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A Class of Chain Ratio-Product Type Estimators for Population Mean Under Double Sampling Scheme in The Presence of Non-Response

Sanjay Kumar and Priyanka Chhaparwal

Department of Statistics, Central University of Rajasthan, Bandarsindri, Kishangarh-305817, Ajmer, Rajasthan.

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ABSTRACT

In this paper, we propose conventional and alternative ratio-product type estimators for population mean using two auxiliary variables in the presence of non-response. The purposed estimators have been found to be more efficient than the relevant estimators for the fixed values of first-phase sample of size $\boldsymbol{n'}$ and sub-sample of size $\boldsymbol{n}(<\boldsymbol{n'})$ taken from the first-phase sample size $\boldsymbol{n'}$ under the specified condition. The purposed estimators are more efficient than the corresponding estimators for population mean $(\bar{\boldsymbol{Y}})$ of a study variable $\boldsymbol{y}$ in the case of fixed cost and have less cost in comparison to the cost incurred by the corresponding relevant estimators for a specified variance. The conditions under which the purposed estimators are more efficient then the relevant estimator have been obtained. An empirical as well as a Monte-Carlo simulation study have been done to demonstrate the efficiencies for the purposed estimators over other relevant estimators.

Introduction

The use of auxiliary information in sample survey have been considered mainly in the field of agricultural, biological, medical and social sciences at the stage of planning, designing, selection of units and devising the estimation procedure. Auxiliary information may be fruitfully utilized to arrive at improved estimators compared to those, not utilizing auxiliary information.

Use of auxiliary information for forming ratio and regression methods of estimators were introduced during 1930's with a comprehensive theory provided by Cochran [1]. In many situations of practical importance, the population mean of the auxiliary variable $(\boldsymbol{x})$ is not known, but the population mean of an additional auxiliary variable $(\boldsymbol{z})$ is known, which is cheaper and less correlated to the study variable $(\boldsymbol{y})$ in comparison to the main auxiliary variable $(\boldsymbol{i.e.}\, \boldsymbol{\rho_{yz}} < \boldsymbol{\rho_{xz}})$. For example, while estimating the total yield of wheat in a district, the yield and area under the crop are likely to be unknown, but the total area of each farm may be known. Then $\boldsymbol{y}, \boldsymbol{x}$ and $\boldsymbol{z}$ are respectively the yield, area under wheat and area under cultivation.

In such a case, Chand [2] and Kiregyera [3, 4] and Srivastava et. al. [5] have purposed chain ratio type estimators using an additional variable with known population mean.

Sometimes, it may not be possible to collect the complete information for all the units selected in the sample due to non-response. The missing observations due to non-response may occur during the investigation, which may be at random and their ignorance of such missing observation may lead to biased estimators, though the amount of the bias may be very negligible. If the missing observations due to non-response is not at random then the amount of bias in estimators will be large. It may increase the error in the estimators and the sampling error will also increase. Estimation of population mean in sample surveys when some observations are missing due to non-response not at random has been consider by Hansen and Hurwitz [6] and Rao [7, 8], Khare and Shrivastava [9, 10], Khare and Kumar [11, 12], Singh et. al [13, 14, 15]. In this paper, we have purposed conventional and alternative ratio-product chain type estimators for population mean of the study variable in the presence of non-response. The expressions for biases and mean square errors of the purposed estimators are obtained and a comparison of purposed estimators has been made with the relevant estimators. The optimum values of the first-phase sample $(\boldsymbol{n'})$, sub-sample $(\boldsymbol{n})$ and sub-sampling fraction $(\boldsymbol{K})$ have been obtained for a fixed cost $\boldsymbol{C} \leq \boldsymbol{C_0}$ and for a specified variance $\boldsymbol{V} = \boldsymbol{V_0}$. A comparative study of the purposed estimators with the relevant estimators has been made with the help of an empirical study as well as a Monte-Carlo simulation study.

The Estimators

Let (Y_i, X_i, Z_i) be the non-negative values for the i^{th} unit of the population $U = (U_1, U_2, \ldots . U_N)$ on the study variable y, the auxiliary variable x and the additional auxiliary variable z with their population means $(\bar{Y}, \bar{X}, \bar{Z})$, coefficient of variations (C_y, C_x, C_z) and correlation coefficients $(\rho_{yx}, \rho_{xz}, \rho_{yz})$. The population U is supposed to be composed of N_1 responding and N_2 non responding units. From the population of size N, a sample of size n is selected by using SRSWOR method of sampling and it was observed that n_1 units respond and n_2 units don't respond. Further, by making extra effort, a sub-sample of size $r = \frac{n_2}{K} (K > 1)$ is drawn from n_2 non responding units by using SRSWOR method of sampling. Hence, we have n_1 units from the respondent group and r units from the non-respondent group of the population in the sample for which the value of the y character is obtained. Hansen and Hurwitz [6] proposed the estimator for $\bar{Y}$, which is given as follows:

$$\bar{y}^* = \frac{n_1}{n}\bar{y}_1 + \frac{n_2}{n}\bar{y}'_2 , \qquad (1)$$

Where $\bar{\boldsymbol{y}}_1$ and $\bar{\boldsymbol{y}}'_2$ denote the sample means of $\boldsymbol{y}$ character based on $\boldsymbol{n_1}$ and $\boldsymbol{r}$ units respectively.

Tele:
E-mail addresses: sanjay.kumar@curaj.ac.in

The estimator $\bar{y}^*$ is unbiased and the $V(\bar{y}^*)$ is given by

$$V(\bar{y}^*) = \frac{f}{n}S_y^2 + \frac{W_2(K-1)}{n}S_{y(2)}^2, \quad (2)$$

where $f = \frac{N-n}{N}, W_i = \frac{N_i}{N}(i = 1,2), S_y^2$ and $S_{y(2)}^2$ are the population mean squares of the character y for the whole population and for the non-responding part of the population. Similarly, the estimator $\bar{x}^*$for the population mean $\bar{X}$ in the presence of non-response is given by

$$\bar{x}^* = \frac{n_1}{n}\bar{x}_1 + \frac{n_2}{n}\bar{x}_2', \quad (3)$$

where, $\bar{x}_1$ and $\bar{x}_2'$ denote the sample means of x character based on n_1 and r units respectively.
The $V(\bar{x}^*)$ is given by

$$V(\bar{x}^*) = \frac{f}{n}S_x^2 + \frac{W_2(K-1)}{n}S_{x(2)}^2 \quad (4)$$

where, S_x^2 and $S_{x(2)}^2$ are the population mean squares of the character x for the whole population and for the non-responding part of the population.

In the case, when population mean $\bar{X}$ is not known, a first- phase sample of size $n'(< N)$ is taken from the population of size N by using simple random sampling without replacement (SRSWOR) scheme of sampling and the population mean $\bar{X}$ is estimated by first-phase sample mean $\bar{x}'$ based on n' units. Again a sub-sample of size n is selected from n' first-phase sample by using SRSWOR scheme of sampling and it has been observed that n_1 units respond and n_2 units do not respond in the sample of size n for the study variable y. It is also assumed that the population of size N is composed of N_1 responding and N_2non-responding units, though they are unknown. Further, a sub-sample of size $r = \frac{n_2}{K}(K > 1)$ from n_2 non-responding units has been drawn by using SRSWOR method of sampling by making extra effort.

In such a situation, the conventional and alternative two-phase sampling ratio (t_{c1}, t_{a1}), product (t_{c2}, t_{a2}) type estimators have been proposed by Khare and Srivastava [9] which are given as follows:

$$t_{c1} = \bar{y}^*\frac{\bar{x}'}{\bar{x}^*}, t_{a1} = \bar{y}^*\frac{\bar{x}'}{\bar{x}}, t_{c2} = \bar{y}^*\frac{\bar{x}^*}{\bar{x}'}, t_{a2} = \bar{y}^*\frac{\bar{x}}{\bar{x}'} \quad (5)$$

where, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\bar{x}' = \frac{1}{n'}\sum_{i=1}^{n'} x_i$.

The bias and the mean square errors of the estimators (t_{c1}, t_{a1}) and (t_{c2}, t_{a2}) are given as

$$B(t_{c1}) = \frac{1}{\bar{X}}\left[\left(\frac{1}{n} - \frac{1}{n'}\right)\{RS_x^2 - S_{yx}\} + \frac{W_2(K-1)}{n}\{RS_{x(2)}^2 - S_{yx(2)}\}\right], \quad (6)$$

$$MSE(t_{c1}) = \left(\frac{1}{n} - \frac{1}{n'}\right)\{S_y^2 + R^2S_x^2 - 2RS_{yx}\} + \frac{W_2(K-1)}{n}\{S_{y(2)}^2 + R^2S_{x(2)}^2 - 2RS_{yx(2)}\} + \left(\frac{1}{n'} - \frac{1}{N}\right)S_y^2, \quad (7)$$

$$B(t_{a1}) = \frac{1}{\bar{X}}\left(\frac{1}{n} - \frac{1}{n'}\right)\{RS_x^2 - S_{yx}\}, \quad (8)$$

$$MSE(t_{a1}) = \left(\frac{1}{n} - \frac{1}{n'}\right)\{S_y^2 + R^2S_x^2 - 2RS_{yx}\} + \frac{W_2(K-1)}{n}S_{y(2)}^2 + \left(\frac{1}{n'} - \frac{1}{N}\right)S_y^2, \quad (9)$$

$$B(t_{c2}) = \frac{1}{\bar{X}}\left[\left(\frac{1}{n} - \frac{1}{n'}\right)S_{yx} + \frac{W_2(K-1)}{n}S_{yx(2)}\right], \quad (10)$$

$$MSE(t_{c2}) = \left(\frac{1}{n} - \frac{1}{n'}\right)\{S_y^2 + R^2S_x^2 + 2RS_{yx}\} + \frac{W_2(K-1)}{n}\{S_{y(2)}^2 + R^2S_{x(2)}^2 + 2RS_{yx(2)}\} + \left(\frac{1}{n'} - \frac{1}{N}\right)S_y^2, \quad (11)$$

$$B(t_{a2}) = \frac{1}{\bar{X}}\left(\frac{1}{n} - \frac{1}{n'}\right)S_{yx}, \quad (12)$$

and

$$MSE(t_{a2}) = \left(\frac{1}{n} - \frac{1}{n'}\right)\{S_y^2 + R^2S_x^2 + 2RS_{yx}\} + \frac{W_2(K-1)}{n}S_{y(2)}^2 + \left(\frac{1}{n'} - \frac{1}{N}\right)S_y^2. \quad (13)$$

where, $S_y^2 = \frac{1}{N-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2, S_x^2 = \frac{1}{N-1}\sum_{i=1}^{n}(X_i - \bar{X})^2,$

