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The Study of Strategic and Extensive form of Non-cooperative Game Theory

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ABSTRACT

In this paper, we consider a wide range of widely-studied models strategic form behavioral game theory. It is standard multi agent settings to assume that agents will adopt Nash equilibrium strategies. This paper gives a brief overview of game theory. Therefore in the first section we want to outline what game theory generally is and where it is applied. In the next section, we introduce some of the most important terms of Non-cooperative game theory such as strategic form (or) normal form games, extensive form and Nash equilibrium.

Introduction

A game is a description of strategic interaction that includes the constraints on the actions that the players *can* take and the player's interests, but does not specify the actions that the players *do* take. A *solution* is a systematic description of the outcomes that may emerge in a family of games. Game theory suggests reasonable solutions for classed of games and examines their properties. Nash equilibrium is one of the most basic concepts in game theory.

The next-most standard approach is to devise new solution concepts that overcome problems with Nash equilibrium, e.g., competitive safety strategies (Tennenholtz, 2002), minimax regret equilibrium (Hyafil and Boutilier, 2004), generalized strategic eligibility (Conitzer and Sandholm, 2005), CURB sets (Benisch, Davis, and Sandholm, 2006), and iterated regret minimization (Halpern and Pass, 2009). Still other work aims to identify strategies that work well without detailed modeling of the opponent. This line of work is perhaps exemplified by the very influential series of Trading Agent Competitions (Wellman, Greenwald, and Stone, 2007). We are most interested in approaches that make explicit predictions about which actions a player will adopt, and that are grounded in human behavior. The relatively new field of behavioral game theory extends game-theoretic models to account for human behavior by taking account of human cognitive biases and limitations (Camerer, 2003). Experimental evidence is a cornerstone of behavioral game theory, and researchers have developed many models of how humans behave in strategic situations based on experimental data. Among these models, the closely related cognitive hierarchy model (Camerer, Ho, and Chong, 2004), and quantal response equilibrium (McKelvey and Palfrey, 1995). Although different studies consider different specific variations, the overwhelming majority of behavioral models of initial play of normal-form games fall broadly into this categorization.

Game Theory

A game is made of three basic components: a set of Players, a set of actions, and a set of preferences. These are collectively known as the rules of the game, and the modeller's objective is to describe a situation in terms of the rules of a game so as to explain what will happen in that situation. Trying to maximize their payoffs, the player will devise plane known as *strategies* that pick actions depending on the information that has arrived at each moment. The combination of strategies chosen by each player is known as the equilibrium. Given an equilibrium, the modeler can see what actions come out of the conjunction of all the players' plans, and this tells him the *outcome* of the game.

The number of players will be denoted by n. Let us label the palyers with the integers 1 to n, and denote the set of players by $N = \{1,2,........n\}$. We assume throughout that there are atleast two players, that is $n \geq 2$. There are three main mathematical models or forms used in the study of games,(i) the extensive form (ii) the strategic or normal form and (iii) the coalitional form.

In the strategic form, many of the details of the game such as position and move are lost; the main concepts are those of a strategy and a payoff. In the strategic form, each player chooses a strategy from a set of possible strategies. We denote the strategy set or action space of player i by A_i, for $i = 1,2,\ldots\ldots\ n$. Each player considers all the other players and their possible strategies, and then chooses a specific strategy from his strategy set. All players make such a choice simultaneously, the choices are revealed and the game ends with each player receiving some payoff. Each player's choice may influence the final outcome for all players. We model the payoffs as taking as numerical values. The mathematical and philosophical justification behind the assumption that each player can replace such payoffs with numerical values is said to be *utility theory*. This theory is treated in detail in the books of Savage(1954) and of Fishburn (1988). We therefore assume that each player receives a numerical payoff that depends on the actions chosen by all the players.

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The strategic form of a game is defined by the three objects
(i) the set, $N = \{1,2,........n\}$ of players.
(ii) the sequence, $A_1, A_2,....A_n$ of strategy sets of the players, and
(iii) the sequence, $f_1(a_1, a_2......a_n)$…….. $f_n(a_1, a_2......a_n)$, of real-valued payoff functions of the players.
A game in strategic form is said to be zero-sum if the sum of the payoffs to the players is zero no matter what actions are by the players. That is, the game is zero-sum if $\sum_{i=1}^{n} f_i(a_1, a_2,....a_n) = 0$ for all $a_1 \in A_1, a_2 \in A_2,.....a_n \in A_n$.

