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Unitarizable and Uniformly Non-Amenable Groups

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ABSTRACT

The group B(m,n) satisfies identity relation $x^n = 1$. Moreover, since F_m^n is a verbal subgroup of group F^m generated with word x^n, the group B(m,n) is free in the variety of all n-periodic groups, i.e. all groups, where the identical relation $x^n = 1$ holds. The group B(m,n) is free in the variety of all n-periodical groups and is called free-periodical or free Burnside group of the period n and rank m.

Introduction

Uniform non-amenability

Let A be a finite subset of a group G, and let S be a finite generating set for G. By the boundary of the subset A with respect to the finite generating set S ⊂ G we mean the set

$$\partial (A) = \{a \in A | ax \notin A \, for \, some \, x \in S\}.$$

Definition 1. By the Folner constant of the group G with respect to the generating set S we mean the number

$$Fol_s(G) = inf \frac{|\partial(A)|}{|A|}$$

Where the infimom is taken over all finite nonempty subsets $A \subset G$ (see [8]).

As is known, a group is amenable if and only if $Fol_s(G) = 0$ for some (and hence for every) finite generating set S (see [8], [12] – [13]). Recall that a group G is said to be amenable if G admits a finitely additive measure μ defined on the $\sigma - algebra$ of all subsets of G and such that $\mu(G) = 1 \, and \, \mu(gA) = \mu(A) \, for \, any \, g \in G \, and \, A \subset G$. As was proved by von Neumann [14], the class of amenable groups is closed with respect to the operations of passage to subgroups, quotient groups, inductive limits, and extensions. On The other hand, every group containing a free subgroup of rank 2 is non-amenable.

In 1977, Adyan conjectured that the m-generated free periodic groups $B(m,n)$ of odd period n≥665 are non-amenable for $m > 1$ (see [15]). Later he confirmed this conjecture in [9]. In [9], Adyan found a sufficient condition for the non-amenability of groups for which the word problem admits a Dehn algorithm solution (finitely presented groups of this kind are said to be hyperbolic (see [17, Theorem 1])). Further, for the relatively free groups $B(m,n)$ of the Burnside variety, where $m > 1$ and n is odd,$n \geq 665$, a system of defining relations is indicated which satisfies both the Dehn condition and the sufficient Non-amenability condition mentioned above.

Thus, the first example of non-amenable group that satisfy a non-trivial identity relation was indicated by S. I. Adian. The well-known theorem of Adian (see [2], Theorem 5) asserts that the free Burnside group $B(m,n) = \langle a_1, a_2, \dots a_m | A^n = 1 for \, all \, words \, A = A(a_1^{\mp 1}, a_2^{\mp 1}, \dots a_m^{\mp 1}).\rangle$ is non-amenable for any odd number $n \geq 665 \, and \, m > 1$. The group $B(m,n)$ does not contain absolutely free groups, since it satisfies the identity $x^n = 1$.

Definition 2. ((see [8] and [12]). The number $Fol_A(G) = inf \, Fol_s(G)$ where the infimum is taken over all finite generating sets S in G, is referred to as a Folner constant of G. A finitely generated group is said to be uniformly non-amenable if $Fol(G) > 0$.

Some classes of uniformly nonamenable groups are known. For instance, every non-elementary hyperbolic group and every large group are uniformly non-amenable [8]. On the other hand, there are non-amenable groups which are not uniformly non-amenable (see [8], [22]). D. Osin [4] proved the uniform non-amenability of the groups $B(m,n) \, for \, m > 1$ and for odd $n > 1078$. This result was obtained earlier in [8]; however, the proof in [22] used a conjecture expressed in [10] and still unproved. In the paper [5] V.Atabekyan proved that for any odd integer n≥1003, every finitely generated noncyclic subgroup H of a free Burnside group B(m,n) is a uniformly non-amenable group. In the next paper [23] has been prove the conjecture expressed in [10].

Unitarizability

Let G be a group, H be a Hilbert space. A representation $\pi: G \to B(H)$ is called unitarizable, if there exists an invertible operator T such that the operator is a unitary operator for any element g ∈G . The group G is called unitarizable, if every uniformly bounded representation $\pi : G \to B(H)$ is unitarizable. J. Dixmier in [17] and M.M. Day in [18] proved that every amenable group is unitarizable. The question of whether the Converse holds has been open since then. The first example of non-unitarizable group is Constructed in [19], where it is shown the non-unitarizability of the group $SL_2(R)$.