$S_{yx} = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \bar{Y})(X_i - \bar{X}), S_{yx(2)} = \frac{1}{N_2-1}\sum_{i=1}^{N_2}(Y_{2i} - \bar{Y}_2)(X_{2i} - \bar{X}_2), R = \frac{\bar{Y}}{\bar{X}}.$

In case, when $\bar{X}$ is unknown, but $\bar{Z}$, the population mean of additional auxiliary variable z (closely related to x) is known, which may be cheaper and less correlated to the study variable (y) in comparison to the main auxiliary variable (x). In such a situation, a first-phase sample of size $n'(> n)$ is selected from the population of size N using SRSWOR and we estimate the population mean $\bar{X}$ by $\bar{x}' = \frac{1}{n'}\sum_{i=1}^{n'} x_i$ using the sample means $\bar{z}' = \frac{1}{n'}\sum_{i=1}^{n'} z_i$ based on n' units and the known additional population mean $\bar{Z}$. We see that $\hat{\bar{X}} = \frac{\bar{x}'}{\bar{z}'}\bar{Z}$ is more precise than first phase sample mean $\bar{x}'$ if $\rho_{xz} > \frac{1}{2}\frac{C_z}{C_x}$.

Now, we propose conventional and alternative two phase sampling ratio-product type estimators (T_1, T_2) for $\bar{Y}$ using available information on two auxiliary variables x and z in the presence of non-response, which are given as follows:

$$T_1 = \bar{y}^*\left[K_1\left(\frac{\bar{x}'}{\bar{x}^*}\right)\left(\frac{\bar{Z}}{\bar{z}'}\right) + K_2\left(\frac{\bar{x}^*}{\bar{x}'}\right)\left(\frac{\bar{z}'}{\bar{Z}}\right)\right] \quad (14)$$

and

$$T_2 = \bar{y}^*\left[\alpha_1\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\bar{Z}}{\bar{z}'}\right) + \alpha_2\left(\frac{\bar{x}}{\bar{x}'}\right)\left(\frac{\bar{z}'}{\bar{Z}}\right)\right] \quad (15)$$

where, K_1, K_2, α_1 and α_2 are constants such that $K_1 + K_2 = 1$ and $\alpha_1 + \alpha_2 = 1$.

The expressions for biases and mean square errors (MSEs) of the estimators

The expressions for biases and MSEs of the proposed estimators $\boldsymbol{T_i}, (\boldsymbol{i = 1, 2})$ up to the terms of order $\boldsymbol{n^{-1}}$ and $\boldsymbol{n'^{-1}}$are given as follows:

Let $\bar{y}^* = \bar{Y}(1+\epsilon_0^*), \bar{x}^* = \bar{X}(1+\epsilon_1^*), \bar{x}' = \bar{X}(1+\epsilon_1'), \bar{x} = \bar{X}(1+\varepsilon_1)$ and $\bar{z}' = \bar{Z}(1+\epsilon_2')$ such that $E(\epsilon_0^*) = E(\epsilon_1^*) = E(\epsilon_1') = E(\epsilon_2') = E(\epsilon_1) = 0, |\epsilon_0^*|, |\epsilon_1^*|, |\epsilon_1'|, |\epsilon_2'|, |\epsilon_1| < 1$.

By using simple random sampling without replacement method of sampling, we have,

$E({\epsilon_0^*}^2) = \frac{1}{\bar{Y}^2}V(\bar{y}^*) = \frac{1}{\bar{Y}^2}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_y^2 + \frac{W_2(K-1)}{n}S_{y(2)}^2\right\}$,

$E({\epsilon_1^*}^2) = \frac{1}{\bar{X}^2}V(\bar{x}^*) = \frac{1}{\bar{X}^2}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_x^2 + \frac{W_2(K-1)}{n}S_{x(2)}^2\right\}$, $E\left({\epsilon_1'}^2\right) = \frac{1}{\bar{X}^2}V(\bar{x}') = \frac{1}{\bar{X}^2}\left(\frac{1}{n'}-\frac{1}{N}\right)S_x^2$,

$E\left({\epsilon_2'}^2\right) = \frac{1}{\bar{Z}^2}V(\bar{z}') = \frac{1}{\bar{Z}^2}\left(\frac{1}{n'}-\frac{1}{N}\right)S_z^2$, $E(\epsilon_1^2) = \frac{1}{\bar{X}^2}V(\bar{x}) = \frac{1}{\bar{X}^2}\left(\frac{1}{n'}-\frac{1}{N}\right)S_x^2$,

$E(\epsilon_0^*, \epsilon_1^*) = \frac{1}{\bar{Y}\bar{X}}Cov(\bar{y}^*, \bar{x}^*) = \frac{1}{\bar{Y}\bar{X}}\left\{\left(\frac{1}{n}-\frac{1}{N}\right)S_{yx} + \frac{W_2(K-1)}{n}S_{yx(2)}\right\}$,

$E(\epsilon_0^*, \epsilon_1') = \frac{1}{\bar{Y}\bar{X}}Cov(\bar{y}^*, \bar{x}') = \frac{1}{\bar{Y}\bar{X}}\left(\frac{1}{n'}-\frac{1}{N}\right)S_{yx}$, $E(\epsilon_0^*, \epsilon_2') = \frac{1}{\bar{Y}\bar{Z}}Cov(\bar{y}^*, \bar{z}') = \frac{1}{\bar{Y}\bar{Z}}\left(\frac{1}{n'}-\frac{1}{N}\right)S_{yz}$,

$E(\epsilon_1^*, \epsilon_2') = \frac{1}{\bar{X}\bar{Z}}Cov(\bar{x}^*, \bar{z}') = \frac{1}{\bar{X}\bar{Z}}\left(\frac{1}{n'}-\frac{1}{N}\right)S_{xz}$,

$E(\epsilon_1^*, \epsilon_1') = \frac{1}{\bar{X}^2}Cov(\bar{x}^*, \bar{x}') = \frac{1}{\bar{X}^2}V(\bar{x}') = \frac{1}{\bar{X}^2}\left(\frac{1}{n'}-\frac{1}{N}\right)S_x^2$

and

$E(\epsilon_1', \epsilon_2') = \frac{1}{\bar{X}\bar{Z}}Cov(\bar{x}', \bar{z}') = \frac{1}{\bar{X}\bar{Z}}\left(\frac{1}{n'}-\frac{1}{N}\right)S_{xz}$.

We have,

$$B(T_1) = \frac{1}{\bar{X}}\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\left\{RK_1S_x^2 + (1-2K_1)S_{yx}\right\} + \frac{W_2(K-1)}{n}\left\{RK_1S_{x(2)}^2 + (1-2K_1)S_{yx(2)}\right\}\right] + \frac{1}{\bar{Z}}\left(\frac{1}{n'}-\frac{1}{N}\right)\left\{R_1K_1S_z^2 + (1-2K_1)S_{yz}\right\}, \quad (16)$$

$$MSE(T_1) = \left(\frac{1}{n}-\frac{1}{n'}\right)\left\{S_y^2 + A^2R^2S_x^2 + 2ARS_{yx}\right\} + \frac{W_2(K-1)}{n}\left\{S_{y(2)}^2 + A^2R^2S_{x(2)}^2 + 2ARS_{yx(2)}\right\} + \left(\frac{1}{n'}-\frac{1}{N}\right)\left\{S_y^2 + A^2R_1^2S_z^2 + 2AR_1S_{yz}\right\}, \quad (17)$$

$$B(T_2) = \frac{1}{\bar{X}}\left(\frac{1}{n}-\frac{1}{n'}\right)\left\{R\alpha_1S_x^2 + (1-2\alpha_1)S_{yx}\right\} + \frac{1}{\bar{Z}}\left(\frac{1}{n'}-\frac{1}{N}\right)\left\{R_1\alpha_1S_z^2 + (1-2\alpha_1)S_{yz}\right\} \quad (18)$$

and

$$MSE(T_2) = \left(\frac{1}{n}-\frac{1}{n'}\right)\left\{S_y^2 + B^2R^2S_x^2 + 2BRS_{yx}\right\} + \frac{W_2(K-1)}{n}S_{y(2)}^2 + \left(\frac{1}{n'}-\frac{1}{N}\right)\left\{S_y^2 + B^2R_1^2S_z^2 + 2BR_1S_{yz}\right\}, \quad (19)$$

where, $A = (1-2K_1)$, $B = (1-2\alpha_1)$, $S_z^2 = \frac{1}{N-1}\sum_{i=1}^N(Z_i-\bar{Z})^2$, $S_x^2 = \frac{1}{N-1}\sum_{i=1}^N(X_i-\bar{X})^2$,

$R_1 = \frac{\bar{Y}}{\bar{Z}}$, $S_{yz} = \frac{1}{N-1}\sum_{i=1}^N(Y_i-\bar{Y})(Z_i-\bar{Z})$.