Now the strategic form is extended to two-person non-zero-sum games. In general, such games do not have values and players do not have optimal strategies. The theory breaks naturally into two parts: (i) Non-cooperative theory (ii) Cooperative theory.
In the non-cooperative theory in which the players, if they may communicate, may not form binding agreements. This is the area of most interest to economists, see Gibbons(1992), and Bierman and Fernandez (1993). In 1994, John Nash, John Harsanyi and Reihard Selten received the Nobel Prize in Economics for work in this area. The main concept, replacing value and optimal strategy is the notion of a strategic equilibrium, also called a Nash equilibrium.

In this Cooperative theory the players are allowed to form binding agreements, and so there is strong incentive to work together to receive the largest total payoff. The problem then is how to split the total payoff between or among the players. This cooperative theory also splits into two parts. If the players measure utility of the payoff in the same units and there is a means of exchange of utility such as side payments, we say the game has transferable utility; otherwise non-transferable utility.

When the number of players grows large, even the strategic form of a game, though less, detailed than the extensive form, becomes too complex for analysis. In the Coalitional form of a game, the notion of a strategy disappears; the main features are those of a coalitional and the value or worth of the coalition. In many-player games, there is a tendency for the players to form coalitions to favor common interests. It is assumed each coalition can guarantee its members a certain amount, called the value of the coalition. The coalition form of a game is a part of cooperative game theory with transferable utility, so it is natural to assume that the grand coalition, consisting of all the players, will form, and it is a question of how the payoff received by the grand coalition should be shared among the players. There we introduce the important concepts of the core of an economy. The core is a set of payoffs to the players where each coalition receives at least its value. We will also look for principles that lead to a unique way to split the payoff from the grand coalition, such as the shapely value and the nucleolus.

Non-Cooperative Games in Extensive Forms and Equilibrium N-Tuples

Non-cooperative

A non –cooperative theory is based on the absence of coalitions in that it is assumed that each participant acts independently, without collaboration or communication without any of the others.

Strategy

The term ‘strategy’ is defined as a complex set of plans of action specifying precisely what the player will do under every possible future contingency that might occur during the play of the game.(i.e) the strategy of a player is the decision rule he was for making a choice from his list of courses of action.
Strategy can be classified as (i) Pure Strategy (ii) Mixed Strategy.
(i) Pure strategy: A Strategy is called pure if one knows inadvance of the play that it is certain to be adopted irrespective of the Strategy the other players might choose.
(ii) Mixed strategy: A mixed strategy of player i will be a collection of non-negative numbers which have unit sum and are in one to one correspondence with his pure strategies.

Nash Equilibrium

Nash equilibrium exists in any game if there is a set of strategies with the property that no player can increase her payoff by changing her strategy while the other players keep their strategies unchanged. These sets of strategies and the corresponding payoffs represent the Nash equilibrium . We can simply see that the action profile (defect,defect) is the Nash equilibrium in the Prisoners dilemma game and the actions profile (ballet,ballet) and (football, football) are the ones for the battle of the sexes game.

Pure and mixed strategy Nash equilibrium

In any game someone will find pure and mixed strategies, a pure strategy has a probability of one, and will be always played. On the other hand, a mixed strategy has multiple purse strategies with probabilities connected to them. A player would only use a mixed strategy when she is indifferent between several pure strategies, and when keeping the challenger guessing is desirable, that is when the opponent can benefit from knowing the next move.

Another reason why a player might decide to play a mixed strategy is when a pure strategy is not dominated by other pure strategies, but dominated by a mixed strategy. Finally, in a game without a pure strategy Nash equilibrium, a mixed strategy may result in a Nash equilibrium.

Normal forms and Mixed strategy equilibria

Although not all finite n-person non-cooperative games have pure strategy equilibria we can ask about the situation if mixed strategies are permitted. His result, which generalizes the Von Neumann minimax theorem, is that main objective of this paper and certainly provides one of the strongest arguments in favor of equilibrium points as a solution concept for n-person non-cooperative games.