It is known, that if all the countable subgroups of a given group G are unitarizable, then the group G is unitarizable itself, and if a group is unitarizable, then all its subgroups and factor groups are unitarizable (see for example [20]). Therefore, from existence of non-unitarizable group it follows that the absolutely free group F_∞ of countable rank is non-unitarizable, and therefore any group that contains a subgroup isomorphic to the free groupF_2 of rank 2 is non-unitarizable. N. Monod and N. Ozawa in joint paper [25] when studying the problem of whether or not the unitarizability of a group implies its amenability (see [17]), obtained an interesting criterion according to which the non-amenability of a given group G is equivalent to non-unitarizability of group $A\, wr\, G$ for all infinite Abelian groups A, where $A\, wr\, G$ is wreath product of groups A and G. Bearing on the mentioned criterion and on Adian's Theorem about non-amenability of groups $B(m,n), N. Monod\ and\ N$. Ozawa proved (see [25], Theorem 2) that the free Burnside groups B(m,n) are non-unitarizable for all composite odd numbers n = $n_1.n_2$, where n_1 ≥665 and $m \geq 2$. V.Atabekyan in paper [21] has strengthened this result. He proved, that for any composite odd number n = $n_1.n_2$, where n_1 ≥ 665 and $m > 1$, any non-cyclic subgroup of free Burnside group B(m,n) is non-unitarizable. This result of paper [21] indicates the new examples of non-unitarizable periodic groups different from Burnside groups.

Actually, comparing it with result by A.Yu. Olshanskii [6] (for $odd\ n > 1078$) and the result by V. Atabekyan [11] (for $odd\ n \geq 1003$), we obtain that any proper normal subgroup of group B(m,n) is not isomorphic to any free Burnside group and at the same time is non-unitarizable group. According to [6] (see corollary 0.11), if all the countable subgroups of a given group G are unitarizable, then the group G is unitarizable itself. The infinite cyclic group is amenable and, therefore, is unitarizable (see [17], [18]). Hence, any absolutely free group of $rank \geq 2$appearing as non-unitarizable contains countable unitarizable subgroup. Another result of the paper [21], states that for every composite odd number n = $n_1.n_2$, where n_1 ≥ 665 and$m > 1$, any infinite subgroup of group $B(m,n)$ is non-unitarizable, and any finite subgroup is unitarizable. Thus, for subgroups of free Burnside groups the unitarizability is equivalent to amenability. In this chapter we prove that there exist finitely generated non-unitarizable periodic groups of restricted period, that are different from free Burnside groups and their non-cyclic subgroups. According to the result of Dixmier-Day, non-unitarizable groups are non-amenable. Constructed below non-unitarizable groups are not only non-amenable, but also uniformly non-amenable.

The construction of pairwise non-isomorphic
Non-unitarizable groups

The well-known theorem by S.I. Adian (see [1]) states, that for $m > 1$ and odd $n \geq 665$ the group $B(m,n)$ is infinite. As it is shown in the work [18], for arbitrary odd $n \geq 1003$ there are continuum many simple 2-generated non-isomorphic groups {Γi}i∈I of the given period n ≥ 1003. Let n = $n_1.n_2$, be arbitrary composite odd number, where n_1 ≥ 665. Then n≥1003. Let's form a direct product $G_i = B(2,n) \times \Gamma i$ of the group B(2,n) with each group $\Gamma_i \in \{\Gamma_i\} i \in I$. Lemma 1. The groups $G_i = B(2,n) \times \Gamma i$ (i ∈ I) are pairwise non-isomorphic non- unitarizable groups. Proof. Consider any two groups G_1 = B (2,n)×Γ_1 and G_2 = B(2,n)×Γ_2, where Γ_1,Γ_2∈ $\{\Gamma_i\}$i∈I are non-isomorphic 2-generated groups of period n, and show that groups G_1 and G_2 are non-isomorphic. Proving by contradiction suppose, that ϕ :$G_1 \rightarrow G_2$ is some isomorphism. It is obvious, that groups $B(2,n)$ and Γi are contained in G_i as subgroups (i = 1,2). Consider the image $\phi(B(2,n))$ of subgroup $B(2,n)$ via isomorphism ϕ . Since the image of normal subgroup via isomorphism is a normal subgroup, then $\phi(B(2,n))$ is normal subgroup in G_2. Let us show, that the intersection of subgroup $\phi(B(2,n))$ with normal subgroupΓ_2, of the group G_2 is trivial. Actually, since the subgroup Γ_2, is simple group, then any normal subgroup containing a non-trivial element of subgroup Γ_2, contains all the elements of that group. Therefore, if $\phi(B(2,n)) \cap \Gamma_2, \neq \emptyset, then\ \phi(B(2,n)) \rhd \Gamma_2,. It\ is\ obvious\ that\ \phi(B(2,n)) \cong B(2,n)$.

