



Free Convection Flow in a Vertical Channel Filled with Porous Matrix for Variable Properties

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ABSTRACT

The perturbation method and Runge-kutta shooting method has been carried out to study the influence of the effect of exponential viscosity-temperature relation, exponential thermal conductivity-temperature and the combined effects of the variable viscosity and the variable thermal conductivity on steady free convection flow in a vertical channel filled with porous medium. The Darcy-Brinkman model is used to predict the flow in porous medium. The walls are maintained at constant but different temperatures. Numerical results are presented for a wide range of parameters of variable viscosity parameter, variable thermal conductivity parameter, wall temperature ratio, buoyancy parameter and porous parameter on the velocity and temperature fields. Furthermore, the effect of the governing parameters on skin friction and Nusselt number are tabulated. The solutions obtained by Runge-Kutta shooting method are compared with perturbation method solutions and the results agree very well in the absence of buoyancy parameter.

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Introduction

The study of heat transfer phenomenon in porous media has been intensively increased during the last four decades. This is because of its important applications in contemporary technologies such as geothermal exploitations, oil recovery, radioactive waste management, transpiration cooling and ground water pollution. A newly explored area in which the heat transfer phenomenon finds interesting applications in the study of flow and heat transfer analysis in partial filled boxes. The boxes may contain oranges, apples, mangoes or some other such food items. In such systems, it is of great importance to investigate the air flow and heat transfer from the atmosphere to the inner side of the boxes to keep the food items fresh. Timely reviews of the literature on heat transfer in porous media have been reported in the excellent monographs by Ingham and Pop [1], Vafai [2] and Nield and Bejan [3]. With interest coming from these broad fields to technical engineering applications, the study of natural convection in cavities filled with porous media, both theoretically and experimentally, has also received considerable attention in the past few decades. The main objectives of these studies are usually the analysis of the resulting flow and heat transfer characteristics on functions of the dimensionless parameters governing the involved phenomenon. Abundant literature is available on convection phenomena, but the literature lacks studies that take into account the convection in a saturated porous medium. Non-Darcy effects on natural convection in porous media, have found many applications, such as in fluid flow in geothermal reservoirs, separation processes in chemical industries, dispersion of chemical contaminants through water saturated soil, solidification of casting, migration of moisture in grain storage systems, crude oil production, etc. A comprehensive survey concerned with this subject can be found in the literature (Pop and Ingham [4], Vafai [2,5], Nield and Bejan [3]).

Raptis et al. [6] studied the hydromagnetic free convection flow through a porous medium between two parallel plates. Aldos et al. [7] studied mixed convection flow from a vertical

plate embedded in a porous medium in the presence of a magnetic field. Umavathi and Malashetty [8] studied the Oberbeck convection flow of couple stressed fluid in a vertical porous channel. Chamkha [9] considered magnetohydrodynamic (MHD) free convection flow from a vertical plate embedded in a thermally stratified porous medium with Hall effects. Makinde and Mhone [10] studied the heat transfer analysis of MHD oscillatory flow in a channel filled with a porous medium. Umavathi et al. [11-14] analyzed the flow and heat transfer characteristic in a porous medium for different geometries. Recently Umavathi and Veerashetty [15,16] studied the non-Darcy mixed convection in a vertical porous channel with boundary conditions of third kind.

Most of these studies are based on constant physical properties, more accurate prediction for the flow and heat transfer can be achieved by taking into account the variation of these properties such as variation of fluid viscosity with temperature (constant thermal conductivity), variation of fluid with conductivity (constant viscosity) and combined effect of variation of fluid viscosity and thermal conductivity with temperature.

Accordingly Garry et al. [17] and Mehta and Sood [18] have concluded that compared to the constant viscosity case, the flow characteristics change substantially when this effect is included. Recently, Kafoussius and Williams [19] and Kafoussius and Rees [20] have used the local non similarity method to investigate the effect of the temperature dependent viscosity on the mixed convection flow past a vertical flat plate in the region near the leading edge. From all these studies they came to a conclusion that the viscosity of the fluid is sensitive to temperature variations, the effect of temperature-dependent viscosity has to be taken into consideration; otherwise considerable errors may occur in the characteristics of the heat transfer process. Hossain et al. [21] have investigated the natural convection flow from a vertical wavy surface and Hossain and Munir [22] investigated the mixed convection flow from a vertical flat plate. Klemp et al. [23] has studied the effect of

temperature dependent viscosity on the entrance flow in a channel in the hydrodynamic case. Attia and Kotb [24] studied the steady MHD fully developed flow and heat transfer between two parallel plates with temperature dependent viscosity.

For the fluids, which are important in the theory of lubrication, the heat generated by the internal friction and the corresponding rise in temperature do affect the viscosity and thermal conductivity of the fluid and they can no longer be regarded as constant. The physical properties of fluids such as viscosity and thermal conductivity may change significantly with temperature (Schlichting, [25]). The temperature-dependent property problem is further complicated by the fact that the properties of different fluids behave differently with temperature. Different relations between the physical properties of fluids and temperature were given by Kays and Crawford [26]. Ockendon and Ockendon [27] presented an analysis for suddenly heated or cooled channel flow of a Newtonian fluid with the viscosity either algebraically or exponentially dependent on temperature. Elbashbesy and Ibrahim [28] analyzed the flow of viscous incompressible fluids along a heated vertical plate, taking into account the variation of the viscosity and thermal diffusivity with temperature.

In the present work, the effect of variable viscosity keeping thermal conductivity constant, the effect of variable thermal conductivity keeping viscosity constant and the effect of both variable viscosity and conductivity with temperature is studied with flow of viscous incompressible permeable fluid in a vertical channel. The fluid is flowing between two infinite plates maintained at two constant but different temperatures. The Darcy-Brinkman model is used to define the porous medium. The viscosity and thermal conductivity is assumed to vary exponentially with temperature. The viscosity and Darcy dissipation terms in the energy equation are taken into consideration. The governing coupled nonlinear ordinary differential equations are solved using the Runge-Kutta shooting method (RKSM) and analytically using perturbation method (PM). The effects of temperature dependent viscosity, temperature dependent thermal conductivity, wall temperature ratio, buoyancy parameter and porous parameter on the velocity and temperature distributions are discussed.