Minimax Principl

This principal minimizes the maximum losses. The maximum losses with respect to different alternatives of player B, irrespective of player A’s alternatives, are obtained first. The minimum of these maximum losses is known as the minimax value and the corresponding alternatives are called as minimax strategy.

Strategic and extensive form games

The strategic form (also called normal form) is the basic type of game studied in non-cooperative game theory. A game in strategic form lists each player's strategies, and the outcomes that result from each possible combination of choices. An outcome is represented by a separate payoff for each player, which is a number (also called utility) that measures how much the player likes the outcome.

The extensive form, also called a game tree, is more detailed than the strategic form of a game. It is a complete description of how the game is played over time. This includes the order in which players take actions, the information that players have at the time they must take those actions, and the times at which any uncertainty in the situation is resolved. A game in extensive form may be analyzed directly, or can be converted into an equivalent strategic form.

By a non-cooperative game is meant a game in which absolutely no preplay communication is permitted between the players and in which players are awarded their due payoff according to the rules of the game.

In particular, agreements to share payoffs, even if this were practicable (and in many instances it is not), are specially forbidden. Thus in a non-cooperative game it is 'all players for themselves'.

We do not assert that transitory strategic cooperation cannot occur in as non-cooperative game if permitted by the rules. Typically, however, such arrangements to cooperative are not 'binding unto death'. For a requirement of this type would possess the limitation of cooperative games (that agreements are binding) without the possibility of preplay negotiation or profit sharing, atleast one of which normally occurs in cooperative games.

An n person non-cooperative game Γ in extensive form can be regarded as a graph theoretic tree of vertices (states) and edges (decisions or choice) with certain properties.

These properties can be summarized as follows:

(i) Γ has a distinguished vertex called the initial state.

(ii) There is a payoff function which assigns to each outcome an n-tuple ($(P_1, P_2, ...P_n)$ where P_i denoted the payoff to the i^{th} player.

(iii) Each non-terminal vertex of Γ is given one of $n+1$ possible labels according to which player makes the choice at that vertex. If the choice is made by chance the vertex is labeled with an N is equipped with a probability distribution over the edges leading from it.

(iv) The vertices of each player, other than nature, are partitioned into disjoint subsets known as information sets. A player is presumed to know which information set he or she is in, but not which vertex of the information set. This has the consequence that (a) Any two vertices in the same information set have identical sets of choices (edges) leading from them.

(b) No vertex can follow another vertex in the same information set.

Player i $(1 \le i \le n)$ is said to have perfect information in Γ if each information set for this player consists of one element.

The game Γ in extensive form is said to have perfect information if every player in Γ has perfect information. By a pure strategy for player i is meant a function which assigns to each of player $i's$ information sets one of the edges leading from a representative vertex of this set.

We denote by S_i the set of all pure strategies for player i. A game in extensive form is finite if it has a finite number of vertices. If Γ has no chance elements the payoff P_i to the i^{th} player is completely determined by an n-tuple $(\sigma_1, \sigma_2, ...\sigma_n)$, where $\sigma_i \in S_i$, that is $P_i = P_i(\sigma_1, \sigma_2, ...\sigma_n)$. If, however, chance of elements are involved then $P_i(\sigma_1, \sigma_2, ...\sigma_n)$ is taken to be the statistical expectation of the payoff function of player i, with respect to the probability distributions specified from property (iv), when the pure strategies $(\sigma_1, \sigma_2, ...\sigma_n)$, $\sigma_i \in S_i$, are chosen.

A game is Zero sum if $\sum_{i=1}^{n} P_i(\sigma_1, \sigma_2, ...\sigma_n) = 0$ for all n-tuples $(\sigma_1, \sigma_2, ...\sigma_n)$, $\sigma_i \in S_i$.