Lemma 2. (see Theorem 1 of paper [7]) Let $B(m,n)$ be a free periodic group of arbitrary rank m with period n. Then for all odd numbers n ≥ 1003 the normalizer of any nontrivial subgroup N of the group B(m,n) coincides with N if the subgroup N is free in the variety of all n-periodic groups. Since the subgroup Γ_2, is a normal subgroup of the group $\phi(B(2,n))$, then by Lemma 2, the normal subgroup Γ_2, is not free n-periodic group.

Lemma 3. (see Theorem 0.1, [3]) For every odd number $n \geq 1003$, every non-cyclic subgroup of the group $B(2,n)$ contains a subgroup isomorphic to the group $B(\infty,n)$. By Lemma 3, subgroup Γ_2, contains free periodic subgroup H of rank 2. But this is impossible since only the non-cyclic subgroup H of the group Γ_2, is the group.Γ_2, Thus, $\phi(B(2,n)) \cap \Gamma_2 = \emptyset$. It is clear that the following isomorphisms are true:

$\Gamma_1 \simeq G_1/B(2,n) \simeq \phi(G_1)/\phi(B(2,n))\ .Since\ \phi(G_1)/\phi(B(2,n)) = G_2/\phi(B(2,n))\ and \phi(B(2,n)) \cap \Gamma_2 = \emptyset$, then the group Γ_2, can be embedded into the group G_2/ϕ(B(2,n))≃Γ_1. But this is again contradiction since the groups Γ_1 andΓ_2 are non-isomorphic infinite groups and any proper subgroup of the group Γ_1 is finite. The contradiction proves that groupsG_1 and G_2 are non-isomorphic. Since our constructed groups$G_i = B(2,n) \times \Gamma_i$ (i ∈ I) contain non-unitarizable subgroup$B(2,n)$, then they are non-unitarizable either.

The upper bound of the lengths of generators of free periodic groups

Lemma 4. Let $n \geq 1003$ be an arbitrary odd number. Then every noncyclic subgroup $\langle X,Y \rangle$ of $B(2,n)$ contains a noncyclic subgroup of the form U[A,C] U^{-1}such that C is an elementary period of some rank α and $C^{B(m,n,\alpha-1)}$= [A^d,$Z^{-1}B^dZ$], where A and B are minimized elementary periods of some ranks γ ≤ β ≤ $\alpha - 1$, d = 191, and the lengths of the words UAU^{-1} and UCU^{-1} with respect to the generators X and Y satisfy the inequalities $|UAU^{-1}|_{\{X,Y\}} < 84^3 n$ $UCU^{-1}|_{\{X,Y\}} < 84^3 n$.