Many estimators turn out as special cases of $T_i, i = 1, 2$, which are given as follows:

(i) For $K_1 = 1$ and $\alpha_1 = 1$, the proposed estimators T_1 and T_2 reduce to conventional and alternative chain ratio type estimators $T_{c1} = \bar{y}^*\left(\frac{\bar{x}'}{\bar{x}^*}\right)\left(\frac{\bar{Z}}{\bar{z}'}\right)$ and $T_{a1} = \bar{y}^*\left(\frac{\bar{x}'}{\bar{x}}\right)\left(\frac{\bar{Z}}{\bar{z}'}\right)$ respectively.

$$B(T_{c1}) = \frac{1}{\bar{X}}\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\left\{RS_x^2 - S_{yx}\right\} + \frac{W_2(K-1)}{n}\left\{RS_{x(2)}^2 - S_{yx(2)}\right\}\right] + \frac{1}{\bar{Z}}\left(\frac{1}{n'}-\frac{1}{N}\right)\left\{R_1S_z^2 - S_{yz}\right\}, \quad (20)$$

$$MSE(T_{c1}) = \left(\frac{1}{n}-\frac{1}{n'}\right)\left\{S_y^2 + R^2S_x^2 - 2RS_{yx}\right\} + \frac{W_2(K-1)}{n}\left\{S_{y(2)}^2 + R^2S_{x(2)}^2 - 2RS_{yx(2)}\right\} + \left(\frac{1}{n'}-\frac{1}{N}\right)\left\{S_y^2 + R_1^2S_z^2 - 2R_1S_{yz}\right\}, \quad (21)$$

$$B(T_{a1}) = \frac{1}{\bar{X}}\left(\frac{1}{n}-\frac{1}{n'}\right)\left\{RS_x^2 - S_{yx}\right\} + \frac{1}{\bar{Z}}\left(\frac{1}{n'}-\frac{1}{N}\right)\left\{R_1S_z^2 - S_{yz}\right\}, \quad (22)$$

$$MSE(T_{a1}) = \left(\frac{1}{n}-\frac{1}{n'}\right)\left\{S_y^2 + R^2S_x^2 - 2RS_{yx}\right\} + \frac{W_2(K-1)}{n}S_{y(2)}^2 + \left(\frac{1}{n'}-\frac{1}{N}\right)\left\{S_y^2 + R_1^2S_z^2 - 2R_1S_{yz}\right\} \quad (23)$$

(ii) For $K_1 = 0$ and $\alpha_1 = 0$, the proposed estimators T_1 and T_2 reduce to conventional and alternative chain product type estimators $T_{c2} = \bar{y}^*\left(\frac{\bar{x}^*}{\bar{x}'}\right)\left(\frac{\bar{z}'}{\bar{Z}}\right)$ and $T_{a2} = \bar{y}^*\left(\frac{\bar{x}}{\bar{x}'}\right)\left(\frac{\bar{z}'}{\bar{Z}}\right)$ respectively.

$$B(T_{c2}) = \frac{1}{\bar{X}}\left[\left(\frac{1}{n}-\frac{1}{n'}\right)S_{yx} + \frac{W_2(K-1)}{n}S_{yx(2)}\right] + \frac{1}{\bar{Z}}\left(\frac{1}{n'}-\frac{1}{N}\right)S_{yz}, \quad (24)$$

$$MSE(T_{c2}) = \left(\frac{1}{n}-\frac{1}{n'}\right)\left\{S_y^2 + R^2S_x^2 + 2RS_{yx}\right\} + \frac{W_2(K-1)}{n}\left\{S_{y(2)}^2 + R^2S_{x(2)}^2 + 2RS_{yx(2)}\right\} + \left(\frac{1}{n'}-\frac{1}{N}\right)\left\{S_y^2 + R_1^2S_z^2 + 2R_1S_{yz}\right\}, \quad (25)$$

$$B(T_{a2}) = \frac{1}{\bar{X}}\left(\frac{1}{n}-\frac{1}{n'}\right)S_{yx} + \frac{1}{\bar{Z}}\left(\frac{1}{n'}-\frac{1}{N}\right)S_{yz} \quad (26)$$

and

$$MSE(T_{a2}) = \left(\frac{1}{n}-\frac{1}{n'}\right)\left\{S_y^2 + R^2S_x^2 + 2RS_{yx}\right\} + \frac{W_2(K-1)}{n}S_{y(2)}^2 + \left(\frac{1}{n'}-\frac{1}{N}\right)\left\{S_y^2 + R_1^2S_z^2 + 2R_1S_{yz}\right\} \quad (27)$$

(iii) For $n' = N$, $K_1 = 1$ and $\alpha_1 = 1$, the proposed estimators T_1 and T_2 reduce to conventional and alternative chain ratio type estimators t_{c1} and t_{a1} respectively.

(iv) For $n' = N, K_1 = 0$ and $\alpha_1 = 0$, the proposed estimators T_1 and T_2 reduce to conventional and alternative chain ratio type estimators t_{c2} and t_{a2} respectively.

Differentiating (17) and (19) w.r.t. K_1 and α_1 respectively, we get the optimum values for K_1 and α_1, which are given as follows:

$K_{1(opt)} = (c+d)/2d = K_1'$ and $\alpha_{1(opt)} = (c_1+d_1)/2d_1 = \alpha_1'$,

where, $c = f_1RS_{yx} + fR_1S_{yz} + \frac{W_2(K-1)}{n}RS_{yx(2)}, d = f_1R^2S_x^2 + fR_1^2S_z^2 + \frac{W_2(K-1)}{n}R^2S_{x(2)}^2$,

$c_1 = f_1RS_{yx} + fR_1S_{yz}$ and $d_1 = f_1R^2S_x^2 + fR_1^2S_z^2$.

After putting these optimum values $K_{1(opt)}$ and $\alpha_{1(opt)}$ in the equations (17) and (19) respectively, we have

$$MSE(T_1)_{min.} = \left(\frac{1}{n}-\frac{1}{n'}\right)\left\{S_y^2 + A_1^2R^2S_x^2 + 2A_1RS_{yx}\right\} + \frac{W_2(K-1)}{n}\{S_{y(2)}^2 + A_1^2R^2S_{x(2)}^2 + A_1RS_{yx(2)}\} + \left(\frac{1}{n'}-\frac{1}{N}\right)\left\{S_y^2 + A_1^2R_1^2S_z^2 + 2A_1R_1S_{yz}\right\}, \quad (28)$$

and

$$MSE(T_2)_{min.} = \left(\frac{1}{n}-\frac{1}{n'}\right)\left\{S_y^2 + B_1^2R^2S_x^2 + 2B_1RS_{yx}\right\} + \frac{W_2(K-1)}{n}S_{y(2)}^2 + \left(\frac{1}{n'}-\frac{1}{N}\right)\left\{S_y^2 + \ B_1^2R_1^2S_z^2 + 2B_1R_1S_{yz}\right\}, \tag{29}$$

where, $A_1 = (1-2K_1')$, $B_1 = (1-2\alpha_1')$.

$$B(T_1) = 0 \text{ if } K_1 = \frac{\frac{1}{\bar{X}}\left\{\left(\frac{1}{n}-\frac{1}{n'}\right)S_{yx} + \frac{W_2(K-1)}{n}S_{yx(2)}\right\} + \frac{1}{\bar{Z}}\left(\frac{1}{n'}-\frac{1}{N}\right)S_{yz}}{\frac{1}{\bar{X}}\left\{\left(\frac{1}{n}-\frac{1}{n'}\right)(RS_x^2 - 2S_{yx}) + \frac{W_2(K-1)}{n}\left(RS_{x(2)}^2 - 2S_{yx(2)}\right)\right\} + \frac{1}{\bar{Z}}\left(\frac{1}{n'}-\frac{1}{N}\right)(R_1S_z^2 - 2S_{yz})} \tag{30}$$

and

$$B(T_2) = 0 \text{ if } \alpha_1 = \frac{\frac{1}{\bar{X}}\left(\frac{1}{n}-\frac{1}{n'}\right)S_{yx} + \frac{1}{\bar{Z}}\left(\frac{1}{n'}-\frac{1}{N}\right)S_{yz}}{\frac{1}{\bar{X}}\left(\frac{1}{n}-\frac{1}{n'}\right)(RS_x^2 - 2S_{yx}) + \frac{1}{\bar{Z}}\left(\frac{1}{n'}-\frac{1}{N}\right)(R_1S_z^2 - 2S_{yz})} \ . \tag{31}$$

If $\frac{\beta_{yx}}{R} \cong \frac{\beta_{yz}}{R_1} \cong \frac{\beta_{yx(2)}}{R}$, then in this case,

$MSE(T_1)_{min.}^* = \left(\frac{1}{n}-\frac{1}{n'}\right)\left\{S_y^2 + A_{11}^2R^2S_x^2 + 2A_{11}RS_{yx}\right\} + \frac{W_2(K-1)}{n}\left\{S_{y(2)}^2 + A_{11}^2R^2S_{x(2)}^2 + \ 2A_{11}RS_{yx(2)}\right\} + \left(\frac{1}{n'}-\frac{1}{N}\right)\left\{S_y^2 + A_{11}^2R_1^2S_z^2 + 2A_{11}R_1S_{yz}\right\}$,

and

$MSE(T_2)_{min.}^* = \left(\frac{1}{n}-\frac{1}{n'}\right)\left\{S_y^2 + B_{11}^2R^2S_x^2 + 2B_{11}RS_{yx}\right\} + \frac{W_2(K-1)}{n}S_{y(2)}^2 + \left(\frac{1}{n'}-\frac{1}{N}\right)\left\{S_y^2 + \ B_{11}^2R_1^2S_z^2 + 2B_{11}R_1S_{yz}\right\}$,

where, $A_{11} = \ -\frac{S_{yx}}{2RS_x^2}$, $B_{11} = \ -\frac{S_{yx}}{2RS_x^2}$.