Definition 3.7. 1 : Equilibrium point *A pure strategy* n*-tuple* $(\sigma_1, \sigma_2, ...\sigma_n)$, $\sigma_i \in S_i$ *, is said to be an equilibrium point of* Γ *if for each* i, $(1 \le i \le n)$, *and any* $\sigma_i' \in S_i$, $P_i\left(\sigma_1, \sigma_2, ...\sigma_i', ...\sigma_n\right) \le P_i(\sigma_1, \sigma_2, ...\sigma_i, ...\sigma_n)$. *Thus an* n*-tuple* $(\sigma_1, \sigma_2, ...\sigma_n)$ *is an equilibrium point if no player has a positive incentive for a unilateral change of strategy.*

We truncate a finite n-person game Γ having perfect information by deleting the initial vertex and the edges leading from it. Because each information set consists of a single vertex, what remains is a finite number of sub games $\Gamma_1, \Gamma_2, ...\Gamma_r$, called the truncations of Γ, each having perfect information. We can also consider the truncation of a given pure strategy $\sigma_i \in S_i$ by restricting it as a function to the vertices of some truncation of Γ.

Theorem 3.7.1

A finite n-person non-cooperative game Γ in extensive form which has perfect information possesses an equilibrium point in pure strategies.

Proof:

Let x_0 be the initial vertex of Γ and let the other ends of the edges from x_0 be the vertices $a_1, a_2, ...a_r$. Then $a_j, 1 \le j \le r$, are the initial vertices of the games $\Gamma_1, \Gamma_2, ...\Gamma_r$ respectively, obtained by truncating Γ.

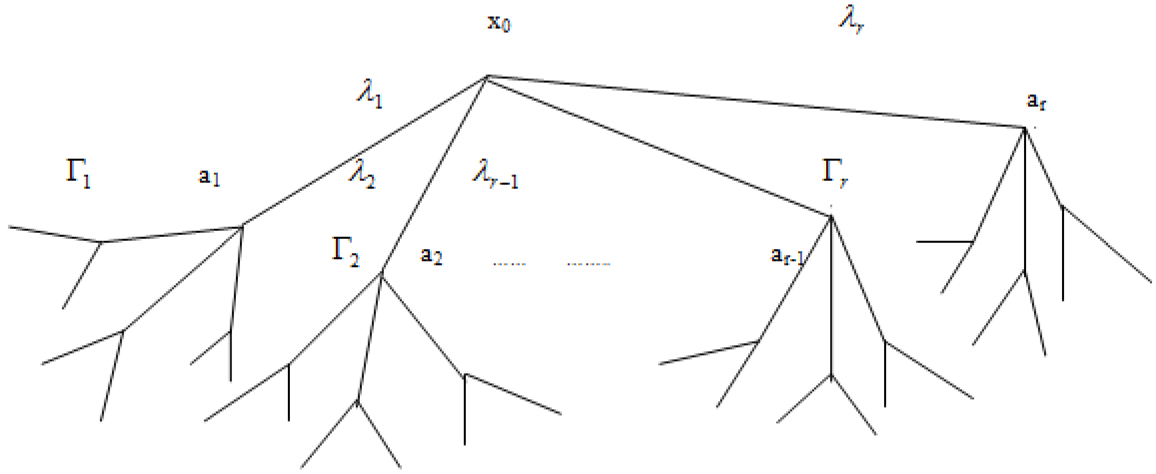

Figure 1. Truncating Γ

Let the longest play in Γ be of length N. We shall prove the theorem by induction on N. Clearly the games $\Gamma_1, \Gamma_2, ...\Gamma_r$ have length at most $N-1$.

Let $(\sigma_{1j}, \sigma_{2j}, ...\sigma_{nj})$ be pure strategies for each player for the game Γ_j $(1 \le j \le r)$.

Let $(\sigma_1, \sigma_2, ...\sigma_n)$ be pure strategies for each player for the Γ. We write $P_i(\sigma_1, \sigma_2, ...\sigma_n)$, $P_i^j(\sigma_{1j}, \sigma_{2j}, ...\sigma_{nj})$ for the payoffs to player i in Γ and Γ_j respectively.

For games of length zero the theorem is trivial (in action is equilibrium), so we assume the existence of equilibrium points for games of perfect information with length at most $N-1$, in particular for $\Gamma_1, \Gamma_2, ...\Gamma_r$.