Proof. Let $\Delta \rightleftharpoons \langle X,Y \rangle_{B(2,n)}$be an arbitrary noncyclic subgroup of $B(2,n)$. It follows from VI.2.4 in [1] and VI.1.2 in [1] that X=$TF^iT^{-1}(B(2,n)T^{-1}YT = Z^{-1}E^jZ(B(2,n))$for some words T and Z and some minimized elementary periods F and E of some ranks σ and ρ respectively. Without loss of generality, we may assume that σ ≤ ρ and, by VI.2.4 and IV.1.13 in [1], we may also assume that Z $\in \mathfrak{M}_\xi \cap \mathcal{A}_{\xi+1}$ for some ξ ≥ ρ. Let $lcd(i,n) = k$, and let r be an integer such that $|r| < n$and $F^{ir} = F^k$. Choosing a number s⇌$[n/3k]$, we obtain $n/5 < sk < n/3$. Therefore, X^{rs} = T $F^{irs}T^{-1}$ = T $F^{ks}T^{-1}$and $186 < ks < \frac{n+1}{2} - 148$ because n≥1003. Thus, for the word $X_1 \rightleftharpoons X^{rs}$, we have X_1 = T $F^{ks}T^{-1}$and $|X_1|_{\{X,Y\}} <$ n|X $|_{\{X,Y\}}$ = n. In a

similar way, we can find an element $Y_1 \in \langle X, Y \rangle_{B(2,n)}$ and numbers t and l such that $T^{-1}Y_1$ T = $Z^{-1}E^{tl}$ Z, 186 <tl<$\frac{n+1}{2} - 148$and $|Y1|_{\{X,Y\}}$< n. By Theorem VI.3.1 in [1], $[X_1, Y_1]$ ≠1. By Lemmas 2, 7.2, and 2.8 in [9], the com- mutator $[X_1, Y_1]$ = T$[F^{ks}, Z^{-1}E^{tl}$Z$]T^{-1}$Z]T is conjugate in $B(m,n)$ to some minimized elementary period D of some rank δ ≥ ρ + 1. Let $T^{-1}[X_1, Y_1]$ T = Z_1^{-1}DZ_1 where$Z_1 \in \mathfrak{M}_\lambda \cap \mathcal{A}_{\lambda+1}$for some λ ≥ δ. In this case, applying Lemmas 3.2, 7.2, and 2.8 of [9] again, we see that the commutator $[X_1, [X_1, Y_1]^d]$= T$[F^{ks}, Z^{-1}1\ D^d Z_1]T^{-1}$is conjugate in $B(2,n)$ to some minimized elementary period B of some rank μ ≥ δ + 1. Assume that $T^{-1}[X_1, [X_1, Y_1]^d]$T =$Z_2^{-1}$ BZ_2.
Thus, the subgroup Δ⇌$\langle X, Y \rangle_{B(2,n)}$contains the elements
$[X_1, Y_1] = TZ_1^{-1}DZ_1T^{-1}and[X_1, [X_1, Y_1]^d] = Z_2^{-1}$ BZ_2T^{-1}

We can assume that $Z_3^{-1} \rightleftharpoons Z_1Z^{-1}2 \in \mathfrak{M}_\nu \cap \mathcal{A}_{\nu+1}$, where ν ≥ μ. By Lemma 3.2 in [9], we can find a reduced form C of the commutator £$[D^d, Z_3^{-1}B^dZ_3]$. By Lemma 7.2 in [9],C is an elementary period of some rank τ ≥ μ + 1. By 3.6 in [9], C=w$[D^d, z_3^{-1}B^dZ_3]w^{-1}(B(2,n,\mu))$where w ∈Θ(D, D_1).
Consider the elementary periods A⇌wD$w^{-1}$and C = $[D^d, z_3^{-1}B^dZ_3]w^{-1}$. By definition A = w$Z_1T^{-1}[X_1, Y_1]$T $Z_1^{-1}W^{-1}$ and C = w$Z_1T^{-1}[[X_1, Y_1]^d, [X_1, [X_1, Y_1]^d]^d]TZ_1^{-1}W$. Thus, if U ⇌T$Z_1^{-1}W^{-1}$, then UA$U^{-1}$∈Δ, UC$U^{-1}$∈Δ and $|UAU^{-1}|_{\{X,Y\}}$ = $[X_1, Y_1]_{\{X,Y\}}$< n$[X, Y]|_{\{X,Y\}}$= 4n, (3.1) $|UCU^{-1}|_{\{X,Y\}}$= $|[[X_1, Y_1]^d, [X_1, [X_1, Y_1]^d]^d]|_{\{X,Y\}}$ < n(8d + 2d(8d + 2)). (3.2) It remains to note that n(8d + 2d(8d + 2)) <84^3n. This proves the Lemma 4.

