one-point determining and suppose $\boldsymbol{G + u\,v}$ is also one-point determining Then $\boldsymbol{N(u)} \cap \boldsymbol{N(v)} = \emptyset$. For, if $\boldsymbol{N(u)} \cap \boldsymbol{N(v)} \neq \emptyset$, then there exist at least one vertex $\boldsymbol{w_i} \in \boldsymbol{N(u)} \cap \boldsymbol{N(v)}$ But $\boldsymbol{w_i}$ is adjacent to $\boldsymbol{w_j}$ for some $\boldsymbol{i} \neq \boldsymbol{j}$. Then $\boldsymbol{uvw_jw_iu}$ form a 4 – cycle and this shows that $\boldsymbol{G + uv}$ is not one-point determining.

Conversely, $\boldsymbol{G}$ is one-point determining. Then for any two distinct vertices $\boldsymbol{v_1}$and $\boldsymbol{v_2}$ the neighborhoods $\boldsymbol{N(v_1)}$and $\boldsymbol{N(v_2)}$ have at most one vertex in common. Suppose there doesn't exist an edge $\boldsymbol{u'v'}$ such that $\boldsymbol{u'} \in \boldsymbol{N(u)}$ and $\boldsymbol{v'} \in \boldsymbol{N(v)}$ Then $\boldsymbol{G + uv}$ is one-point determining. For, if not, there exist $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\boldsymbol{N(u)}$ and $\boldsymbol{N(v)}$ contains at least two vertices in common, say, $\boldsymbol{u'}$ and $\boldsymbol{v'}$. Then $\boldsymbol{u}$ is adjacent to both $\boldsymbol{u'}$ and $\boldsymbol{v'}$and $\boldsymbol{v}$ is adjacent to both $\boldsymbol{u'}$ and $\boldsymbol{v'}$. Thus $\boldsymbol{u\,u'\,v\,v'\,u}$ form a 4 – cycle. Hence, $\boldsymbol{u'} \in \boldsymbol{N(u)}$ *and* $\boldsymbol{v'} \in \boldsymbol{N(v)}$ – a contradiction. Therefore $\boldsymbol{G + uv}$ is also one-point determining.

Definition 2. Let $\boldsymbol{G = (V, E)}$, be a graph and let $\boldsymbol{U}$ be a subset of $\boldsymbol{V}$. A subgraph $\boldsymbol{H}$ of $\boldsymbol{G}$ is said to be *normal edge induced* by $\boldsymbol{U}$ if $\boldsymbol{H}$ is maximal subgraph of $\boldsymbol{G}$ that contains $\boldsymbol{U}$ and all edges of $\boldsymbol{G}$ that are incident on a vertex in $\boldsymbol{U}$.

Example 2. Consider the path graph of order 6 shown in figure 2 (*top*).

Let $\boldsymbol{U = \{v_2, v_4\}}$. Then the subgraph $\boldsymbol{H}$shown in figure 2 (*bottom*) is normal edge induced by *U*.

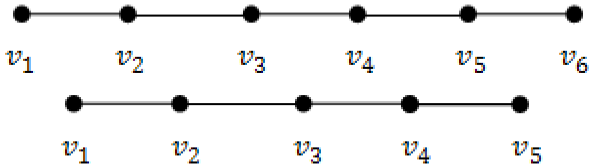

Figure 2. Example of a normal edge induced subgraph

Example 3. Consider the graph $\boldsymbol{G}$ shown in Figure 3 (*left*). Let $\boldsymbol{U = \{v_1, v_2\}}$. Then the subgraph $\boldsymbol{H}$ shown in Figure 3 (*right*) is normal edge induced by *U*.