Let $(\sigma_{1j}^0, \sigma_{2j}^0, ...\sigma_{nj}^0)$ be such a point for Γ_j that is for every i, $(1 \le i \le n)$.

$$P_i^j(\sigma_{1j}^0, \sigma_{2j}^0, ...\sigma_{ij}^{'}, ...\sigma_{nj}^0) \le P_i^j(\sigma_{1j}^0, \sigma_{2j}^0, ...\sigma_{ij}^0, ...\sigma_{nj}^0) \quad (1)$$

We shall construct an equilibrium point $(\sigma_1^0, \sigma_2^0, ...\sigma_n^0)$ for the game Γ.

Case 1 : x_0 is labelled N

Let $\lambda_1, \lambda_2, ...\lambda_r$, $0 \le \lambda_j \le 1$, $\sum \lambda_j = 1$, denote the probabilities for the vertices $a_1, a_2, ...a_r$ to be selected. Let x be any vertex of Γ.

If $x = x_0$ we do not need to define $\sigma_i^0(x)$, nor do we need to define it if x is any other vertex labeled with an N.

Otherwise $x \in \Gamma_j$ for some j and is labeled with an i, $(1 \le i \le n)$. We then define $\sigma_i^0(x) = \sigma_{ij}^0(x)$.

For any pure strategies $(\tau_1, \tau_2, ...\tau_n)$ of Γ we denote the restriction of τ_i to Γ_j by τ_i / Γ_j.

We plainly have $P_i(\tau_1, \tau_2, ...\tau_n) = \sum_{j=1}^{r} \lambda_j P_i^j(\tau_1/\Gamma_j, \tau_2/\Gamma_j, ...\tau_n/\Gamma_j)$ and $\sigma_i^0/\Gamma_j = \sigma_{ij}^0$.

Thus for $(1 \le i \le n)$,

$$P_i(\sigma_1^0, \sigma_2^0, ...\sigma_n^0) = \sum_{j=1}^{r} \lambda_j P_i^j(\sigma_{1j}^0, \sigma_{2j}^0, ...\sigma_{ij}^0, ...\sigma_{nj}^0)$$

$$\ge \sum_{j=1}^{r} \lambda_j P_i^j(\sigma_{1j}^0, \sigma_{2j}^0, ...\sigma_{ij}^{'}, ...\sigma_{nj}^0) \text{ from } \quad (1)$$

But $\sum_{j=1}^{r} \lambda_j P_i^j(\sigma_{1j}^0, \sigma_{2j}^0, ...\sigma_{ij}^{'}, ...\sigma_{nj}^0) = P_i(\sigma_1^0, \sigma_2^0, ...\sigma_i^{'}....\sigma_n^0)$

so that, for each i, $(1 \le i \le n)$

$P_i(\sigma_1^0, \sigma_2^0, ...\sigma_i^{'}....\sigma_n^0) \le P_i(\sigma_1^0, \sigma_2^0, ...\sigma_i^0....\sigma_n^0)$, that is, $\left(\sigma_1^0, \sigma_2^0, ...\sigma_n^0\right)$ is an equilibrium point for Γ.

Case 2 : x_0 is labeled with a player index.

Without loss of generality we can suppose x_0 is labeled with a 1.

If $x = x_0$ we define $\sigma_1^0(x)$ to be that choice of $j = \alpha$ for which $\underset{1\le j\le r}{Max}\ P_i^j\left(\sigma_{1j}^0, \sigma_{2j}^0, ...\sigma_{nj}^0\right)$ is attained, that is $\sigma_1^0(x_0) = \alpha.$

For any other vertex $x \neq x_0$, $\sigma_i^0(x)$ is defined, where necessary, as in case 1.

Then $P_1\left(\sigma_1^0, \sigma_2^0, ...\sigma_n^0\right) = P_1^\alpha\left(\sigma_{1\alpha}^0, \sigma_{2\alpha}^0, ...\sigma_{n\alpha}^0\right) \geq P_1^j\left(\sigma_{1j}^0, \sigma_{2j}^0, ...\sigma_{nj}^0\right)$, for $1 \leq j \leq r$.