Lemma5. (Proposition [3]) Let n ≥ 1003 be an arbitrary odd number. Then every noncyclic subgroup ⟨X,Y⟩ B(2,n) contains a noncyclic subgroup of the form U[A,C]U^{-1}such that the commutator $[A^d, Z_1^{-1}B^dZ]$ is equal in the group B(2,n,α−1) to an elementary period C of rank α, where A is an elementary period of rank γ, B is an elementary period of rank β, Z ∈$M_{\alpha-1}$, γ ≤ β ≤ α − 1, d = 191, n ≥ 1003 is an arbitrary odd number, and the words A^q and B^q enter some words in the sets $M_{\gamma-1}$ and$M_{\beta-1}$, respectively. In this case, the words u = C^{200}A$C^{200}A^2 \cdots A^{n-1}C^{200}$, v = C^{300}A$C^{300}A^2 \cdots A^{n-1}C^{300}$form a basis of a free Burnside subgroup of rank 2 in $B(2,n)$. Moreover, $|u|_{\{X,Y\},|}v|_{\{X,Y\},|}$< $(57n)^3$. Proof. Since the subgroups $\langle u, v \rangle_{B(2,n)}$ and $\langle UuU^{-1}, UvU^{-1} \rangle_{B(2,n)}$ are isomorphic, it follows from Proposition 1 of paper [26] that the elements UuU^{-1}, UvU^{-1}form a basis of a free Burnside subgroup of rank 2 in B(2,n). Obviously, UuU^{-1} = $(UCU^{-1})^{200}(UAU^{-1})$· · · $(UAU^{-1})^{n-1}(UCU^{-1})^{200}$, Uu$V$ = $(UCU^{-1})^{300}(UAU^{-1})$· · · $(UAU^{-1})^{n-1}(UCU^{-1})^{300}$. By Lemma 1, we have Uu$U^{-1}$, Uv$U^{-1} \in \langle X, Y \rangle_{B(m,n)}$. Using inequalities (3.1) and (3.2), we obtain $|UuU^{-1}|_{\{X,Y\}}$ <$\frac{n(n-1)\,4n}{2}$+ $200n^2$(8d + 2d(8d + 2)) $|UvU^{-1}|_{\{X,Y\}}$ <$\frac{n(n-1)\,4n}{2}$+ $300n^2$ (8d + 2d(8d + 2)) Note that $|UuU^{-1}|_s \le |UuU^{-1}|_{\{X,Y\}}$, $|UvU^{-1}|_s \le |UvU^{-1}|_{\{X,Y\}}$, because {X,Y}⊆S. It remains to note that $2\boldsymbol{n}^2$(n−1) + $300\boldsymbol{n}^2$ (8d + 2d(8d + 2)) < $(57n)^3$. This completes the proof of lemma.

Some estimates of the Folner's constant

In this section we will prove of some inequality about Folner'sconstant, that will be used to prove the main result of this chapter.
Lemma 6. Let G be a finitely generated group, S a finite generating system, and g ∈G. Let $\boldsymbol{S_1}$ = S∪g. Then Fo$\boldsymbol{l_S}$(G) ≥Fo$\boldsymbol{l_{s_1}G}$. Proof. The Cayley graph of G with respect to $\boldsymbol{S_1}$ is the same as the one with respect to S, but at each vertex v there is an extra edge labelled g leaving v and an extra edge labelled g arriving at v. Consider a non-empty finite subset A. Obviously, adding edges to a Cayley graph cannot move a boundary point of A to the interior. The only thing that can happen is that an interior point now becomes a boundary point if its corresponding edge g or g−1 has its other endpoint outside A. So the boundary with respect to S_1 is at least as large as the boundary with respect to S.
Lemma 7. Let G be a group, and let S = $\{x_1,...,x_n\}$ be a finite generating set of G. Let m≤n, and let H be the subgroup of G generated by the set S_1=$\{x_1,...,x_m\}$. Then, Fol_S(G) ≤Fo$l_{s_1}H$.
Proof. Let A be a non-empty finite subset of G, and choose $y_1,...,y_k$ elements of G in such a way that y_iH∩y_jH = ∅ if i≠ j, and A∩y_iH ≠ ∅. Namely, they_i are representatives of the cosets of H which intersect A. Let$A_i = A \cap y_iH$. The Cayley graph of H with respect to Y sits inside the Cayley graph of G with respect to X. Considering only the edges labelled in Y, the cosets for H form disjoint parallel copies of the Cayley graph of H. Note that Ai is a finite subgraph of the component corresponding to the cosetyiH.
Clearly, by the definition of the Folner constant, we have

$$\frac{|\partial_{s_1}A_i|}{|A_i|} = \frac{|\partial_{s_1}(y_i^{-1}A_i)|}{|y_i^{-1}A_i|} \ge \mathrm{Fo}l_{s_1}H$$