The estimators of $\widehat{B}(T_1), \widehat{B}(T_2), M\widehat{S}E(T_1)$ and $M\widehat{S}E(T_2)$

Population parameters are unknown in the population. So, we take a sample of size $\boldsymbol{n}$ and the parameters are estimated on the basis of the sample and the estimated $\hat{B}(T_1), \hat{B}(T_2), M\hat{S}E(T_1)$ and $M\hat{S}E(T_2)$ are given as follows:

$\hat{B}(T_1) = \ \frac{1}{\bar{x}}\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\left\{\hat{R}\widehat{K}_1s_x^2 + \left(1-2\widehat{K}_1\right)s_{yx}\right\} + \frac{W_2(K-1)}{n}\left\{\hat{R}\widehat{K}_1s_{x(2)}^2 + \left(1-2\widehat{K}_1\right)s_{yx(2)}\right\}\right] + \ \frac{1}{\bar{z}}\left(\frac{1}{n'}-\frac{1}{N}\right)\left\{\hat{R}_1\widehat{K}_1s_z^2 + \left(1-2\widehat{K}_1\right)s_{yz}\right\}$,

$M\hat{S}E(T_1)_{min.} = \left(\frac{1}{n}-\frac{1}{n'}\right)\left\{s_y^2 + \hat{A}_1^2\hat{R}^2s_x^2 + 2\hat{A}_1\hat{R}s_{yx}\right\} + \frac{W_2(K-1)}{n}\left\{s_{y(2)}^2 + \hat{A}_1^2\hat{R}^2s_{x(2)}^2 + \ 2\hat{A}_1\hat{R}s_{yx(2)}\right\} + \left(\frac{1}{n'}-\frac{1}{N}\right)\left\{s_y^2 + \hat{A}_1^2\hat{R}_1^2s_z^2 + 2\hat{A}_1\hat{R}_1s_{yz}\right\}$,

$\hat{B}(T_2) = \ \frac{1}{\bar{x}}\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\left\{\hat{R}\hat{\alpha}_1s_x^2 + (1-2\hat{\alpha}_1)s_{yx}\right\} +\right] + \frac{1}{\bar{z}}\left(\frac{1}{n'}-\frac{1}{N}\right)\left\{\hat{R}_1\hat{\alpha}_1s_z^2 + (1-2\hat{\alpha}_1)s_{yz}\right\}$

and

$M\hat{S}E(T_2)_{min} = \left(\frac{1}{n}-\frac{1}{n'}\right)\left\{s_y^2 + \hat{B}_1^2\hat{R}^2s_x^2 + 2\hat{B}_1\hat{R}s_{yx}\right\} + \frac{W_2(K-1)}{n}s_{y(2)}^2 + \left(\frac{1}{n'}-\frac{1}{N}\right)\left\{s_y^2 + \hat{B}_1^2\hat{R}_1^2s_z^2 + \qquad \hat{B}_1\hat{R}_1s_{yz}\right\}$,

where, $\hat{A}_1 = (1-2\widehat{K}_1')$, $\hat{B}_1 = (1-2\hat{\alpha}_1')$, $\widehat{K}_1' = \frac{\hat{c}+\hat{d}}{2\hat{d}}$, $\bar{\alpha}_1' = \frac{\hat{c}_1+\hat{d}_1}{2\hat{d}_1}$, $\hat{R} = \frac{\bar{y}}{\bar{x}}$, $\hat{R}_1 = \frac{\bar{y}}{\bar{z}}$,

$\hat{c} = f_1\hat{R}s_{yx} + f\hat{R}_1s_{yz} + \frac{W_2(K-1)}{n}\hat{R}s_{yx(2)}$, $\hat{d} = f_1\hat{R}^2s_x^2 + f\hat{R}_1^2s_z^2 + \frac{W_2(K-1)}{n}\hat{R}^2s_{x(2)}^2$,

$\hat{c}_1 = f_1\hat{R}s_{yx} + f\hat{R}_1s_{yz}$, $\hat{d}_1 = f_1\hat{R}^2s_x^2 + f\hat{R}_1^2s_z^2$, $s_y^2 = \frac{1}{n-1}\sum_{i=1}^n(y_i-\bar{y})^2$,

$s_z^2 = \frac{1}{n-1}\sum_{i=1}^n(z_i-\bar{z})^2$, $s_x^2 = \frac{1}{n-1}\sum_{i=1}^n(x_i-x)^2$, $s_{yx} = \frac{1}{n-1}\sum_{i=1}^n(y_i-\bar{y})(x_i-\bar{z})$,

$s_{yz} = \frac{1}{n-1}\sum_{i=1}^n(y_i-\bar{y})(z_i-\bar{z})$, $s_{y(2)}^2 = \frac{1}{r-1}\sum_{i=1}^r(y_{i2}'-\bar{y}_2')^2$, $s_{x(2)}^2 = \frac{1}{r-1}(x_{i2}'-\bar{x}_2')^2$,

$s_{yx(2)} = \frac{1}{r-1}\sum_{i=1}^r(y_{i2}'-\bar{y}_2')(x_{i2}'-\bar{x}_2')$ and (y_i, x_i, z_i) are the values of the i^{th} unit of the sample of size n for variables y, x and z respectively. (y_{i2}', x_{i2}') are the values of the i^{th} unit in the sub-sample of size r drawn from n_2 non-responding units for the variables y and x respectively by using SRSWOR sampling scheme.

The estimators $\hat{B}(T_1), \hat{B}(T_2), M\hat{S}E(T_1)$ and $M\hat{S}E(T_2)$ are almost unbiased estimators of $B(T_1), B(T_2), MSE(T_1)$ and $MSE(T_2)$ respectively up to terms of order (n^{-1}) and the $MSE(.)$ will be of order (n^{-2}) and will be dependent upon the values of higher order terms (> 2) involved in it.

Comparison of the proposed estimators with the relevant estimators

On comparing the $MSE(T_i)$, with $MSE(\bar{y}^*)$, $MSE(t_{ci})$, $MSE(t_{ai})$, $MSE(T_{ci})$ and $MSE(T_{ai})$, $i = 1, 2$, we see that,

$$MSE(T_1) - MSE(y^*) \le 0 \text{ if } \frac{b_1-\sqrt{b_1^2-4a_1c_1}}{2a_1} \le K_1 \le \frac{b_1+\sqrt{b_1^2-4a_1c_1}}{2a_1}, \tag{32}$$

$$MSE(T_1) - MSE(T_{c1}) \le 0 \ \ If \ \frac{b_2-\sqrt{b_2^2-4a_2c_2}}{2a_2} \le K_1 \le \frac{b_2+\sqrt{b_2^2-4a_2c_2}}{2a_2}, \tag{33}$$

$$MSE(T_1) - MSE(T_{c2}) \le 0 \ \ If \ \ 0 \le K_1 \le \frac{b_3}{a_3}, \tag{34}$$

$$MSE(T_1) - MSE(t_{c1}) \le 0 \ \ If \ \frac{b_4-\sqrt{b_4^2-4a_4c_4}}{2a_4} \le K_1 \le \frac{b_4+\sqrt{b_4^2-4a_4c_4}}{2a_4}, \tag{35}$$

$$MSE(T_1) - MSE(t_{c2}) \le 0 \ \ If \ \frac{b_5-\sqrt{b_5^2-4a_5c_5}}{2a_5} \le K_1 \le \frac{b_5+\sqrt{b_5^2-4a_5c_5}}{2a_5}, \tag{36}$$

$$MSE(T_2) - MSE(y^*) \le 0 \ \ If \ \frac{b_6-\sqrt{b_6^2-4a_6c_6}}{2a_6} \le \alpha_1 \le \frac{b_6+\sqrt{b_6^2-4a_6c_6}}{2a_6}, \tag{37}$$

$$MSE(T_2) - MSE(t_{a1}) \le 0 \ \ If \ \frac{b_7-\sqrt{b_7^2-4a_7c_7}}{2a_7} \le \alpha_1 \le \frac{b_7+\sqrt{b_7^2-4a_7c_7}}{2a_7}, \tag{38}$$

$$MSE(T_2)-MSE(t_{a2})\le 0 \quad If \quad \frac{b_8-\sqrt{b_8^2-4a_8c_8}}{2a_8}\le \alpha_1 \le \frac{b_8+\sqrt{b_8^2-4a_8c_8}}{2a_8}, \tag{39}$$

$$MSE(T_2)-MSE(T_{a1})\le 0 \quad If \quad \frac{b_9-\sqrt{b_9^2-4a_9c_9}}{2a_9}\le \alpha_1 \le \frac{b_9+\sqrt{b_9^2-4a_9c_9}}{2a_9} \tag{40}$$

and

$$MSE(T_2)-MSE(T_{a2})\le 0 \quad If \quad 0\le \alpha_1 \le \frac{b_{10}}{a_{10}}, \tag{41}$$

where, $a_1=4\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2\}+\left(\frac{W_2(K-1)}{n}\right)\{R^2S_{x(2)}^2\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2\}\right]$,

$b_1=4\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2+RS_{yx}\}+\left(\frac{W_2(K-1)}{n}\right)\{R^2S_{x(2)}^2+RS_{yx(2)}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+R_1S_{yz}\}\right]$,

$c_1=\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2+2RS_{yx}\}+\left(\frac{W_2(K-1)}{n}\right)\{R^2S_{x(2)}^2+RS_{yx(2)}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+R_1S_{yz}\}$,

$a_2=\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2\}+\left(\frac{W_2(K-1)}{n}\right)\{R^2S_{x(2)}^2\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2\}\right]$,