Since $\left(\sigma_{1j}^0, \sigma_{2j}^0, ...\sigma_{nj}^0\right)$ is an equilibrium point for Γ_j,

$P_1^j\left(\sigma_{1j}^{'}, \sigma_{2j}^0, ...\sigma_{nj}^0\right) \leq P_1^j\left(\sigma_{1j}^0, \sigma_{2j}^0, ...\sigma_{nj}^0\right).$

Now any pure strategy $\sigma_1^{'}$ for player 1 in Γ will truncate to some pure strategy $\sigma_{1j}^{'}$ in Γ_j for any j, $1 \leq j \leq r$.

Thus $P_1\left(\sigma_1^{'}, \sigma_2^0, ...\sigma_n^0\right) = P_1^j\left(\sigma_{1j}^{'}, \sigma_{2j}^0, ...\sigma_{nj}^0\right)$, where $\sigma_1^{'}(x_0) = j$. Hence

$P_1\left(\sigma_1^{'}, \sigma_2^0, ...\sigma_n^0\right) \leq P_1^\alpha\left(\sigma_{1\alpha}^0, \sigma_{2\alpha}^0, ...\sigma_{i\alpha}^0, ...\sigma_{n\alpha}^0\right) = P_1\left(\sigma_1^0, \sigma_2^0, ...\sigma_n^0\right).$ (2)

If $i \neq 1$ since $\left(\sigma_{1\alpha}^0, \sigma_{2\alpha}^0, ...\sigma_{n\alpha}^0\right)$ is an equilibrium point for Γ_α

$$\begin{aligned}
P_i\left(\sigma_1^0, \sigma_2^0, ...\sigma_i^{'}, ...\sigma_n^0\right) &= P_i^\alpha\left(\sigma_1^0/\Gamma_\alpha, \sigma_2^0/\Gamma_\alpha, ...\sigma_i^{'}/\Gamma_\alpha, ...\sigma_n^0/\Gamma_\alpha\right)\\
&= P_i^\alpha\left(\sigma_{1\alpha}^0, \sigma_{2\alpha}^0, ...\sigma_{i\alpha}^{'}, ...\sigma_{n\alpha}^0\right) \leq P_i^\alpha\left(\sigma_{1\alpha}^0, \sigma_{2\alpha}^0, ...\sigma_{i\alpha}^0, ...\sigma_{n\alpha}^0\right)\\
&= P_i\left(\sigma_1^0, \sigma_2^0, ...\sigma_i^0, ...\sigma_n^0\right)
\end{aligned}$$

Since $\sigma_1^0(x_0) = \alpha$. Hence if $i \neq 1$, $P_i\left(\sigma_1^0, \sigma_2^0, ...\sigma_i^{'}, ...\sigma_n^0\right) \leq P_i\left(\sigma_1^0, \sigma_2^0, ...\sigma_i^0, ...\sigma_n^0\right)$ (3)

But (2) and (3) together assert that $\left(\sigma_1^0, \sigma_2^0, ...\sigma_n^0\right)$ is an equilibrium point for Γ, and this complete the proof.

Definition 3.7.2

A non-cooperative game Γ (n person game) in normal form is a collection $\Gamma = \left\langle I, \{X_i\}_{i\in I}, \{P_i\}_{i\in I}\right\rangle$, in which the set of player is I, the set of strategies for player i is X_i, and the payoff to player i is given by $P_i : \prod_{i\in I} X_i \to R$. Here the sets X_i could be taken to be sets of pure or mixed strategies.

Theorem 3.7.2: *Every game with complete information and a finite tree has atleast one equilibrium point.*

Definition 3.7.3 :

A mixed strategy n-tuple $x = (x_1, x_2, ...x_n), x_i \in X_i$, is an equilibrium point of an n-person non-cooperative game Γ if each i, $1 \leq i \leq n$, and any $x_i^{'} \in X_i$, $P_i\left(x \| x_i^{'}\right) \leq P_i(x)$.