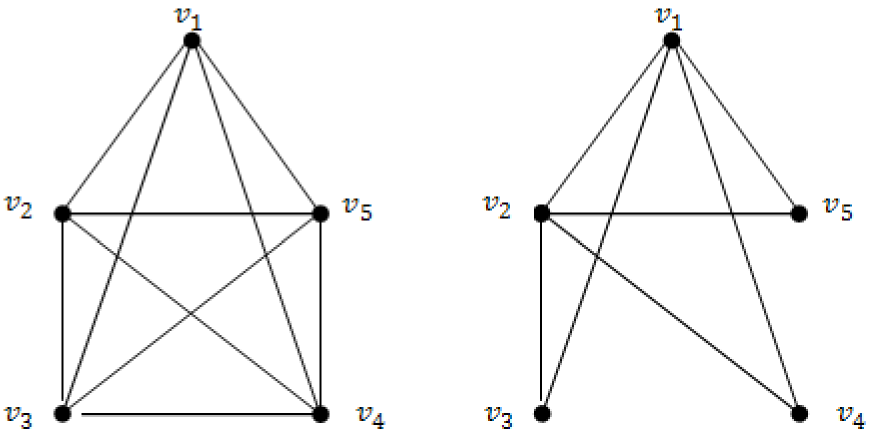

Figure 3. Normal edge induced subgraph

Nucleus of a One Point Determining Graph

Definition 3. Let $\boldsymbol{G}$ be a one-point determining graph. Then, the set $\boldsymbol{G^{01}} = \{\boldsymbol{v}: \boldsymbol{G/v}$ is one point determining$\}$ is called the nucleus of a one point determining graph **G.**

Example 4. Consider the path graph of order 4 shown in Figure 3

Figure 4. For this graph the nucleus is $\boldsymbol{G^{01}} = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\} = \boldsymbol{V(G)}$

Example 5. Consider the complete graph $\boldsymbol{K_3}$ shown in Figure 4

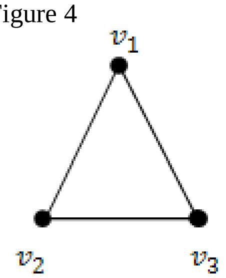

Figure 5 . $\boldsymbol{K_3}$The nucleus is $\boldsymbol{K_3}^{\boldsymbol{01}} = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\} = (\boldsymbol{V(K_3)}$

Example 6. Consider the petersen graph, $\boldsymbol{G}$, shown in Figure 5.

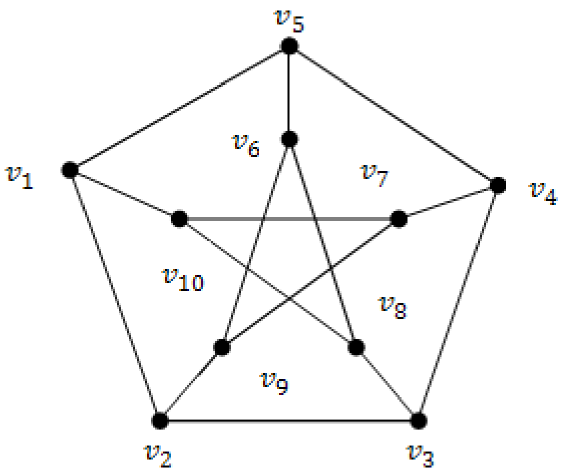

Figure 6 . $\boldsymbol{G}$Then for this graph the nucleus is $\boldsymbol{G^{01}} = \{\boldsymbol{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}}\} = \boldsymbol{V(G)}$

Remark: If $\boldsymbol{G}$ is a tree (or a path) then $\boldsymbol{G^{01}} = \boldsymbol{V(G)}$. But $\boldsymbol{G^{01}} = \boldsymbol{V(G)}$ doesnot imply that $\boldsymbol{G}$ is a tree.

Theorem 5*: Let* $\boldsymbol{G}$ *be a one point determining graph with* $\boldsymbol{G^{01}} = \boldsymbol{V(G)}$ and every normal edge induced subgraph **H** , with $\sum \boldsymbol{d_{H(v)}} \geq \sum \boldsymbol{d_{G(v)}} - \boldsymbol{1}$, *H* is a tree. Then **G** is a tree.

Proof: Suppose $\boldsymbol{G}$ is one-point determining and not a tree. Then there is a cycle, except $\boldsymbol{C_4}$, in G. If $\boldsymbol{H}$ is a normal edge induced subgraph of $\boldsymbol{G}$ with $\sum \boldsymbol{d_{H(v)}} \geq \sum \boldsymbol{d_{G(v)}} - \boldsymbol{1}$ and $\boldsymbol{G^{01}} = \boldsymbol{V(G)}$ must shows that there is a cycle in $\boldsymbol{H}$ also. Thus $\boldsymbol{H}$ is not a tree. This contradiction shows that $\boldsymbol{G}$ is a tree.