Now, using the argument of the Lemma 6, it is clear that the boundary for A using only elements of Y is smaller than the X-boundary of A. Then,
$\frac{|\partial_S A|}{|A|} \ge \frac{|\partial_{s_1}A|}{|A|} = \frac{\sum_i |\partial_{s_1}A|}{\sum_i |A_i|} \ge \mathrm{Fo}l_y$H.

The following lemma is proved in [8]. Lemma 8. (see Theorem 7.1 in [8]). Let G be a finitely generated group with the set of generators S = $x_1,...,x_n$ and let H be a subgroup of G with the set of generators S_1=$y_1,...,y_k$. Denote by L the maximal length of the elements$y_1...y_k$ with respect to the generators S = $x_1,...,x_s$ Then Fol_s(G)≥$\frac{1}{1+kL}$Fo$l_{s_1}H$
Proof. Let A be a non-empty finite subset of G. As in Lemma 7, we consider A as a finite union of intersections A_i of A with right cosets of H, and we write ∂_{s_1}A =$\cup_i \partial_{s_1}A_i$, viewing each A_i as existing inside a copy of the Cayley graph of H with respect toS_i. We have Fol_{s_1}H ≤ $\frac{|\partial_{s_1}A|}{|A|}$even if A is not a subset of H.

By definition, every element τ ∈$\partial_{s_1}A_i$ can be joined with a point outsideA_i, and so outside A, by multiplication by somey_j, which we think of as a path labelled w_j in the generators S. If τ ∈∂S_A, then this path begins at τ. If τ ∉∂S_A, then the path must necessarily pass through a vertex in ∂S_A, which is not the final vertex of the path, just before leaving A. Consider a vertex ν ∈∂S_A it may be that ν ∈∂_{s_1}A. Otherwise, there are at most l(w_j)−1 ≤ L−1 ways in which a path labelled $\boldsymbol{w_j}$ may pass through ν in such a way that ν is neither the initial nor the final vertex. Thus a vertex $v \in \partial S_A$ corresponds to at most $1 + \sum_{j=1}^{k}(l(w_j) - 1)\ |\partial_S A| \leq (1 + Kl)\ |\partial_S A|$ different vertices in $\partial_{s_1}A$ (and each vertex in $\partial_{s_1}A$ has at least one corresponding vertex in $\partial_S A$). It follows that

$||\partial_{s_1}A| \leq (1 + (\sum_{j=1}^{k}(l(w_j) - 1)\ |\partial_S A|)$

$\leq (1 + Kl)\ |\partial_S A|$

Since the previous inequality is valid for any non-empty finite subset of G, we deduce the

Result

Lemma 9. (See Theorem 4.1 [8]) Let G be a finitely generated group and let X be a finite generating system for G. Let N be a normal subgroup of G, π the canonical homomorphism of G onto G/N and $X' = \pi(X)$. Then, $Fol_S(G) \geq Fol_{X'}\pi(G/N)$ and hence, $Fol(G) \geq Fol(G/N)$.

Lemma 10. Let G be a finitely generated group with the set of generators S, let $\pi : G \to G'$ be an isomorphism of groups, and let $S' = \pi(S)$. Then $Fol_S(G) = Fol_{S'}(G0)$.

Proof. The proof is follows of Lemma 9.