Theorem 3.7.3: *A mixed strategy n-tuple $x = (x_1, x_2, ...x_n)$ is an equilibrium point of a finite game Γ if and only if for each player index i, $P_i(x\|\sigma_i) \leq P_i(x)$ for every pure strategy $\sigma_i \in S_i$.*

Proof:

If x is an equilibrium point of Γ, for each i, $1 \leq i \leq n$, the inequality $P_i(x\|\sigma_i) \leq P_i(x)$ is irreducible from $P_i\left(x\|x_i^{'}\right) \leq P_i(x)$. Since a pure strategy is a particular case of a mixed strategy.

To prove that the condition is sufficient to ensure that x is an equilibrium point, choose an arbitrary mixed strategy $x_i^{'} \in X_i$,

$$x_i^{'}(\sigma_i) P_i(x\|\sigma_i) \leq x_i^{'}(\sigma_i) P_i(x)$$

$$\sum_{\sigma_i \in s_i} x_i^{'}(\sigma_i) P_i(x\|\sigma_i) \leq \sum_{\sigma_i \in s_i} x_i^{'}(\sigma_i) P_i(x)$$

From $P_i(x_1, x_2 .....x_n) = \sum_{\sigma_1 \in s_1} \text{……..} \sum_{\sigma_n \in s_n} P_i(\sigma_1, \sigma_2, ...\sigma_n) \prod_{j=1}^{n} x_j(\sigma_j)$

And $P_i(x\|\sigma_i) = \sum_{\sigma_1 \in s_1} \text{………} \sum_{\sigma_{i-1} \in s_{i-1}} \sum_{\sigma_{i+1} \in s_{i+1}} \text{………} \sum_{\sigma_n \in s_n} P_i(\sigma) \prod_{\substack{j=1\\ j\neq 1}}^{n} x_j(\sigma_j)$ where $\sigma = (\sigma_1, ...\sigma_n)$

We get $P_i(x\|\sigma_i) \le P_i(x) \sum_{\sigma_i \in s_i} x_i^{'}(\sigma_i)$

Which is $P_i(x\|\sigma_i) \le P_i(x)$, since $\sum_{\sigma_i \in s_i} x_i^{'}(\sigma_i)$ =1

This theorem gives an effective procedure for checking a possible equilibrium point.

Theorem 3.7.4: *For any mixed strategy n-tuple* $x = (x_1, x_2, ...x_n)$ *each player* i, $1 \le i \le n$, *possesses a pure strategy* σ_i^k *such that* $x_i(\sigma_i^k) > 0$ *and* $P_i(x\|\sigma_i^k) \le P_i(x)$.

Theorem 3.7.5: Nash Theorem

Any finite n-person non-cooperative game Γ *has atleast one mixed strategy equilibrium point.*

Based on the four theorem the following problem solution takes the informative one.

Research Article

Consider the non-cooperative n-person game in which each player $i \in I$ *has exactly two pure strategies, either* $\sigma_i = 1$ *or* $\sigma_i = 2$. *The payoff is* $P_i(\sigma_1, \sigma_2 ....\sigma_n) = \sigma_i \prod_{j \ne i}(1 - \delta(\sigma_i, \sigma_j))$, $i \in I$, *where* δ *is the kronecker* δ *given by*

$$\delta(\sigma_i, \sigma_j) = \begin{cases} 1, if \sigma_i = \sigma_j \\ 0, otherwise. \end{cases}$$

If player i *uses a mixed strategy in which pure strategy 1 is chosen with probability* $p_i(i \in I)$. *Prove that* $(p_1, p_2, ...p_n)$ *defines an equilibrium point if and only if* $\prod_{j \ne i}(1 - p_j) = 2\prod_{j \ne i} p_j$ *for every* $i \in I$. *Deduce that a mixed strategy equilibrium is given by* $p_i = \dfrac{1}{\left(1 + 2^{\frac{1}{n-1}}\right)}, \forall i \in I,$ *and that for n=2,3 this is the only equilibrium point.*

Solution:

Player 1: If $\sigma_1 = 1$ then $P_1 = 0$ unless $\sigma_2 = ... = \sigma_n = 2,$ in which case $P_1 = 1$.

If $\sigma_1 = 2$ then $P_1 = 0$ unless $\sigma_2 = ... = \sigma_n = 1,$ in which case $P_1 = 2$.