$b_2=\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2+RS_{yx}\}+\left(\frac{W_2(K-1)}{n}\right)\{R^2S_{x(2)}^2+RS_{yx(2)}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+R_1S_{yz}\}\right]$,

$c_2=\left(\frac{1}{n}-\frac{1}{n'}\right)\{S_{yx}\}+\left(\frac{W_2(K-1)}{n}\right)\{RS_{yx(2)}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1S_{yz}\}$,

$a_3=\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2\}+\left(\frac{W_2(K-1)}{n}\right)\{R^2S_{x(2)}^2\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2\}\right]$,

$b_3=\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2+RS_{yx}\}+\left(\frac{W_2(K-1)}{n}\right)\{R^2S_{x(2)}^2+RS_{yx(2)}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+R_1S_{yz}\}\right]$,

$a_4=4\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2\}+\left(\frac{W_2(K-1)}{n}\right)\{R^2S_{x(2)}^2\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2\}\right]$,

$b_4=4\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2+RS_{yx}\}+\left(\frac{W_2(K-1)}{n}\right)\{R^2S_{x(2)}^2+RS_{yx(2)}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+R_1S_{yz}\}\right]$,

$c_4=4\left(\frac{1}{n}-\frac{1}{n'}\right)\{RS_{yx}\}+4\left(\frac{W_2(K-1)}{n}\right)\{RS_{yx(2)}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+2R_1S_{yz}\}$,

$a_5=4\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2\}+\left(\frac{W_2(K-1)}{n}\right)\{R^2S_{x(2)}^2\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2\}\right]$,

$b_5=4\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2+RS_{yx}\}+\left(\frac{W_2(K-1)}{n}\right)\{R^2S_{x(2)}^2+RS_{yx(2)}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+R_1S_{yz}\}\right]$,

$c_5=\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+2R_1S_{yz}\}$, $a_6=4\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2\}\right]$,

$b_6=4\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2+RS_{yx}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+R_1S_{yz}\}\right]$,

$c_6=\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2+2RS_{yx}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+R_1S_{yz}\}$,

$a_7=4\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2\}\right]$,

$b_7=4\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2+RS_{yx}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+R_1S_{yz}\}\right]$,

$c_7=\left(\frac{1}{n}-\frac{1}{n'}\right)\{4RS_{yx}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+2R_1S_{yz}\}$,

$a_8=4\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2\}\right]$,

$b_8=4\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2+RS_{yx}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+R_1S_{yz}\}\right]$, $c_8=\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+2R_1S_{yz}\}$,

$a_9=\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2\}\right]$, $b_9=\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2+RS_{yx}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+R_1S_{yz}\}\right]$,

$c_9=\left(\frac{1}{n}-\frac{1}{n'}\right)\{RS_{yx}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1S_{yz}\}$, $a_{10}=\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2\}\right]$,

and

$b_{10}=\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\{R^2S_x^2+RS_{yx}\}+\left(\frac{1}{n'}-\frac{1}{N}\right)\{R_1^2S_z^2+R_1S_{yz}\}\right]$.

Confidence limits

For large samples, the estimates of the population mean may be assumed to follow approximately normal distribution. The 95% confidence limits for the population mean is written as: $\hat{\hat{Y}}\pm 1.96\sqrt{variance(\hat{\hat{Y}})}$,

where, $\hat{\hat{Y}}$ is the estimated population mean and 1.96 is the value of the normal variate for a 95% level of confidence coefficient.

Determination of n', n and K for fixed cost $C\le C_0$

Let C_0 be the total coat (fixed) of the survey apart from overhead cost. The cost function C_0 can be written as

$$C_0=(C_1'+C_2')n'+C_1n+C_2n_1+C_3\frac{n_2}{K}. \tag{42}$$

Since C_0 will vary from sample to survey, so the expected cost can be written as

$$C=E(C_0)=(C_1'+C_2')n'+n(C_1+C_2W_1+C_3\frac{W_2}{K}), \tag{43}$$

where,

C_1' = The cost per unit of identifying and observing main auxiliary character x at the first-phase.

C_2' = The cost per unit of identifying and observing additional auxiliary character z at the first-phase.

C_1 = The cost per unit of mailing questionnaire/visiting the unit at the second phase.

C_2 = The cost per unit of collecting and processing data from n_1 responding units.

C_3 = The cost per unit of obtaining and processing data after extra effort from the sub-sampled units.
$W_i = \frac{N_i}{N} (i = 1,2)$denote the response and non-response rate in the population respectively.
It is to be noted that $C_2' < C_1' < C_1 < C_2 < C_3$. (44)
The expression for $MSE\{T(i)\}, i = 1, 2, 3,4, 5,6, 7, 8, 9, 10$ can be written as follows:

$$MSE\{T(i)\} = \frac{M_{0i}}{n} + \frac{M_{1i}}{n'} + \frac{K}{n} M_{2i} - \frac{1}{N} M_{3i}, \quad (45)$$

where, $T(1) = T_1, T(2) = T_2, T(3) = T_{c1}, T(4) = T_{a1}, T(5) = T_{c2}, T(6) = T_{a2}, T(7) = t_{c1},\ T(8) = t_{a1}, T(9) = t_{c2}, T(10) = t_{a2}$ and M_{0i}, M_{1i}, M_{2i} and M_{3i} are the coefficients of the terms of $\frac{1}{n}, \frac{1}{n'}, \frac{K}{n}$ and $\frac{1}{N}$ respectively in the expression of $MSE\{T(i)\}$.
To find the optimum values of n', n, K and minimum values of $MSE\{T(i)\}$ in case of the fixed cost $C \le C_0$, let us define a function φ which is given by

$$\varphi = MSE\{T(i)\} + \delta_i \left\{(C_1'+C_2')n' + n\left(C_1 + C_2W_1 + C_3\frac{W_2}{K}\right) - C_0\right\}, i = 1, 2, 3, 4, 5, 6, 7, 8, 9,10 \quad (46)$$

where, δ_i is the Lagrange's multiplier.
Now, differentiating φ with respect to n', n and K and equating them to zero, we get,

$$n' = \sqrt{\frac{M_{1i}}{\delta_i(C_1'+C_2')}} \quad (47)$$

$$n = \sqrt{\frac{M_{0i}+KM_{2i}}{\delta_i\left(C_1+C_2W_1+C_3\frac{W_2}{K}\right)}} \quad (48)$$

and

$$\frac{n}{K} = \sqrt{\frac{M_{2i}}{\delta_i C_3 W_2}} \quad (49)$$

Now, putting the value of n in (15), we get,

$$K_{opt.} = \sqrt{\frac{C_3W_2M_{0i}}{(C_1+C_2W_1)M_{2i}}} \quad (50)$$

Putting the values of n', n and $K_{opt.}$ from (47), (48) and (49) in (43), we get

$$\sqrt{\delta_i} = \frac{1}{C_0}\left[\sqrt{(C_1' + C_2')M_{1i}} + \sqrt{\left[C_1 + C_2W_1 + \frac{C_3W_2}{K_{opt.}}\right]\left[M_{0i} + K_{opt.}M_{2i}\right]}\right] \quad (51)$$

It has also been seen that the determinant of the matrix of second order derivative of φ with respect to n', n and $K_{opt.}$ is negative for the optimum values of n', n and $K_{opt.}$, which shows that the solution for n', n given by (47) and (48) and the optimum value of K for $C \le C_0$ minimizes the $MSE\{T(i)\}$. The minimum value of $MSE\{T(i)\}$ for the optimum value of n', n and $K_{opt.}$are given as follows:
Neglecting the term of order N^{-1}, we have

$$MSE\{T(i)\}_{min.} = \frac{1}{C_0}\left[\sqrt{(C_1' + C_2')M_{1i}} + \sqrt{\left(C_1 + C_2W_1 + \frac{C_3W_2}{K_{opt.}}\right)\left(M_{0i} + K_{opt.}M_{2i}\right)}\right]^2 - \frac{1}{N}M_{3i} \quad (52)$$

Neglecting the term of order N^{-1}, we have

$$MSE\{T(i)\}_{min.} = \frac{1}{C_0}\left[\sqrt{(C_1' + C_2')M_{1i}} + \sqrt{\left(C_1 + C_2W_1 + \frac{C_3W_2}{K_{opt.}}\right)\left(M_{0i} + K_{opt.}M_{2i}\right)}\right]^2 \quad (53)$$

In the case of $\bar{y}^*$, neglecting the term of order N^{-1}, we have

$$MSE(\bar{y}^*)_{min.} = \frac{1}{C_0}\left[\sqrt{\left(C_1 + C_2W_1 + \frac{C_3W_2}{K_{opt.}}\right)\left(M_0 + K_{opt.}M_2\right)}\right]^2 \quad (54)$$

where, M_0 and M_2 are the coefficients of the terms of $\frac{1}{n}$ and $\frac{K}{n}$ respectively in the expression of $V(\bar{y}^*) = \frac{f}{n}S_y^2 + \frac{W_2(K-1)}{n}S_{y(2)}^2$.