Mathematical formulation

Consider a steady laminar, fully developed flow of an incompressible viscous fluid between two parallel plates. The distance between the plates is $2b$ and the origin of coordinate axis is located in the mid-plane of the channel. The two plates are kept at two constant temperatures T_1 for the left plate and T_2 for the right plate. The channel is assumed to occupy the region of space $-b \leq Y \leq b$. A fluid rises in the channel driven by buoyancy forces. The no-slip boundary condition is imposed on the parallel plates for the velocity, and since the plates are infinite in the X -direction, the physical variables are invariant in these directions and the problem is essentially one-dimensional with velocity component $U(Y)$ along the X -axis. The physical properties characterizing the fluid except density, viscosity and thermal conductivity are assumed to be constant. As customary, the Boussinesq approximation and the equation of state

$$\rho = \rho_0[1 - \beta(T - T_0)]. \quad (1)$$

will be adopted. The flow and heat transfer of viscous fluid is examined considering the following three cases.

In Case 1 we consider only variation of viscosity with constant thermal conductivity, in Case 2 keeping viscosity constant, vary the thermal conductivity and in Case 3 we study the combination of both variable viscosity and thermal conductivity.

Case 1: Effect of variable viscosity

The governing equations of motion for variable viscosity become (Nield and Bejan, [3])

$$\frac{d}{dY} \left(\mu \frac{dU}{dY} \right) - \frac{\mu U}{\kappa} + \rho_0 g \beta (T - T_0) = 0 \quad (2)$$

$$K_0 \frac{d^2 T}{dY^2} + \mu \left(\frac{dU}{dY} \right)^2 + \frac{\mu}{\kappa} U^2 = 0 \quad (3)$$

where U is the velocity of the fluid, T the temperature of the fluid, ρ_0 the static density, β the coefficient of thermal expansion, g the acceleration due to gravity, μ the viscosity and K_0 the thermal conductivity of the fluid.

The boundary conditions on the velocity and temperature fields are given as

$$U = 0 \text{ at } Y = \pm b \quad (4)$$

$$T = T_1 \text{ at } Y = -b, \quad T = T_2 \text{ at } Y = b \quad (5)$$

where b is the characteristic length.

The fluid viscosity μ is assumed to vary with temperature as (Saravanan and Kandaswamy, [29] and Attia, [30])

$$\mu = \mu_0 e^{-a(T - T_0)} \quad (6)$$

where the subscript 0 denotes the reference state and a an empirical constant.

In Eq. (6) the viscosity μ is assumed to depend on temperature exponentially. The parameter a may take positive values for liquids such as water, benzene or crude oil. In some gases like air, helium or methane a may be negative, i.e., the coefficient viscosity increases with temperature (Sutton and Sherman, [31]; Schlichting, [32]). This type of model can find applications in many processes where preheating of the fuel is used as a means to enhance heat transfer effects. In addition, for many fluids such as lubricants, polymers, and coal slurries where viscous dissipation is substantial, an appropriate constitutive relation where viscosity is a function of temperature should be used.

Equations (2)-(5) determine the velocity and temperature distribution, they can be written in a dimensionless form by means of the following dimensionless parameters

$$u = \frac{\mu_0}{\rho_0 g \beta b^2 \Delta T} U, \quad \theta = \frac{T - T_0}{\Delta T}, \quad y = \frac{Y}{b}, \quad N = \frac{\rho_0^2 b^4 g^2 \beta^2 \Delta T}{\mu_0 K_0} \quad (7)$$

Using Eqs. (6) and (7) the dimensionless governing Eqs. (2) - (5) reduces to

$$\frac{d^2 u}{dy^2} - b_v \frac{d\theta}{dy} \frac{du}{dy} - \sigma^2 u + \theta + b_v \theta^2 = 0 \quad (8)$$

$$\frac{d^2 \theta}{dy^2} + N \left(\frac{du}{dy} \right)^2 - b_v N \theta \left(\frac{du}{dy} \right)^2 + \sigma^2 N u^2 - b_v \sigma^2 N \theta u^2 = 0 \quad (9)$$

and the non dimensional boundary conditions reduces to

$$u = 0 \text{ at } y = \pm 1 \quad (10)$$

$$\theta = 1 + m \text{ at } y = -1, \theta = 1 \text{ at } y = 1. \quad (11)$$

where $b_v = a\Delta T$ is the variable viscosity parameter,

$$m = \frac{T_1 - T_2}{\Delta T} \text{ the wall temperature ratio and } N \text{ is the}$$

buoyancy parameter.

The Eq. (6) can be approximated by expanding μ in terms of a truncated Taylor's series about $x = 0$ and consider only first two terms in the series.

Case 2: Effect of variable conductivity

In this case we consider the steady laminar fully developed flow of a viscous fluid (constant viscosity) with the variable conductivity and the effect of viscous permeable dissipation. The governing equations of this motion are

$$\mu_0 \frac{d^2 U}{dY^2} - \frac{\mu_0 U}{\kappa} + \rho_0 g \beta (T - T_0) = 0 \quad (12)$$

$$\frac{d}{dY} \left(K \frac{dT}{dY} \right) + \mu_0 \left(\frac{dU}{dY} \right)^2 + \frac{\mu_0}{\kappa} U^2 = 0 \quad (13)$$

The boundary conditions are same as in Eqs. (4) and (5).

The thermal conductivity of the fluid is assumed as (Attia, [30])

$$K = K_0 e^{-b(T-T_0)} = K_0 (1 + b(T_0 - T)) \quad (14)$$

The thermal conductivity of the fluid is assumed to vary with temperature as can be seen in Eq. (14) where the parameter b may be positive for some fluids such as air or water vapor or negative for others like liquid water or benzene (Schlichting, [32] and White, [33]).

The thermal conductivity changes approximately linearly with temperature in the range from 0 °F to 400 °F (Kays, [34]). The above governing Eqs. (12) and (13) are written in dimensionless form by using Eqs. (7) and (14) as

$$\frac{d^2 u}{dy^2} - \sigma^2 u + \theta = 0 \quad (15)$$

$$\frac{d^2 \theta}{dy^2} - b_k \left(\frac{d\theta}{dy} \right)^2 + N \left(\frac{du}{dy} \right)^2 + b_k N \theta \left(\frac{du}{dy} \right)^2 + \sigma^2 Nu^2 + \sigma^2 N b_k \theta u^2 = 0 \quad (16)$$

The corresponding boundary conditions are same as in Eqs. (10) and (11), where $b_k = b\Delta T$ is the variable conductivity parameter.

Case 3: Combined effect of variable viscosity and thermal conductivity

The momentum equations governing the motion of an incompressible fluid in the presence of viscous dissipation with the variable viscosity and variable thermal conductivity are given by

$$\frac{d}{dY} \left(\mu \frac{dU}{dY} \right) - \frac{\mu U}{\kappa} + \rho_0 g \beta (T - T_0) = 0 \quad (17)$$

$$\frac{d}{dY} \left(K \frac{dT}{dY} \right) + \mu \left(\frac{dU}{dY} \right)^2 + \frac{\mu}{\kappa} U^2 = 0 \quad (18)$$

The expressions for viscosity μ and thermal conductivity K are given in Eqs. (6) and (14).