Similarly for the other players. Consider now the mixed strategy n-tuple $x = (x_1, x_2, ...x_n)$, where $x_i = (p_i, 1 - p_i)$ for $1 \le i \le n,$ and $p_i(i \in I)$ is the probability of choosing $\sigma_i = 1$.

From the above observation we obtain $P_i(x) = p_i \prod_{j \ne i}(1 - p_j) + 2(1 - p_j)\prod_{j \ne i} p_j$.

Also $P_i(x\|\sigma_i) = \prod_{j \ne i}(1 - p_j)$ if $\sigma_i = 1$

$P_i(x\|\sigma_i) = 2\prod_{j \ne i}(p_j)$ if $\sigma_i = 2$

According to Theorem 2 "A mixed strategy n-tuple $x = (x_1, x_2, ...x_n)$ is an equilibrium point of a finite game Γ if and only if for each player index i, $P_i(x\|\sigma_i) \le P_i(x)$ for every pure strategy $\sigma_i \in S_i$."

x is an equilibrium point if and only if

$$\prod_{j \ne i}(1 - p_j) \le p_i \prod_{j \ne i}(1 - p_j) + 2(1 - p_j)\prod_{j \ne i} p_j \qquad (1)$$

And $2\prod_{j \ne i}(p_j) \le p_i \prod_{j \ne i}(1 - p_j) + 2(1 - p_j)\prod_{j \ne i} p_j$ for every $i \in I$. (2)

Rearranging equation (1) we have

$$(1 - p_i)\prod_{j \ne i}(1 - p_j) \le 2(1 - p_j)\prod_{j \ne i} p_j$$

That is, $\prod_{j \ne i}(1 - p_j) \le 2\prod_{j \ne i} p_j$ (3)

Similarly rearranging equation (2) gives

$$(2 - 2 + 2p_i)\prod_{j \ne i} p_j \le p_i \prod_{j \ne i}(1 - p_j)$$

$$(2p_i)\prod_{j \ne i} p_j \le p_i \prod_{j \ne i}(1 - p_j)$$

$$2\prod_{j \ne i} p_j \le \prod_{j \ne i}(1 - p_j) \qquad (4)$$

From equation (3) and equation (4) it follows that x is an equilibrium point if and only if

$$\prod_{j\neq i}\left(1-p_j\right)=2\prod_{j\neq i}p_j \text{ for every } i\in I \qquad (5).$$

For n=2 or 3 the system of equation (5) has no solution with any $p_i =$ 0 or 1, but for n=4 these are several such solutions, for example $p_1 = p_4 = 1$, $p_2 = p_3 = 0$.

If n $\geq$ 5we can find solution with $p_1 = p_4 = 1$, $p_2 = p_3 = 0$ and the remaining $n-4 p_i$
Arbitrary. To complete the analysis suppose $0 < p_i < 1$ for every $i \in I$.

Consider the equation (5) for $i = k, i = l$ where $k \neq l$. This gives $\prod_{j\neq k}\left(1-p_j\right)=2\prod_{j\neq k}p_j$ and $\prod_{j\neq l}\left(1-p_j\right)=2\prod_{j\neq l}p_j$.

If we put A= $\Pi\left(1-p_j\right)$, B= $\Pi\, p_j$, since $0 < p_i < 1$, all i, we can write these as $\frac{A}{1-p_k}=\frac{2B}{p_k}$, $\frac{A}{1-p_l}=\frac{2B}{p_l}$.

Since A $\neq$ 0 and B $\neq$ 0 we easily see that $p_k = p_l$. But k and l were arbitrary, so that every player must use the same mixed strategy in x.

Condition (5) therefore becomes simply $\left(1-p\right)^{n-1} = 2p^{n-1}$.

Solving for p we obtain $\left(1-p\right)=2^{\frac{1}{n-1}}p$

$$1=p\left(1+2^{\frac{1}{n-1}}\right)$$

$$\therefore \quad p=\frac{1}{1+2^{\frac{1}{n-1}}} \text{ as required.}$$

Conclusion

The process of finding equilibrium points in a bimatrix game consists in carrying out a finite number of rational operations on the values of the payoff matrix. For n $\geq$ 3 the above value of p is irrational, which shows that the situation for n=2 is untypical.

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