Determination of n', n and K for fixed variance $V = V_0$

From(50), we see that the optimum value of K is independent of the total cost or specified precision. Let V_0 be the variance of the estimator $T(i), i = 1,2,3,4,5, 6, 7, 8, 9, 10$ fixed in advanced. From(50), we see that the optimum value of K is independent of the total cost or specified precision. Let V_0 be the variance of the estimator $T(i), i = 1,2,3,4,5, 6, 7, 8, 9, 10$ fixed in advance, then we have,

$$V_0 = \frac{M_{0i}}{n} + \frac{M_{1i}}{n'} + \frac{K}{n}M_{2i} - \frac{1}{N}M_{3i}. \quad (55)$$

The total cost apart from overhead cost is minimized by obtaining the optimum values ofn', n and K for specified precision $V = V_o$. For this purpose, we defined a function φ_1 which is given as follows:

$$\varphi_1 = (C_1'+C_2')n' + n\left(C_1 + C_2W_1 + C_3\frac{W_2}{K}\right) + \mu_i[MSE\{T(i)\} - V_o], \quad (56)$$

$i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, where, μ_i is the Lagrange's multiplier.
After differentiating φ_1 with respect to n', n and K and equating them to zero, we get,

$$n' = \sqrt{\frac{\mu_i M_{1i}}{(C_1'+C_2')}}, \quad (57)$$

$$n = \sqrt{\frac{\mu_i(M_{0i}+KM_{2i})}{\left(C_1+C_2W_1+C_3\frac{W_2}{K}\right)}}, \quad (58)$$

and

$$\frac{n}{K} = \sqrt{\frac{\mu_i M_{2i}}{C_3W_2}} \quad (59)$$

Now, putting the value of n in (59), we get,

$$K_{opt.} = \sqrt{\frac{C_3W_2M_{0i}}{(C_1+C_2W_1)M_{2i}}} \tag{60}$$

Putting the values of n', n and $K_{opt.}$ from (57), (58) and (60) in (55), we get

$$\sqrt{\mu_i} = \frac{1}{V_0+\frac{1}{N}M_{3i}}\left[\sqrt{(C_1'+C_2')M_{1i}} + \sqrt{\left[C_1+C_2W_1+\frac{C_3W_2}{K_{opt.}}\right]\left[M_{0i}+K_{opt.}M_{2i}\right]}\right] \tag{61}$$

It has also been seen that the determinant of the matrix of second order derivative of $\boldsymbol{\varphi}$ with respect to $\boldsymbol{n'}, \boldsymbol{n}$ and $\boldsymbol{K_{opt.}}$ is negative for the optimum values of n', n and $K_{opt.}$, which shows that the solution for n', n given by (57) and (58) and the optimum value of K for $C \leq C_0$ minimizes the $MSE\{T(i)\}$. Now, putting the values of $\sqrt{\mu_i}$ from (61) and K_{opt} from (60) in (57) and (58), we can obtain the value of n' and n for which the estimator $MSE\{T(i)\}, i = 1,2,3,4,5,6,7,8,9,10$ attains the variance $V_0$with expected cost given by

$$C[MSE\{T(i)\}]_{\min} = \frac{\left[\sqrt{(C_1'+C_2')M_{1i}}+\sqrt{\left[C_1+C_2W_1+\frac{C_3W_2}{K_{opt.}}\right]\left[M_{0i}+K_{opt.}M_{2i}\right]}\right]^2}{\left(V_0+\frac{1}{N}M_{3i}\right)}, i = 1,2,3,4,5,6,7,8,9,10.$$

Neglecting the term of order $\frac{1}{N}$, we have

$$C[MSE\{T(i)\}]_{\min} = \frac{\left[\sqrt{(C_1'+C_2')M_{1i}}+\sqrt{\left[C_1+C_2W_1+\frac{C_3W_2}{K_{opt.}}\right]\left[M_{0i}+K_{opt.}M_{2i}\right]}\right]^2}{V_0}, i = 1,2,3,4,5,6,7,8,9,10.$$

Empirical study

The data on physical growth of upper socio-economic group of 100 school going children of Varanasi under an ICMR study, Department of Pediatrics, B.H.U., during 1983-84 has been taken under study. The last 25% (i.e. 25 children) of units have been considered as non-responding units. The study variable $(\boldsymbol{y})$, auxiliary variable $(\boldsymbol{x})$ and the additional auxiliary variable $(\boldsymbol{z})$ are taken as follows:

y- weight (in kg) of the children,

x- chest circumference (in cm) of the children,

z- skull circumference (in cm) of the children.

The values of the parameters of the y, x and z variables for the given data are taken as follows:

$\bar{X} = 55.76, \bar{Y} = 19.46, \bar{Z} = 51.09, S_x^2 = 10.63, S_y^2 = 8.92, S_z^2 = 2.42,$

$\rho_{yx} = 0.85, \rho_{yz} = 0.30, \rho_{xz} = 0.31, S_{yx} = 8.31, S_{yz} = 1.44.$

The non-response rate in the population is considered to be 25%. So, the values of the population parameters based on the non-responding parts, which are taken as the last 25% units of the population are given as follows:

$\bar{X}_2 = 54.96, \bar{Y}_2 = 18.84, S_{x(2)}^2 = 4.85, S_{y(2)}^2 = 3.95, S_{yx(2)} = 3.25.$

Table 1. Bias, percent relative efficiencies (with respect to $\bar{y}^*$) and 95% confidence intervals of different estimators $\bar{y}^*, t_{c1}, T_{c1}, T_1, t_{a1}, T_{a1}$ and T_2 for the fixed values of n', n and different values of K ($N = 100, n' = 70, n = 60$)

Estimators		$K = 2$	$K = 3$	$K = 4$
	$\bar{y}^*$	100.00 (0.07593)* (18.89469-19.97483)**	100.00 (0.09238) (18.68593-19.8774)	100.00 (0.10884) (18.82421-20.11746)
Conventional	t_{c1}	130.44(0.05821) (19.03922-19.98496)	136.51(0.0677) (18.77558-19.79534)	141.09(0.07714) (18.93173-20.02049)
	T_{c1}	138.02(0.05501) (18.93341-19.85281)	143.28(0.06448) (18.84218-19.83756)	147.19(0.07395) (18.9342-20.00016)
	T_1	157.55(0.04819) (18.92183-19.78236) $(K_{1(opt)} = 1.4918)$	166.21(0.05558) (18.93399-19.85816) $(K_{1(opt)} = 1.4836)$	172.85(0.06297) (18.97184-19.95552) $(K_{1(opt)} = 1.4787)$
Alternative	t_{a1}	116.45(0.06520) (18.94100-19.94193)	113.14(0.08166) (18.8212-19.94136)	110.93(0.09811) (18.73077-19.95864)
	T_{a1}	122.46(0.06200) (18.83488-19.81096)	117.75(0.07846) (18.88695-19.98497)	114.67(0.09492) (18.73199-19.93969)
	T_2	132.62(0.05725) (18.77847-19.63908) $(\alpha_{1(opt)} = 1.5088)$	125.34(0.07371) (19.13016-20.05453) $(\alpha_{1(opt)} = 1.5088)$	120.71(0.09017) (18.70597-19.68999) $(\alpha_{1(opt)} = 1.5088)$

*, **Figures in parenthesis give mean square errors and confidence interval (95%) of the estimators

From table 1, we observe that for the fixed values of n', n and different values of $K = 2, 3, 4$, the proposed estimators (T_1) and (T_2) has less mean square error than the corresponding estimators (t_{c1}, T_{c1}), (t_{a1}, T_{a1}) and $\bar{y}^*$. The estimator (T_1) has less mean square error than the estimator (T_2). As sub-sampling fraction K increases, mean square error of the estimators also increases.

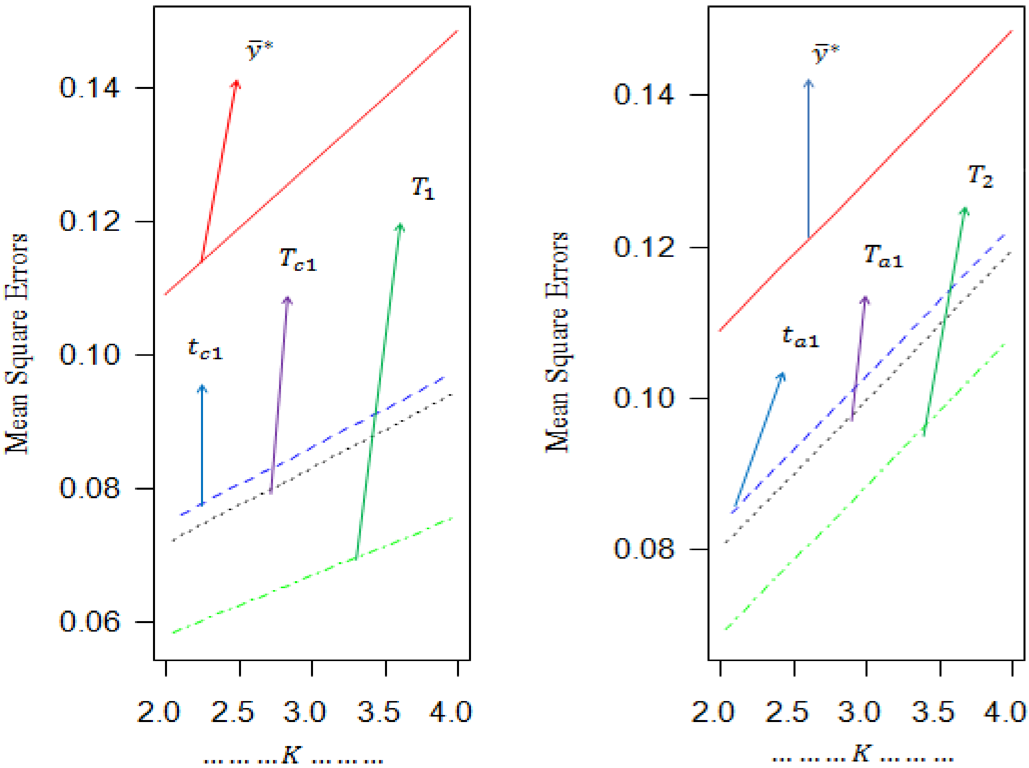

Figure 1. Mean square errors of different estimators ($\bar{y}^*, t_{c1}, t_{a1}, T_{c1}, T_{a1}, T_1, T_2$) for $N = 100, n' = 70, n = 50$ and for different values of K.