In terms of the non dimensional variables as in Eq. (7), Eqs. (17) and (18) take the form

$$\frac{d^2 u}{dy^2} - b_v \frac{d\theta}{dy} \frac{du}{dy} - \sigma^2 u + \theta + b_v \theta^2 = 0 \quad (19)$$

$$\begin{aligned} & \frac{d^2 \theta}{dy^2} - b_k \left(\frac{d\theta}{dy} \right)^2 + N \left(\frac{du}{dy} \right)^2 + N \sigma^2 u^2 \\ & + (b_k - b_v) N \theta \left(\frac{du}{dy} \right)^2 + (b_k - b_v) N \theta \sigma^2 u^2 \\ & - b_k b_v \theta^2 N \left(\frac{du}{dy} \right)^2 - b_k b_v \theta^2 N \sigma^2 u^2 = 0 \end{aligned} \quad (20)$$

and the corresponding boundary conditions are given in Eqs. (10) and (11).

Equations (19)-(20) show that the dimensionless velocity and temperature fields depend on four parameters: the viscosity parameter b_v , the conductivity parameter b_k , the buoyancy parameter N , the porous parameter σ and the wall temperature ratio m .

Solutions

Perturbation method

The solutions of the governing equations of motion are found using perturbation method for all the three cases.

Case 1: Effect of variable viscosity

Equations (8) and (9) are coupled nonlinear equations because of variable viscosity and viscous dissipation and it is difficult, in general, to solve analytically. When neglecting the viscous dissipative heating ($N = 0$), Eqs. (8) and (9) become linear and solutions can easily be obtained.

In many practical applications cited above, N can not be zero ($N \neq 0$), but in many situations it can take small values. For example, for mercury in a channel of half-width 2cm, and with $T_1 - T_0 = 20^\circ \text{C}$, N takes the value of 0.128. Small values of N (< 1) facilitate finding analytical solutions of Eqs. (8) and (9) in the form

$$u = u_0 + Nu_1 + \dots \quad (21)$$

$$\theta = \theta_0 + N\theta_1 + \dots \quad (22)$$

where the second and higher order terms on the right-hand side give a correction to u_0 , θ_0 accounting for the dissipative effects. Substituting Eqs. (8)-(11) into Eqs. (21) and (22) and equating like powers of N to zero, we obtain

Zeroth order equations

$$\frac{d^2 u_0}{dy^2} - b_v \frac{d\theta_0}{dy} \frac{du_0}{dy} - \sigma^2 u_0 + \theta_0 + b_v \theta_0^2 = 0 \quad (23)$$

$$\frac{d^2 \theta_0}{dy^2} = 0 \quad (24)$$

The corresponding boundary conditions are

$$u_0 = 0 \text{ at } y = \pm 1 \quad (25)$$

$$\theta_0 = 1 + m \text{ at } y = -1, \theta_0 = 1 \text{ at } y = 1 \quad (26)$$

First order equations

$$\frac{d^2 u_1}{dy^2} - b_v \frac{d\theta_0}{dy} \frac{du_1}{dy} - b_v \frac{d\theta_1}{dy} \frac{du_0}{dy} - \sigma^2 u_1 + \theta_1 + 2b_v \theta_0 \theta_1 = 0 \quad (27)$$

$$\frac{d^2\theta_1}{dy^2} + \left(\frac{du_0}{dy}\right)^2 - b_v\theta_0\left(\frac{du_0}{dy}\right)^2 \quad (28)$$

$$+ \sigma^2 u_0^2 - b_v \sigma^2 \theta_0 u_0^2 = 0$$

The corresponding boundary conditions are

$$u_1 = 0 \text{ at } y = \pm 1; \theta_1 = 0 \text{ at } y = \pm 1 \quad (29)$$

Case 2: Effect of variable conductivity

The method of solution is similar to the Case 1. Substituting Eqs. (21) and (22) into Eqs. (12)-(14) and equating like powers of N to zero, we obtain

Zeroth order equations

$$\frac{d^2 u_0}{dy^2} - \sigma^2 u_0 + \theta_0 = 0 \quad (30)$$

$$\frac{d^2 \theta_0}{dy^2} - b_k \left(\frac{d\theta_0}{dy}\right)^2 = 0 \quad (31)$$

The corresponding boundary conditions are same as in Eqs. (25) and (26).

First order equation

$$\frac{d^2 u_1}{dy^2} - \sigma^2 u_1 + \theta_1 = 0 \quad (32)$$

$$\frac{d^2 \theta_1}{dy^2} - 2b_k \frac{d\theta_0}{dy} \frac{d\theta_1}{dy} + \left(\frac{du_0}{dy}\right)^2 + b_k \theta_0 \left(\frac{du_0}{dy}\right)^2 \quad (33)$$

$$+ \sigma^2 u_0^2 + \sigma^2 b_k \theta_0 u_0^2 = 0$$

The corresponding boundary conditions are same as in Eqs. (29). We shall further perform a perturbation analysis of the Eqs. (30)-(33) considering variable conductivity parameter as a perturbation parameter. The solutions of Eqs. (30)-(33) is assumed as follows

$$u_0 = u_{00} + b_k u_{01}; \theta_0 = \theta_{00} + b_k \theta_{01};$$

$$u_1 = u_{10} + b_k u_{11}; \theta_1 = \theta_{10} + b_k \theta_{11} \quad (34)$$

Using Eqs. (34) into Eqs. (25), (26), (29) and (30)-(33) and equating like powers of b_k , we obtain the following boundary value problems

$$\frac{d^2 u_{00}}{dy^2} - \sigma^2 u_{00} + \theta_{00} = 0 \quad (35)$$

$$\frac{d^2 \theta_{00}}{dy^2} = 0 \quad (36)$$

$$\frac{d^2 u_{01}}{dy^2} - \sigma^2 u_{01} + \theta_{01} = 0 \quad (37)$$

$$\frac{d^2 \theta_{01}}{dy^2} - \left(\frac{d\theta_{00}}{dy}\right)^2 = 0 \quad (38)$$

$$\frac{d^2 u_{10}}{dy^2} - \sigma^2 u_{10} + \theta_{10} = 0 \quad (39)$$

$$\frac{d^2 \theta_{10}}{dy^2} + \left(\frac{du_{00}}{dy}\right)^2 + \sigma^2 u_{00}^2 = 0 \quad (40)$$

$$\frac{d^2 u_{11}}{dy^2} - \sigma^2 u_{11} + \theta_{11} = 0 \quad (41)$$

$$\frac{d^2 \theta_{11}}{dy^2} - 2 \frac{d\theta_{00}}{dy} \frac{d\theta_{10}}{dy} + 2 \frac{du_{00}}{dy} \frac{du_{01}}{dy} \quad (42)$$

$$+ \theta_{00} \left(\frac{du_{00}}{dy}\right)^2 + 2\sigma^2 u_{00} u_{01} + \sigma^2 \theta_{00} u_{00}^2 = 0$$