Remark: A graph $\boldsymbol{G}$ is said to be one-point distinguishing graph, if for any two distinct vertices $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$, the closed neighborhoods $\boldsymbol{N[v_1]}$ and $\boldsymbol{N[v_2]}$ have at most one vertex in common. Only trivial graphs are in this type. And so, there is no relevance to one-point distinguishing graph

Two-point distinguishing graphs

Definition 4: A graph $\boldsymbol{G}$ is said to be *two-point distinguishing graph*, if for any two distinct vertices $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$, the closed neighborhoods $\boldsymbol{N[v_1]}$ and $\boldsymbol{N[v_2]}$, have at most two vertices in common.

Example 7: Consider the path graph of order 5 shown in figure 5

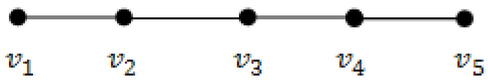

Figure 7. Example of a two point distinguishing graph

Example 8: Consider the following cycle graph $\boldsymbol{C_3}$, $\boldsymbol{C_4}$ and $\boldsymbol{C_6}$ shown in figure 6

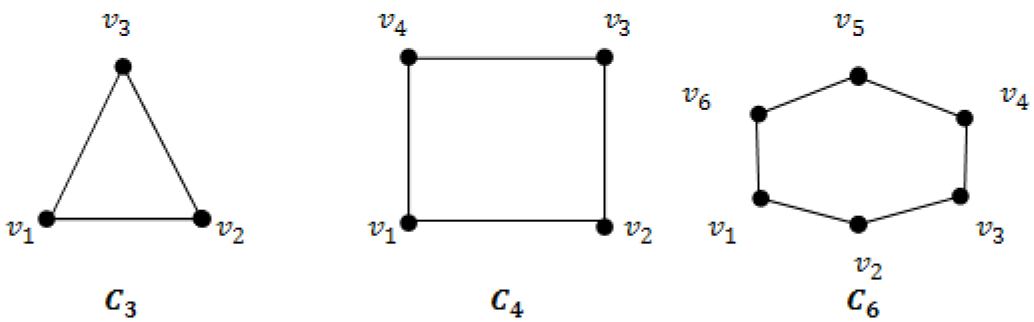

Figure 8

In $\boldsymbol{C_3}$, $\boldsymbol{N[v_1] = N[v_2] = N[v_3]} = \{\boldsymbol{v_1, v_2, v_3}\}$. Hence, $\boldsymbol{C_3}$ is not two-point distinguishing graph. In $\boldsymbol{C_4}$, $\boldsymbol{N[v_1]} = \{\boldsymbol{v_1, v_2, v_4}\}$, $\boldsymbol{N[v_2]} = \{\boldsymbol{v_1, v_2, v_3}\}$, $\boldsymbol{N[v_3]} = \{\boldsymbol{v_2, v_3, v_4}\}$ and $\boldsymbol{N[v_4]} = \{\boldsymbol{v_1, v_3, v_4}\}$. **H**ence, $\boldsymbol{C_4}$ is two-point distinguishing graph. In $\boldsymbol{C_6}$, $\boldsymbol{N[v_1]} = \{\boldsymbol{v_1, v_2, v_6}\}$, $\boldsymbol{N[v_2]} = \{\boldsymbol{v_1, v_2, v_3}\}$, $\boldsymbol{N[v_3]} = \{\boldsymbol{v_2, v_3, v_4}\}$, $\boldsymbol{N[v_4]} = \{\boldsymbol{v_3, v_4, v_5}\}$, $\boldsymbol{N[v_5]} = \{\boldsymbol{v_4, v_5, v_6}\}$, $\boldsymbol{N[v_6]} = \{\boldsymbol{v_1, v_5, v_6}\}$. **H**ence, $\boldsymbol{C_6}$ is two-point distinguishing graph

Theorem 6: A graph **G** is two-point distinguishing graph if and only if **G** doesn't contain either $\mathbf{C_3}$ or $K_{2,3}$