The main results

In this section we prove the following main Theorem. Theorem 1. Suppose $n = \boldsymbol{n_1 n_2}$ is arbitrary composite odd number, where $\boldsymbol{n_1} \geq 665$. There are continuum many non-isomorphic 4-generated groups $\boldsymbol{G_i}$ that satisfy the identity $\boldsymbol{x^n} = 1$, each one of which is non-unitarizable and at the same time uniformly non-

amenable, the Folner's constant for which satisfies the inequality

$$F\emptyset\boldsymbol{l}(\boldsymbol{G_i}) \geq \frac{\boldsymbol{F\emptyset l_{\{a,b\}}(B(2,n))}}{\boldsymbol{1+2(57n)^3}}$$

where $\boldsymbol{F\emptyset l_{\{a,b\}}}$(B(2,n)) is the Folner's constant of the group B(2,n) with respect to the generating set {a, b}.

Proof. According to Lemma 1 the family of groups $\{G_i = B(2,n) \times \Gamma_i\}_{i \in I}$ consist of continuum 4-generated pairwise non-isomorphic non-unitarizable groups. In order to proof of Theorem 1 it suffices to prove that each group G_i ($i \in I$) is uniformly non-amenable, the Folner's constant for which satisfies the inequality

$$F\emptyset L(G_i) \geq \frac{F\emptyset l_{\{a,b\}}(B(2,n))}{1+2(57n)^3}$$

where $F\emptyset l_{\{a,b\}}$(B(2,n)) is the Folner's constant of the group B(2,n) with respect to the generating set {a, b}. Lemma 11. (see Corollary 1 of the paper [5])For any odd number n≥1003 the group B(2,n) is uniformly non-amenable.

Lemma 12. For any odd number n ≥ 1003 the group B(2,n) is uniformly non- amenable, the Folner's constant for which satisfy the inequality $F\emptyset L_{B(2,n)}(H) \geq \frac{b}{1+2(57n)^3}$,

where b is the Folner's constant of the group $B(2,n)$.

Proof. By Lemma 9, if a group has a uniformly non-amenable homomorphic image, then the group is uniformly non-amenable itself (see [8], Theorem 4.1).

The factor-group of group $\boldsymbol{G_i}$ by the normal closure of subgroup $\boldsymbol{\Gamma_i}$ is isomorphic to group $\boldsymbol{B(2,n)}$. Thus, by Lemma 11 the groups $\boldsymbol{G_i}$ ($i \in I$) are uniformly non-amenable either, since they have a uniformly non-amenable factor-group.

To prove Lemma 12, suppose that S is an arbitrary finite set of generating elements of a noncyclic subgroup H in $B(m,n)$. By Lemma 5, there are elements $u, v \in H$ such that {u, v}is a basis of a free Burnside subgroup of rank 2 and the lengths of the elements u and v with respect to the generating set S satisfy the inequalities $|u|_s < (57n)^3$ and $|v|_s < (57n)^3$, where the number $(57n)^3$ does not depend on the choice of the set S. By Lemma 3.8, we have

$$FOL_s(H) \geq \frac{C}{1+2(57n)^3} > FOl_{\{u,v\}}(\langle u, v\rangle_{B(m,n)})$$

By Lemma 10, the number $C = FOl_{\{u,v\}}(\langle u, v\rangle_{B(m,n)})$ does not depend on the choice of the pair of free generators u and v. Since, by another theorem of Adyan (Theorem 5 in [2]), the group $B(2,n)$ is non-amenable, it follows that C > 0. Thus, $FOL_s(H) \geq \frac{C}{1+2(57n)^3} > 0$ for any set of generators S of the noncyclic subgroup H, which implies the inequalities $Fol(H) = inf FOL_s(H) \geq \frac{C}{1+2(57n)^3} > 0$

Thus, the proof of Theorem1 immediately follows by Lemmas 1 and 12. Corollary 1. For arbitrary composite odd number $n = \boldsymbol{n_1 n_2}$, where $\boldsymbol{n_1} \geq 665$, the group B(4,n) has continuum non-isomorphic factor-groups, each one of which is non-unitarizable and uniformly non-amenable.

Proof. It is sufficient to notice that in each group $\boldsymbol{G_i}$, constructed during the proof of Theorem 1, the identity $\boldsymbol{x^n} = 1$ holds. Add that it is yet another example of non-unitarizable periodic group was constructed by D. Osin in the paper [24], but the group constructed by him do not have bounded exponent, i.e. the orders of the elements constructed by him group increase unboundedly

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