The corresponding boundary conditions are

$$\left. \begin{aligned} u_{00} &= 0 \text{ at } y = \pm 1; \\ \theta_{00} &= 1 + m \text{ at } y = -1, \theta_{00} = 1 \text{ at } y = 1; \\ u_{01} &= 0 \text{ at } y = \pm 1; \theta_{01} = 0 \text{ at } y = \pm 1; \\ u_{10} &= 0 \text{ at } y = \pm 1; \theta_{10} = 0 \text{ at } y = \pm 1; \\ u_{11} &= 0 \text{ at } y = \pm 1; \theta_{11} = 0 \text{ at } y = \pm 1 \end{aligned} \right\} \quad (43)$$

Case 3: Effect of combined variable viscosity and thermal conductivity

The method of solutions of Eqs. (19) and (20) is similar to as in case 2, and we obtain the following equations

$$\frac{d^2 u_{00}}{dy^2} - b_v \frac{d\theta_{00}}{dy} \frac{du_{00}}{dy} - \sigma^2 u_{00} + \theta_{00} + b_v \theta_{00}^2 = 0 \quad (44)$$

$$\frac{d^2 \theta_{00}}{dy^2} = 0 \quad (45)$$

$$\frac{d^2 u_{01}}{dy^2} - b_v \frac{d\theta_{01}}{dy} \frac{du_{00}}{dy} - b_v \frac{d\theta_{00}}{dy} \frac{du_{01}}{dy} \quad (46)$$

$$- \sigma^2 u_{01} + \theta_{01} + 2b_v \theta_{00} \theta_{01} = 0$$

$$\frac{d^2 \theta_{01}}{dy^2} - \left(\frac{d\theta_{00}}{dy}\right)^2 = 0 \quad (47)$$

The solutions of the linear ordinary differential equations in all the above cases are found and present in graphs and tabular form.

Numerical solutions

The analytical solutions obtained in the above section are valid for small values of perturbation parameters. Further it is seen in the above section that it is not possible to find solutions of even the first order in all the cases. Hence we resort to solve the governing equations by numerical methods using Runge-Kutta shooting method (RKSM). The validity of RKSM is justified by comparing the solutions with the results obtained by the perturbation method and the values are displayed in tables. The perturbation method and RKSM solutions agree very well in the absence of perturbation parameter.

Skin friction and Nusselt number

In addition to the velocity and temperature fields, the following physical quantities can be defined:

The dimensionless skin friction at each boundary can be defined as

$$\tau_1 = \frac{du}{dy} \Big|_{y=-1} \text{ and } \tau_2 = \frac{du}{dy} \Big|_{y=1} \quad (48)$$

The dimensionless Nusselt number at each boundary can be defined as follows:

$$Nu_1 = \frac{d\theta}{dy} \Big|_{y=-1} \text{ and } Nu_2 = \frac{d\theta}{dy} \Big|_{y=1} \quad (49)$$

The above equations are solved and the results are tabulated in Tables 1-3 for different governing parameters.

Results and Discussion

The problem under study consists of the numerical and analytical investigation of flow and heat transfer in a vertical channel. The channel is filled with porous medium. The Darcy-Brinkman model is used to define the governing equations. Three cases are considered for the study, in the first case keeping thermal conductivity constant and exponential dependence of the varying viscosity on temperature is analyzed. Keeping the viscosity constant and the exponential dependence of thermal conductivity on temperature is discussed in case 2. The exponential dependent of both the viscosity and thermal conductivity on temperature is studied in case 3.

Due to the variation of viscosity and temperature, results in decomposing the viscous force term in the momentum equation and conducting term into two terms. The variations of these resulting terms with the variable viscosity parameter b_v and thermal conductivity parameter b_k and their relative magnitude have an importance on the flow and temperature fields. The major parameter such as wall temperature ratio m , buoyancy parameter N and porous parameter σ on the flow for positive and negative values of viscosity parameter b_v and the conductivity parameter b_k is numerically evaluated and depicted graphically. The governing equations which are highly nonlinear and coupled cannot be solved in a closed-form therefore the approximate solutions are found analytically using perturbation method (PM) whose solutions are restricted for small values of perturbation parameter. These restrictions are relaxed by finding the solutions numerically using Runge-Kutta shooting method (RKSM).

The validity of Runge-Kutta shooting method is justified by comparing the solutions of RKSM with PM. For temperature dependent viscosity the buoyancy parameter N is used as the perturbation parameter. The analytical solutions are found up to the first order ($\theta = \theta_0 + N\theta_1$). For temperature dependent thermal conductivity, the buoyancy parameter N and thermal conductivity parameter b_k are used as the perturbation parameter. Here also the analytical solutions are found up to first order ($u = u_{00} + b_k u_{01} + N(u_{10} + b_k u_{11})$), but the solution of u_{11} is not found. While finding out the perturbations solutions for temperature dependent viscosity and thermal conductivity, only zeroth order solutions are found ($u = u_{00} + b_k u_{01} + N(u_{10} + b_k u_{11})$). That is to say that the solutions of u_{00}, u_{01} are found, however u_{10}, u_{11} are not found. However the numerical solutions are found for any values of the governing parameter using RKSM method.

For temperature dependent viscosity (case 1), the effects of viscosity variation parameter b_v on the velocity and temperature fields are seen in Figs. 1 and 2 respectively. As the viscosity variation parameter b_v increases, flow increases and the profiles for constant viscosity ($b_v = 0$) lies above $b_v < 0$ and below $b_v > 0$ on velocity and temperature fields. The variable viscosity parameter b_v on the flow was the similar result observed by Attia [35] on the MHD channel flow of dusty fluids. The wall temperature ratio m fixing the values of N , b_v and σ can be viewed in Figs. 3 and 4. It is seen that as the

wall temperature ratio m increases the flow is enhanced, it is interesting to note that for negative values of m there is a flow reversal at the left wall. Since the wall temperature boundary condition are taken as $1+m$ at left wall and fixed as 1 at the right wall, the temperature profiles are varied at the left wall and remains constant at the right wall.