Proof: Let $\boldsymbol{G}$ contains $\boldsymbol{C_3}$ with $\boldsymbol{v_1, v_2, v_3}$ as vertices. Then $\boldsymbol{N[v_1]} \cap \boldsymbol{N[v_2]} = \{\boldsymbol{v_1, v_2, v_3}\}$ so that *G* is not two-point distinguishing graph. Also, if $\boldsymbol{G}$ contains $\boldsymbol{K_{2,3}}$ with $\boldsymbol{V(K_{2,3})} = \{\boldsymbol{u_1, u_2, v_1, v_2, v_3}\}$ then, $\boldsymbol{N[u_1]} \cap \boldsymbol{N[u_2]} = \{\boldsymbol{v_1, v_2, v_3}\}$ so that $\boldsymbol{G}$ is not two-point distingu- ishing graph. Hence the condition is sufficient. Conversely, if $\boldsymbol{G}$ is not a two-point distinguishing graph, then there exists $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\boldsymbol{N[u]}$ and $\boldsymbol{N[v]}$ have more than two vertices in common. In that case $\boldsymbol{G}$ contains a $\boldsymbol{C_3}$ or $\boldsymbol{K_{2,3}}$ .Thus, the condition is necessary.

Observations

Observation 1: All trees are one-point determining graph.

If a tree $\boldsymbol{T}$ is not one-point determining graph, then there exists vertices $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ such that $\boldsymbol{N(u_1)}$ and $\boldsymbol{N(u_2)}$ contains at least two vertices in common say $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. Thus, the vertices $\boldsymbol{u_1 v_1 u_2 v_2\ u_1}$form a cycle in $\boldsymbol{T}$ – contradiction.

Observation 2: The only complete graphs that are both point determining graph. and one-point determining graph are K_2 and K_3

For K_2 and K_3, let $V(K_2) = \{u_1, u_2\}$ and $V(K_3) = \{v_1, v_2, v_3\}$, $N(u_1) = \{u_2\}$ and $N(u_2) = \{u_1\}$. Also, $N(v_1) = \{v_2, v_3\}$, $N(v_2) = \{v_1, v_3\}$ and $N(v_3) = \{v_1, v_2\}$. Hence K_2 and K_3 are both point determining graph and one-point determining graph.

Observation 3: The complete bipartite graph $K_{m,n}$ is one-point determining graph, only when $m = 1$or $n = 1$. Suppose m and $n > 1$. Then there is a C_4 in $K_{m,n}$ and hence by theorem 1, the result follows.

Observation 4: The complete graph K_n, is one-point determining graph, only when $n \leq 3$. For the complete graph K_n, $n \geq 4$, there exists a C_4, by theorem 1 and by observation 2, show that K_n, $n \geq 4$ are not one-point determining graph

Observation 5: All cycle graphs C_n, except C_4 are one-point determining graph. In the cycle C_n, $n \neq 4$, there exists distinct vertices v_i and v_j, $1 \leq i \leq n, 1 \leq j \leq n$ such that $N(v_i)$ and $N(v_j)$ contains at most one vertex in common. Therefore, all cycle graphs C_n, except C_4 are one-point determining graph.

Observation 6: C_4 is the only graph that is not point determining graph and not one-point determining graph.

Observation 7: All path graphs P_n, are one-point determining graph. For a path P_n there exists distinct vertices v_i and v_j, $1 \leq i \leq n, 1 \leq j \leq n$ such that $N(v_i)$ and $N(v_j)$ contains one vertex in common. Hence P_n, is one-point determining graph.

Observation 8: The path which is one-point determining graph and not point determining graph is P_3 only. For P_3, let $V(P_3) = \{v_1, v_2, v_3\}$. Then $N(v_1) = \{v_2\} = N(v_3)$ and by observation 7, the result follows.

Observation 9: All one-point determining graph graphs are point determining graph except P_3. In point determining graph, distinct vertices have all distinct open neighborhoods. At the same time in one-point determining graph, distinct vertices have at most one open neighborhood. Since one-point determining graph is stronger condition than point determining graph, and by observation 8, the result follows.

Observation 10: For any n, ${P_n}^{01}$contains all vertices of P_n, that is $|{P_n}^{01}| = n$. Also, for a tree T of order n, $|T^{01}| = n$. Since all path graphs are one-point determining graph, $|{P_n}^{01}| = n$ and the result follows for tree also.

Observation 11: All trees are two-point distinguishing graph. If a tree T is not two-point distinguishing graph, then there exists vertices v_1 and v_2 such that $N(v_1)$ and $N(v_2)$ contains at least three vertices in common say u, v **and** w. Thus, the vertices $v_1\ u\ v_2\ w\ v_1$form a cycle in T – contradiction.