From figure1, we see that the proposed estimators T_1 have minimum mean square errors than that of the relevant estimators $\bar{y}^*, t_{c1}$and T_{c1}. Similarly, the proposed estimators T_2 has minimum mean square error than that of the relevant estimators $\bar{y}^*, t_{a1}$and T_{a1}. Also, as sub-sampling fraction K increases, mean square errors of the estimators increases.

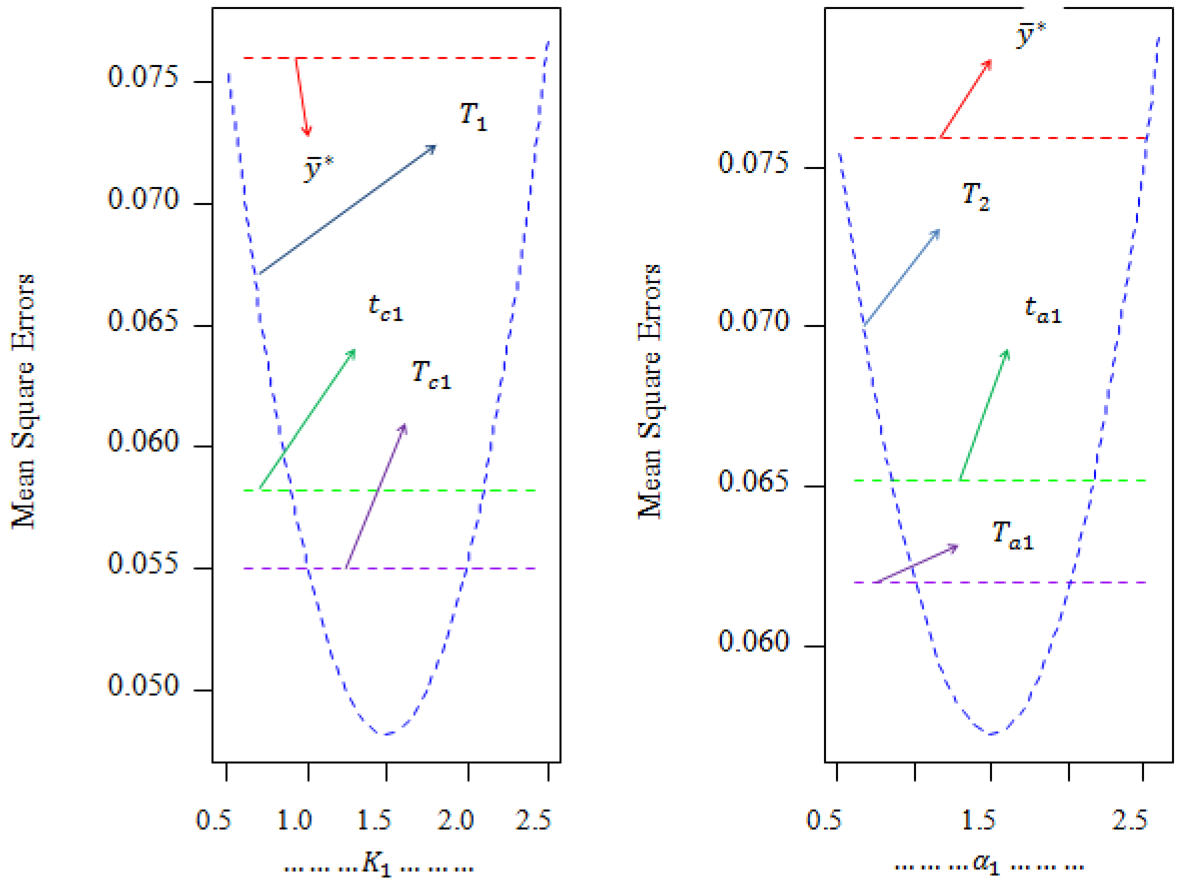

Figure 2. Mean square errors of different estimators ($\bar{y}^*, t_{c1}, T_{c1}, T_1$) for different values of K_1 and Mean square errors of different estimators ($\bar{y}^*, t_{a1}, T_{a1}, T_2$) for different values of α_1 and ($N = 100, n' = 70, n = 60, K = 2.0$)

From figure 2, we see that the proposed estimators $\boldsymbol{T_1}$ have minimum mean square errors than that of the relevant estimators $\bar{\boldsymbol{y}}^*, \boldsymbol{t_{c1}}$and $\boldsymbol{T_{c1}}$ for a wide range of $\boldsymbol{K_1}$ Similarly, the proposed estimators $\boldsymbol{T_2}$ has minimum mean square error than that of the relevant estimators $\bar{\boldsymbol{y}}^*, \boldsymbol{t_{a1}}$and $\boldsymbol{T_{a1}}$ for a wide range of $\boldsymbol{\alpha_1}$.

Monte Carlo simulation study

Case 1: In the present Monte Carlo simulation study, we consider the same data set as described in the previous section-8. From the population of 100 schools going children of Varanasi, two first-phase samples of different sizes 70 and 60 are taken by simple random sampling without replacement and the values of $\bar{x}'$and $\bar{z}'$ based on 70 and 60 units are calculated. Again, we take two sub-samples of different sizes 50 and 36 from each first-phase sample of size 70 and 60 respectively using simple random sampling without replacement scheme. In each sub-sample, the last 25% (12 and 9 children respectively) of units have been considered as non-responding units. We again take a sub-sample of r units from non-responding units with simple random sampling without replacement and collect all information on $r = \frac{n_2}{K}$ units. Here, $n_2 = 12, 09$ is the non-responding unit in each sub-sample and $K = 2, 3, 4$ respectively. The above process is replicated 1000 times.

Simulated absolute bias and simulated mean square error of $T(i), i = 1,2,3,4,7,8$ are calculated as follows:

$Absolute\ B\{T(i)\} = \frac{1}{1000} | \sum_{j=1}^{1000} T(i)_j - \bar{Y} |$

$$MSE\{T(i)\} = \frac{1}{1000} \sum_{j=1}^{1000} \{T(i)_j - \bar{Y}\}^2$$

Table 2. Simulated absolute bias and simulated percent relative efficiencies (with respect to $\bar{y}^*$) of different estimators $\bar{y}^*, t_{c1}, T_{c1}, T_1, t_{a1}, T_{a1}$ and T_2 for the fixed values of n', n and different values of K ($N = 100, n' = 70, n = 50$)

	B(.) MSE(.)	K = 2	K = 3	K = 4
	$B(\bar{y}^*)$ $MSE(\bar{y}^*)$	0.0193 100.00(0.1343)*	0.0246 100.00(0.1642)	0.0383 100.00(0.2163)
Conventional	$B(t_{c1})$ $MSE(t_{c1})$	0.0133 154.72(0.0868)	0.0200 161.85(0.1014)	0.0299 169.95(0.1243)
	$B(T_{c1})$ $MSE(T_{c1})$	0.0132 160.87(0.0835)	0.0199 166.52(0.0986)	0.0297 173.35(0.1248)
	$B(T_1)$ $MSE(T_1)$	0.0070 214.80(0.0625) $K_{1(opt)} = 1.5371$	0.0154 227.17(0.0723) $K_{1(opt)} = 1.5218$	0.0217 248.51(0.0870) $K_{1(opt)} = 1.5115$
Alternative	$B(t_{a1})$ $MSE(t_{a1})$	0.0133 124.10(0.1082)	0.0186 117.64(0.1395)	0.0322 112.78(0.0322)
	$B(T_{a1})$ $MSE(T_{a1})$	0.0131 128.07(0.1049)	0.0185 120.07(0.1367)	0.0320 114.21(0.1894)
	$B(T_2)$ $MSE(T_2)$	0.0064 143.53(0.0936) $\alpha_{1(opt)} = 1.5627$	0.0117 128.59(0.1277) $\alpha_{1(opt)} = 1.5627$	0.0253 119.55(0.1809) $\alpha_{1(opt)} = 1.5627$

*Figures in parenthesis give mean square errors of the estimators.

Table 3. Simulated absolute bias, simulated percent relative efficiencies (with respect to $\bar{y}^*$) and 95% confidence intervals of different estimators $\bar{y}^*, t_{c1}, T_{c1}, T_1, t_{a1}, T_{a1}$ and T_2 for the fixed values of n', n and different values of K ($N = 100, n' = 60, n = 36$)

	$B(.)$ $R.E.(.)(MSE(.))$ $C.I.(.)$	$K = 2$	$K = 3$	$K = 4$
	$B(\bar{y}^*)$ $MSE(\bar{y}^*)$ $C.I.(\bar{y}^*)$	0.0250 100.00 (0.1999)* (19.0962-20.9160)	0.03400 100.00 (0.2731) (19.3558-21.4498)	0.03897 100.00 (0.3627) (18.6968-21.0754)
Conventional	$B(t_{c1})$ $MSE(t_{c1})$ $C.I.(t_{c1})$	0.0170 153.66(0.1301) (19.2837-20.7420)	0.0261 164.21(0.1663) (19.4220-21.0482)	0.0329 168.88(0.2148) (19.0544-20.8470)
	$B(T_{c1})$ $MSE(T_{c1})$ $C.I.(T_{c1})$	0.0164 158.21(0.1264) (19.2564-20.6922)	0.0255 167.85(0.1627) (19.4220.21.0482)	0.0323 172.15(0.2107) (19.0244-20.8001)
	$B(T_1)$ $MSE(T_1)$ $C.I.(T_1)$	0.0079 203.87(0.0981) (19.3270-20.5515) $K_{1(opt)} = 1.5504$	0.0174 235.13(0.1162) (19.2940-20.6532) $K_{1(opt)} = 1.5355$	0.0266 246.45(0.1472) (19.2117-20.6676) $K_{1(opt)} = 1.5249$
Alternative	$B(t_{a1})$ $MSE(t_{a1})$ $C.I.(t_{a1})$	0.0191 129.37(0.1545) (19.2378-20.8197)	0.0281 119.73(0.2281) (19.4757-21.3760)	0.0329 112.55(0.3223) (18.8137-21.0036)
	$B(T_{a1})$ $MSE(T_{a1})$ $C.I.(T_{a1})$	0.0185 132.42(0.1510) (18.6506-20.2076)	0.0275 121.66(0.2245) (18.6771-20.5409)	0.0324 113.87(0.3186) (18.5074-20.6606)
	$B(T_2)$ $MSE(T_2)$	0.0118	0.0205	0.0252