The buoyancy parameter also enhances the flow for increasing values of buoyancy parameter N as seen in Figs. 5 and 6. This result is due to the fact that physically an increase in the numerical values of buoyancy parameter N implies an increase of buoyancy force which will help in increasing the dissipations.

Therefore the temperature is increased and hence the velocity is also enhanced. The effect of N on the flow is the similar result observed by Umavathi [36]. The effect of porous parameter σ (increase of Darcy number) is to suppress the flow for fixed values of b_v, N, m . This is also an expected result because physically, large values of porous parameter σ correspond to densely packed porous medium and hence flow rate will be reduced. The effect of porous parameter σ on the flow is similar result observed by Umavathi and Veershetty [15] for constant properties.

The values of skin friction and Nusselt number on the variation of viscosity variation parameter b_v , wall temperature ratio m , buoyancy parameter N and porous parameter σ are tabulated in table 1. One can view that as b_v increases skin friction and Nusselt number increases at the left wall (hot wall) and decreases at the right wall. For increase in wall temperature ratio m , skin friction increases at the left wall and decreases at the right wall whereas the Nusselt number decreases at both the walls.

The buoyancy parameter N increases the skin friction and Nusselt number at the left wall and decreases at the right wall. The porous parameter σ decreases the skin friction and Nusselt number at the left wall and increases at the right wall.

Keeping viscosity constant and considering temperature dependent thermal conductivity, the effects of the governing parameters on the flow are analyzed. It is seen from Figs. 9 and 10 that as the thermal conductivity parameter increases both the velocity and temperature fields are suppressed. The profiles for constant thermal conductivity ($b_k = 0$) lies above $b_k > 0$ and below $b_k < 0$. The effect of thermal conductivity parameter b_k (Figs. 9 and 10) is in contrast with the effect of variable viscosity parameter b_v (Figs. 1 and 2). The effect of b_k is the similar result observed by Attia [37] and Palani and Kim [38]. The effects of wall temperature ratio m , buoyancy parameter N and porous parameter σ for variable thermal conductivity (constant viscosity) shows the similar results observed for variable viscosity (constant thermal conductivity) and hence not shown pictorially.

Table 2 displays the effects of variable thermal conductivity, wall temperature ratio, buoyancy parameter and porous parameter on skin friction and Nusselt number. It is seen that as the variable thermal conductivity parameter b_k increases, skin friction and Nusselt number decreases at the left wall and increases at right wall, which is in contrast with the effects of variable viscosity parameter with constant thermal conductivity.

The effect of wall temperature ratio m , buoyancy parameter N and porous parameter σ on skin friction and Nusselt number are the similar result observed for variable viscosity (table 1).

To understand the flow nature, both the viscosity and temperature are taken as temperature dependent and the results are drawn. Keeping the value for variable thermal conductivity parameter b_k fixed ($b_k = -0.2$), the effect of variable viscosity parameter b_v on the velocity and temperature fields are shown in Figs. 11 and 12.

The effect of b_v is to increase the velocity and temperature fields which is the similar result observed for constant thermal conductivity (Figs. 1 and 2). Keeping the value of variable viscosity parameter b_v fixed ($b_v = -0.2$), the effect of variable thermal conductivity parameter b_k on the flow is to suppress the velocity and temperature fields as seen in Figs. 13 and 14. This is also the similar result observed for constant viscosity (Figs. 9 and 10). The effects of wall temperature m and buoyancy parameter N and porous parameter σ for variations of viscosity and thermal conductivity on velocity and temperature reveal the similar results as observed for variation of thermal conductivity keeping the viscosity and thermal conductivity constant. The effect of b_k and b_v on the flow was the similar result observed by Attia [30] for hydro magnetic channel flow of dusty fluid.

The effect of variable viscosity, variable thermal conductivity, wall temperature ratio m , buoyancy parameter N and porous parameter σ on skin friction and Nusselt number is shown in table 3. Fixing $b_v = -0.2$, increasing the values of b_k decreases the skin friction and Nusselt number at the left wall and increases at the right wall.

Fixing $b_k = -0.2$ and increasing the values of b_v shows that skin friction and Nusselt number increases at the left wall and decreases at the right wall. The effects of m , N and σ on the skin friction and Nusselt number are the similar results as observed for variable viscosity with constant temperature as seen in table 1.

The analytical solutions obtained by regular perturbation methods are valid only for small values of perturbation parameter. To overcome this restriction the governing equations are solved using Runge-Kutta shooting method. The validity of Runge-Kutta shooting method is justified by comparing the results obtained by perturbation method and Runge-Kutta shooting method in the absence of buoyancy parameter N and displayed in tables 4, 5 and 6 for variable viscosity (constant thermal conductivity), variable thermal conductivity (constant viscosity) and combined effect of variable viscosity and thermal conductivity. It is viewed from tables 4 and 5 that the analytical and numerical solutions are exact for $N = 0$ and the error increases as buoyancy parameter N increases.

The computation to evaluate first order solutions for the variable viscosity and combined effect of variable viscosity and thermal conductivity was very tedious and hence table 4 (velocity) and table 6 gives the comparison for $N = 0$ only. Numerical solutions were obtained for the effects of buoyancy parameter N on the flow.