Observation 12: The complete graphs K_n is two-point distinguishing graph, only when $n \leq 2$. For the complete graph K_n, $n \geq 3$, $N(v_i) = \{v_1, v_2, v_3, \dots, v_n\}$ where, $V(K_n) = \{v_1, v_2, v_3, \dots, v_n\}$, $i = 1, 2, \dots, n$. Thus K_n is two-point distinguishing graph only when $n \leq 2$.

Observation 13: The complete bipartite graphs $K_{m,n}$ are two-point distinguishing graph only when m or $n \leq 2$. Suppose, m and $n \geq 3$. Then there exists a $K_{2,3}$ in $K_{m,n}$ and hence by theorem 6, the result follows.

Observation 14: All cycle graphs C_n, except C_3 are two-point distinguishing graph. In the cycle C_n, $n \neq 3$, there exists distinct vertices u_i and u_j, $1 \leq i \leq n, 1 \leq j \leq n$ such that $N(u_i)$ and $N(u_j)$ contains at most two vertices in common. Hence, all cycle graphs C_n, except C_3 are two-point distinguishing graph.

Observation 15: The cycle which is not point distinguishing graph and not two-point distinguishing graph is C_3 only.

Observation 16: All path graphs P_n, are two-point distinguishing graph. For a path P_n there exists distinct vertices u_i and u_j, $1 \leq i \leq n, 1 \leq j \leq n$ such that $N(u_i)$ and $N(u_j)$ contains at most two vertices in common. Thus P_n, is two-point distinguishing graph.

Observation 17: If G is an one-point determining graph containing an isolated vertex v, then $v \in G^{01}$

Observation 18: Wheels are not one-point determining, while wheels except $W_{1,4}$ are point determining. Since every wheel $W_{1,n}$ $n \geq 3$, contains a C_4, by theorem 1, wheels are not one-point determining. For, the wheel $W_{1,4}$, $V(W_{1,4}) = \{v_0, v_1, v_2, v_3, v_4\}$, there exists vertices v_1 and v_3 such that $N(v_1) = \{v_0, v_2, v_4\} = N(v_3)$ shows that $W_{1,4}$ is not point determining. But in $W_{1,n}$, $n \neq 4$, distinct vertices have distinct open neighborhoods.

Observation 19: For any positive integer n, there exists a one-point determining, with $|G^{01}| = n$. Let G be a one-point determining with $n(G) = n$.. Deleting each vertex v once, the resulting graph, $\{ G/v\}$ is also 1PDt. Thus $|G^{01}| = n$.

Observation 20: $|G^{01}| = V(G)$ for any one-point determining. But in point determining, $|G^{01}| \neq V(G)$ **[6]** By result 19, the first part follows. For the later part consider the graph P_4 . It is point determining. but $|{P_4}^{01}| \neq V(P_4)$.

Observation 21: If G is one-point determining, $G + x$ is not one-point determining, unless G is not point determining. If G is one-point determining and $G + x$ is not one-point determining, then $G + x$ contains a C_4 and G does not contain a C_4 must implies that distinct vertices of G have distinct open neighborhoods. Thus G is point determining.

Observation 22: Wheels are not two-point distinguishing graph, but $W_{1,3}$ is the only wheel graph that is not two-point distinguishing.

Every wheel $W_{1,n}$, $n \geq 3$, contains a C_3 , by theorem 6, wheels are not two-point distinguishing graph. For, the wheel graph $W_{1,3}$, $V(W_{1,3}) = \{v_0, v_1, v_2, v_3\}$, $N[v_0] = N[v_1] = N[v_2] = N[v_3] = \{v_0, v_1, v_2, v_3\}$.

References

[1] R.Balakrishnan & K.Ranganathan; A text book of Graph Theory, Springer (2000) Publication.
[2] J.A Boundy & U.S.R Murthy; Graph Theory with Application, Macmillan (1976) Publication.
[3] F. Harray; Graphy Theory, Addison-Wesley (1969) publication.
[4] D.P Sumner; Point determining in graphs, Discrete Math.5 (1973) 179-187.
[5] R.C Entringer & L.D Gassman; Line-Critical Point determining and point distinguishing graphs, Discrete_Math.10 (1974) 43-55.
[6] D P Sumner; The Nucleus of Point determining graph, Discrete_Math.14 (1976) 91-97.