	$C.I.(T_2$	149.31(0.1339) (18.7128-20.1279) $\alpha_{1(opt)} = 1.5726$	131.46(0.2078) (18.7381-20.4623) $\alpha_{1(opt)} = 1.5726$	118.19(0.3069) (18.5460-20.6044) $\alpha_{1(opt)} = 1.5726$

*Figures in parenthesis give mean square errors of the estimators

Table 4. Simulated percent relative efficiencies (with respect to $\bar{y}^*$) of different estimators $\bar{y}^*, t_{c1}, T_{c1}, T_1, t_{a1}, T_{a1}$ and T_2 for the fixed values of n', n and different values of K ($N = 100, n' = 70, n = 50$)

Estimators		$K = 2$	$K = 3$	$K = 4$
	$\bar{y}^*$	100.00(0.1334)*	100.00(0.1739)	100.00(0.2241)
Conventional	t_{c1}	156.27(0.0854)	165.12(0.1053)	171.44(0.1307)
	T_{c1}	162.35(0.0821)	170.24(0.1021)	175.76(0.1275)
	T_1	223.71(0.0596) $K_{1(opt)} = 1.2607$	254.67(0.0683) $K_{1(opt)} = 1.3266$	292.41(0.0766) $K_{1(opt)} = 1.4110$
Alternative	t_{a1}	124.17 (0.1074)	117.30 (0.1482)	112.93(0.1984)
	T_{a1}	127.98 (0.1042)	119.87 (0.1451)	114.79(0.1952)
	T_2	219.55(0.0607) $\alpha_{1(opt)} = 1.3983$	240.96(0.0722) $\alpha_{1(opt)} = 1.6171$	263.01(0.08520) $\alpha_{1(opt)} = 1.6268$

*Figures in parenthesis give mean square errors of the estimators.

Table 5. Simulated error of different estimators $\bar{y}^*, t_{c1}, T_{c1}, T_1, t_{a1}, T_{a1}$ and T_2 for the fixed values of n', n and different values of K ($N = 100, n' = 70, n = 50$)

Estimators		$K = 2$	$K = 3$	$K = 4$
	$\lvert M\hat{S}E(\bar{y}^*) - MSE(\bar{y}^*)\rvert$	0.0009	0.1642	0.2163
Conventional	$\lvert M\hat{S}E(t_{c1}) - MSE(t_{c1})\rvert$	0.0014	0.1014	0.1243
	$\lvert M\hat{S}E(T_{c1}) - MSE(T_{c1})\rvert$	0.0014	0.0986	0.1248
	$\lvert M\hat{S}E(T_1) - MSE(T_1)\rvert$	0.0029	0.004	0.0104
Alternative	$\lvert M\hat{S}E(t_{a1}) - MSE(t_{a1})\rvert$	0.0008	0.1395	0.0322
	$\lvert M\hat{S}E(T_{a1}) - MSE(T_{a1})\rvert$	0.0007	0.1367	0.1894
	$\lvert M\hat{S}E(T_2) - MSE(T_2)\rvert$	0.0329	0.0555	0.0957

Table 6. Relative efficiencies (with respect to $\bar{y}^*$) of different estimators $\bar{y}^*, t_{c1}, T_{c1}, T_1, t_{a1}, T_{a1}$ and T_2 (for the fixed cost $C \leq C_0 = Rs. 100, C_1' = Rs. 0.30, C_2' = Rs. 0.10, C_1 = 1.00, C_2 = 2.00, C_3 = 4.00$)

Estimators		K_{opt}	n'_{opt}	n_{opt}	R. E. (.) in %
	$\bar{y}^*$	1.79	---	33	100.00(0.2967)
Conventional	t_{c1}	1.65	65	24	109.85(0.2701)
	T_{c1}	1.65	60	24	114.93(0.2582)
	T_1	1.61	64	23	119.94(0.2474)
Alternative	t_{a1}	1.18	64	22	107.11(0.2770)
	T_{a1}	1.18	60	23	111.99(0.2649)
	T_2	1.11	63	22	116.90(0.2538)

*Figures in parenthesis give mean square errors of the estimators.

Table 7. Expected cost of different estimators $\bar{y}^*, t_{c1}, T_{c1}, T_1, t_{a1}, T_{a1}$ and T_2 (for the fixed cost $V = V_0 = 0.5000, C_1' = Rs. 0.30, C_2' = Rs. 0.10, C_1 = 1.00, C_2 = 2.00, C_3 = 4.00$)

Estimators		K_{opt}	n'_{opt}	n_{opt}	Expected cost (Rs.)
	$\bar{y}^*$	1.79	---	19	59.34
Conventional	t_{c1}	1.65	35	13	54.02
	T_{c1}	1.65	31	13	51.63
	T_1	1.61	31	12	49.47
Alternative	t_{a1}	1.18	35	12	55.40
	T_{a1}	1.18	32	12	52.98
	T_2	1.11	32	11	50.76

From table 2, we observe that for the fixed values of n', n and different values of $K = 2, 3, 4$, the simulated bias of the proposed estimators (T_1) and (T_2) is less than the corresponding estimators (t_{c1}, T_{c1}), (t_{a1}, T_{a1}) and $\bar{y}^*$. We also observe that the proposed estimators (T_1) and (T_2) has less mean square error than the corresponding estimators (t_{c1}, T_{c1}), (t_{a1}, T_{a1}) and $\bar{y}^*$. The 95% confidence intervals of the estimators are also obtained.

The estimator (T_1) has less mean square error than the estimator (T_2). As sub-sampling fraction K increases, mean square error of the estimators also increases.

From table 3, we observe that for the fixed values of n', n and different values of $K = 2, 3, 4$, the simulated bias of the proposed estimators (T_1) and (T_2) is less than the corresponding estimators (t_{c1}, T_{c1}), (t_{a1}, T_{a1}) and $\bar{y}^*$. We also observe that the proposed estimators (T_1) and (T_2) has less mean square error than the corresponding estimators (t_{c1}, T_{c1}), (t_{a1}, T_{a1}) and $\bar{y}^*$. The simulated 95% confidence intervals of the estimators are also obtained.

The estimator (T_1) has less mean square error than the estimator (T_2). Mean square errors of the estimators also increase as sub-sampling fraction K increases.

Case 2: For the simulation study of the estimators $\hat{B}(.)$ and $M\hat{S}E(.)$. In this case, we have taken the sample of 70 units from the population of 100 school going children of Varanasi with simple random sampling without replacement and calculated $s_y^2, s_x^2, s_z^2, s_{yx}, s_{yz}$ and s_{xz} based on 70 units. In sample of 70 units, the last 25% (18 children) of units have been considered as non-responding units. Again, we take a sub-sample of $r = \frac{18}{K}$ units from non-responding units in sample of 50 units with simple random sampling without replacement and calculate $s_{y(2)}^2, s_{x(2)}^2,\ s_{yx(2)}$ based on r units. Here, we take $K = 2, 3, 4$. After putting the values of $s_y^2, s_x^2, s_z^2, s_{yx}, s_{yz}$ and $s_{y(2)}^2, s_{x(2)}^2,\ s_{yx(2)}$ in the expression of the estimators $\hat{B}(.)$ and $M\hat{S}E(.)$ and replicating above process 1000 times, we find the simulated values of the estimators $\hat{B}(.)$ and $M\hat{S}E(.)$.

From table 4, we observe that for the fixed values of $n' = 70, n = 50$ and different values of $K = 2, 3, 4$, we observe that the proposed estimators (T_1) and (T_2) are more efficient than the corresponding estimators (t_{c1}, T_{c1}), (t_{a1}, T_{a1}) and $\bar{y}^*$.

The estimator (T_1) is more efficient than the estimator (T_2). Mean square error of the estimators increases as sub-sampling fraction K increases.From table 5, we observe that the difference between the estimated value $M\hat{S}E(.)$ and the true value of $MSE(.)$ based on simulation technique are very small and can be neglected. From table 6, we observe that for the fixed cost the proposed estimators (T_1) and (T_2) are more efficient and have less mean square error than the corresponding estimators (t_{c1}, T_{c1}), (t_{a1}, T_{a1}) and $\bar{y}^*$. The estimator (T_1) is more efficient than the estimator (T_2). From table 7, we observe that for the specified variance, the expected cost is minimum for the proposed estimators (T_1) and (T_2) than the corresponding estimators (t_{c1}, T_{c1}), (t_{a1}, T_{a1}) and $\bar{y}^*$. The estimator (T_1) has less cost than the estimator (T_2).

Conclusion

In this work, we have proposed conventional and alternative ratio-product estimators for the population mean in the presence of non-response. Here, we conclude that the using information on an additional auxiliary variable is fruitful in increasing the precision of the estimators compared to those, not utilizing such information. For the support of the problem, an empirical study as well as a Monte Carlo simulation study has been made. The results obtained from the Monte Carlo simulation study based on the empirical data are found to be similar to the results based on the empirical study. On the basis of the empirical study, we observe that use of an additional information in the proposed estimators for population mean in the presence of non-response is found to be more useful in increasing the precision of the proposed estimators with respect to the relevant estimators for the fixed cost $\boldsymbol{C \leq C_0}$. The total cost for the proposed estimators is also less than the relevant estimators for the specified variance $\boldsymbol{V = V_0}$.

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