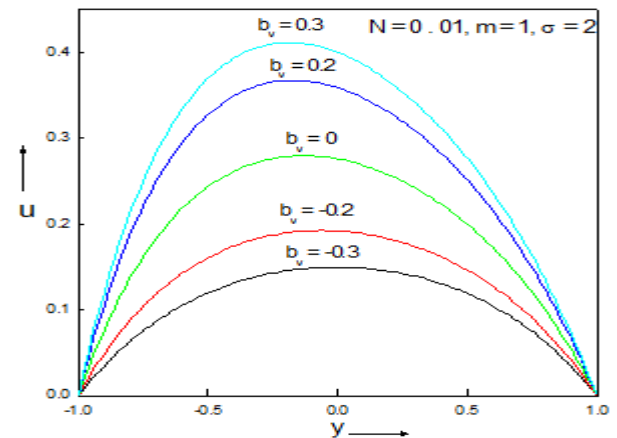


Fig 1: Velocity profiles of variable viscosity for different values of b_v

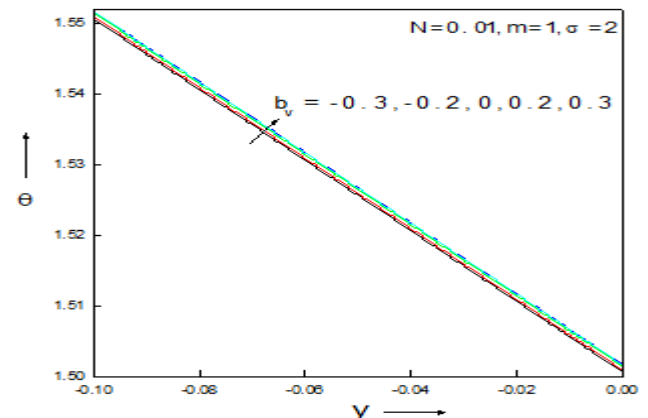


Fig 2: Temperature profiles of variable viscosity for different values of b_v

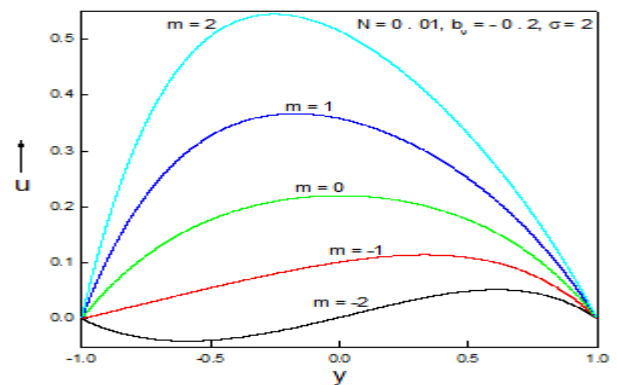


Fig 3: Velocity profiles of variable viscosity for different values of m

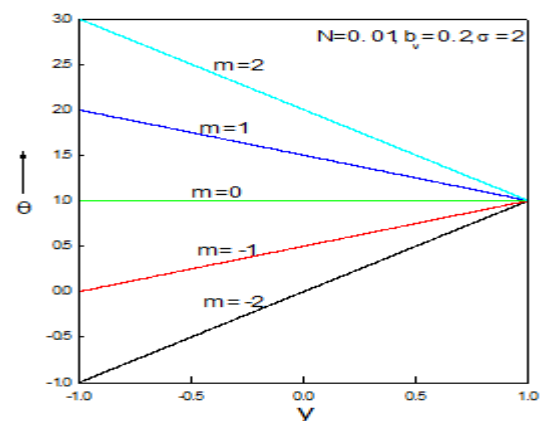


Fig 4: Temperature profiles of variable viscosity for different values of m

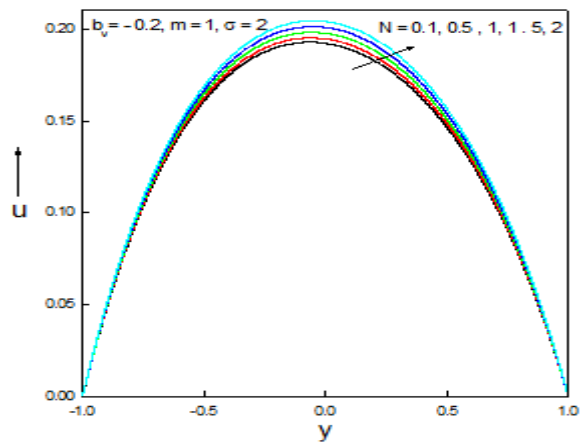


Fig5: Velocity profiles of variable viscosity for different values of N

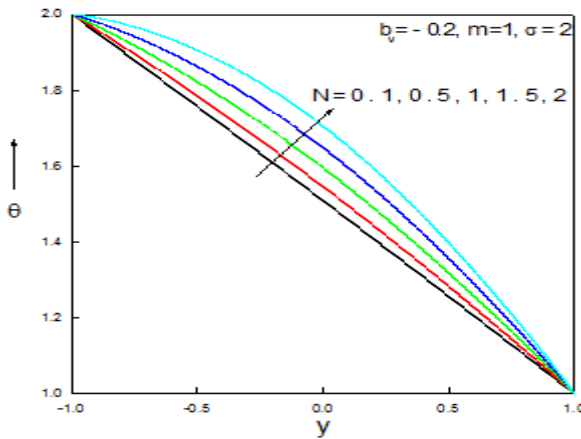


Fig6: Temperature profiles of variable viscosity for different values of N

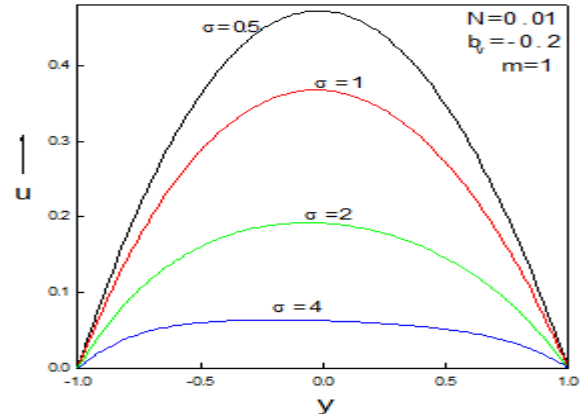


Fig7: Velocity profiles of variable viscosity for different values of σ

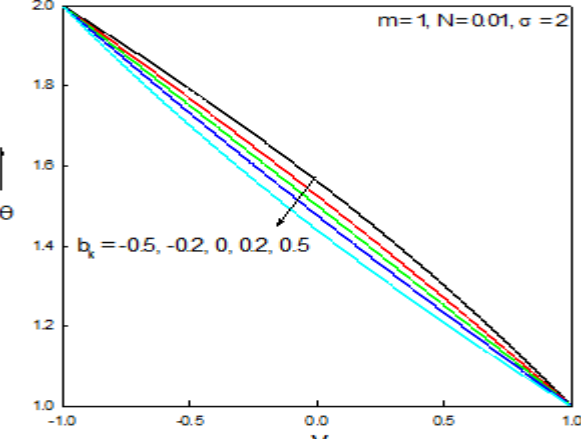


Fig10: Temperature profiles of variable conductivity for different values of b_k

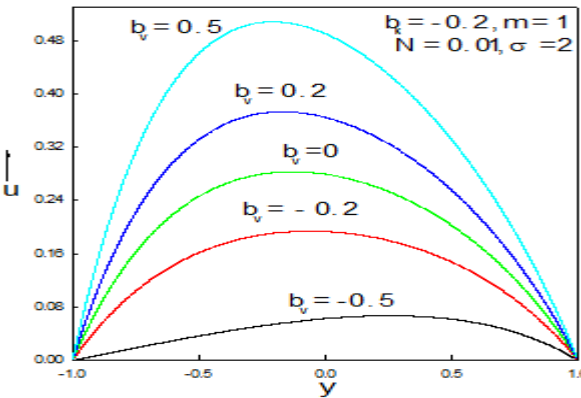


Fig11: Velocity profiles of combined effect for different values of b_v

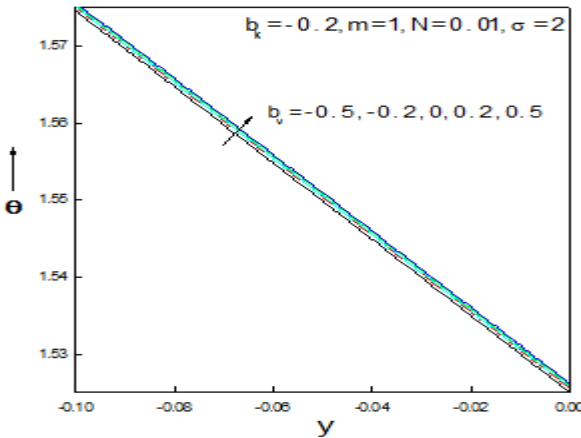


Fig12: Temperature profiles of combined effect for different values of b_k

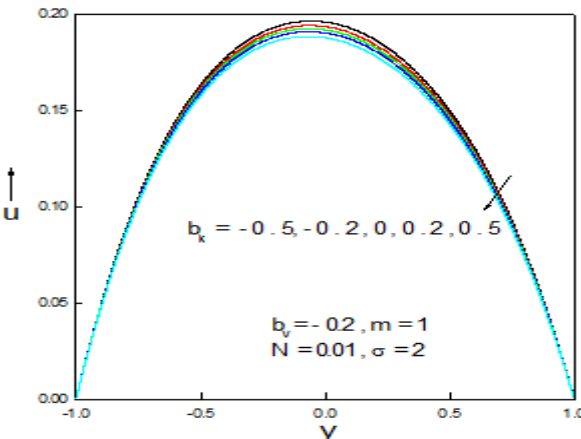


Fig13: Velocity profiles of combined effect for different values of b_k

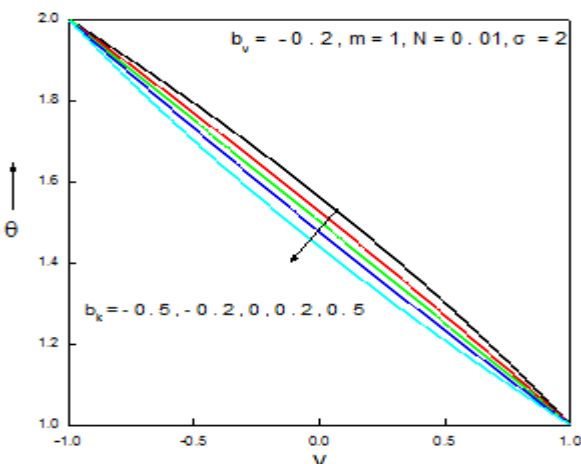


Fig14: Temperature profiles of combined effect for different values of b_k

Table 1. Computations showing the effect of parameter variations on skin friction and Nusselt number

	$\left. \frac{du}{dy} \right _{y=-1}$	$\left. \frac{du}{dy} \right _{y=1}$	$\left. \frac{d\theta}{dy} \right _{y=-1}$	$\left. \frac{d\theta}{dy} \right _{y=1}$
b_v	$m = 1, N = 0.01, \sigma = 2$			
-0.3	0.38145648	-0.38045140	-0.49872118	-0.50118111
-0.2	0.53735320	-0.45207723	-0.49801823	-0.50168887
0	0.85772612	-0.58904407	-0.49671324	-0.50263006
0.2	1.18950790	-0.71776138	-0.49617634	-0.50315186
0.3	1.35964416	-0.77907158	-0.49644120	-0.50313956
m	$b_v = -0.2, N = 0.01, \sigma = 2$			
-2	-0.31868740	-0.22038197	1.00017222	0.99985499
-1	0.10213422	-0.30768770	0.50022067	0.49959903
0	0.38569853	-0.38569853	0.00099517	-0.00099517
1	0.53735320	-0.45207723	-0.49801823	-0.50168887
2	0.56248705	-0.50429840	-0.99734663	-1.00224655
N	$b_v = -0.2, m = 1, \sigma = 2$			
0.1	0.53833444	-0.45332656	-0.48007239	-0.51699784
0.5	0.54280161	-0.45904030	-0.39784495	-0.58749532
1.0	0.54863447	-0.46656707	-0.28904914	-0.68163884
1.5	0.55474880	-0.47454173	-0.17301734	-0.78309566
2.0	0.56114336	-0.48298043	-0.04915279	-0.89255342
σ	$b_v = -0.2, m = 1, N = 0.01$			
0.5	0.98951126	-0.92049039	-0.49541455	-0.50398148
1	0.82485783	-0.74637653	-0.49638414	-0.50312954
2	0.53735320	-0.45207723	-0.49801823	-0.50168887
4	0.28856613	-0.21991601	-0.49926877	-0.50060036

Table 2. Computations showing the effect of parameter variations on skin friction and Nusselt number

	$\left. \frac{du}{dy} \right _{y=-1}$	$\left. \frac{du}{dy} \right _{y=1}$	$\left. \frac{d\theta}{dy} \right _{y=-1}$	$\left. \frac{d\theta}{dy} \right _{y=1}$
b_k	$m = 1, N = 0.01, \sigma = 2$			
-0.5	0.87257251	-0.60594900	-0.39311522	-0.64970441
-0.2	0.86391767	-0.59558818	-0.45109205	-0.55561438
0	0.85772612	-0.58904407	-0.49671324	-0.50263006
0.2	0.85120720	-0.58280944	-0.54894555	-0.45620265
0.5	0.84084685	-0.57405990	-0.64205481	-0.39694231
m	$b_k = 0.2, N = 0.01, \sigma = 2$			
-2	-0.29322463	-0.24151604	0.82435821	1.22939321
-1	0.10040558	-0.36875190	0.45340570	0.55296946
0	0.48220599	-0.48220600	0.00155574	-0.00155576
1	0.85120720	-0.58280944	-0.54894555	-0.45620265
2	1.20647129	-0.67152976	-1.21990999	-0.82898216
N	$b_k = 0.2, m = 1, \sigma = 2$			
0.1	0.85573035	-0.58675207	-0.50715941	-0.48400270
0.5	0.87814478	-0.60633636	-0.30312623	-0.62092645
1.0	0.91349890	-0.63736834	0.00912889	-0.83407020
1.5	0.96317581	-0.68121641	0.43023283	-1.12776538
σ	$b_k = -0.2, m = 1, N = 0.01$			
0.5	1.56619081	-1.23875413	-0.44845841	-0.55857233
1	1.31132138	-0.99871531	-0.44942879	-0.55746176
2	0.86391767	-0.59558818	-0.45109205	-0.55561438
3	0.61314993	-0.38951362	-0.45195603	-0.55469459

Table 3. Computations showing the effect of parameter variations on skin friction and Nusselt number

	$\left. \frac{du}{dy} \right _{y=-1}$	$\left. \frac{du}{dy} \right _{y=1}$	$\left. \frac{d\theta}{dy} \right _{y=-1}$	$\left. \frac{d\theta}{dy} \right _{y=1}$
b_k	$b_v = -0.2, m = 1, N = 0.01, \sigma = 2$			
-0.5	0.54418143	-0.46279409	-0.39325272	-0.64935035
-0.2	0.54022858	-0.45621055	-0.45192555	-0.55485191

0	0.53735320	-0.45207723	-0.49801823	-0.50168887
0.2	0.53428415	-0.44815616	-0.55073615	-0.45513300
0.5	0.52932132	-0.44267898	-0.64462066	-0.39574303
b_v	$b_k = -0.2, m = 1, N = 0.01, \sigma = 2$			
-0.5	0.07459621	-0.22948084	-0.45303962	-0.55376416
-0.2	0.54022858	-0.45621055	-0.45192555	-0.55485191
0	0.86391767	-0.59558818	-0.45109205	-0.55561438
0.2	1.19823334	-0.72586383	-0.45075153	-0.55603305
0.5	1.71946733	-0.90478127	-0.45227197	-0.55536292
m	$b_k = 0.2, b_v = 0.2, N = 0.01, \sigma = 2$			
-2	-0.24074670	-0.28719689	0.82433799	1.22938804
-1	0.10398294	-0.43692084	0.45343125	0.55291329
0	0.57868947	-0.57868948	0.00179261	-0.00179261
1	1.18037885	-0.70999399	-0.54819769	-0.45680450
2	1.90684981	-0.82954237	-1.21896865	-0.83003453
N	$b_k = 0.2, b_v = -0.2, m = -1, \sigma = 2$			
0.1	0.09645532	-0.30281524	0.45520172	0.54892797
0.5	0.09775298	-0.30422213	0.46352609	0.53025827
1.0	0.09944919	-0.30605406	0.47443394	0.50607824
1.5	0.10123522	-0.30797460	0.48595136	0.48088145
2	0.10312020	-0.30999252	0.49814136	0.45457006
σ	$b_k = 0.2, b_v = 0.2, m = 1, N = 0.01$			
0.5	2.12204243	-1.48305886	-0.54169170	-0.46205112
1	1.78082160	-1.19412800	-0.54406232	-0.46006281
2	1.18037885	-0.70999399	-0.54819769	-0.45680450
4	0.64909861	-0.33722557	-0.55149582	-0.45443597

Table 4. Comparison of velocity and temperature for various values of $b_v = 0.2, m = 1, \sigma = 2$ with Runge-Kutta shooting

method						
Velocity			Temperature			
	$N = 0$		$N = 0$		$N = 0.5$	
	Analytical	Numerical	Analytical	Numerical	Analytical	Numerical
1	0.00000000	0.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.2	0.32908640	0.32908640	1.40000000	1.40000000	1.47926354	1.48738864
0.0	0.35809109	0.35809109	1.50000000	1.50000000	1.58394643	1.59244325
-0.2	0.36648480	0.36648479	1.60000000	1.60000000	1.68136412	1.68950259
-1	0.00000000	0.00000000	2.00000000	2.00000000	2.00000000	2.00000000

Table 5. Comparison of velocity and temperature for various values of $b_k = -0.2, m = 1, \sigma = 2$ with Runge-Kutta shooting

method						
Velocity						
y	$N = 0$		$N = 0.01$		$N = 0.1$	
	Analytical	Numerical	Analytical	Numerical	Analytical	Numerical
1	0.00000000	0.00000001	0.00000000	0.00000000	0.00000000	0.00000000
0.2	0.26017279	0.26019434	0.26038334	0.26034619	0.26227833	0.26172587
0.0	0.27927980	0.27927518	0.27950263	0.27943401	0.28150817	0.28087694
-0.2	0.28185957	0.28182969	0.28207384	0.28198063	0.28400227	0.28335176
-1	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
Temperature						
y	$N = 0$		$N = 0.01$		$N = 0.1$	
	Analytical	Numerical	Analytical	Numerical	Analytical	Numerical
1	1.00000000	0.99999999	1.00000000	1.00000000	1.00000000	1.00000001
0.2	1.42400000	1.42428367	1.42533220	1.42525560	1.43732203	1.43408585
0.0	1.52500000	1.52495844	1.52641585	1.52596774	1.53915850	1.53513602
-0.2	1.62400000	1.62364607	1.62537770	1.62460586	1.63777699	1.63332361
-1	2.00000000	2.00000000	2.00000000	2.00000000	2.00000000	2.00000000

Table 6. Comparison of velocity and temperature for various values of $N = 0, b_v = 0.5, m = 1, \sigma = 2$ with

Runge-Kutta shooting method				
Velocity				
	$b_k = 0$		$b_k = 0.5$	
	Analytical	Numerical	Analytical	Numerical
1	0.00000000	0.00000000	0.00000000	0.00000000
0.2	0.43527624	0.43527623	0.41409899	0.41483107
0.0	0.48032956	0.48032955	0.45709881	0.45751414
-0.2	0.49845168	0.49845167	0.47563857	0.47561650
-1	0.00000000	0.00000000	0.00000000	0.00000000
Temperature				
	$b_k = 0$		$b_k = 0.5$	
	Analytical	Numerical	Analytical	Numerical
1	1.00000000	1.00000000	1.00000000	1.00000000
0.2	1.40000000	1.40000000	1.34000000	1.34249675
0.0	1.50000000	1.50000000	1.43750000	1.43814039
-0.2	1.60000000	1.60000000	1.54000000	1.53858861
-1	2.00000000	2.00000000	2.00000000	2.00000000

Conclusion

The problem of free convective flow in a vertical channel filled with porous medium was analyzed for the variation of viscosity (constant thermal conductivity), variation of thermal conductivity (constant viscosity) and for the variation of both viscosity and thermal conductivity on the temperature was studied. The analytical solution were found by perturbation parameter method valid for small values of perturbation parameter and numerical solutions were found by Runge-Kutta shooting method valid for any values of governing parameters. The Runge-Kutta shooting method and perturbation method show good agreement in the absence of buoyancy parameter. The following results were drawn.

1. Increase in the variable viscosity enhances the flow and heat transfer whereas increase in the variable thermal conductivity suppresses the flow and heat transfer for variable viscosity, variable thermal conductivity and their combined effect.
2. The wall temperature ratio and buoyancy parameter enhances the flow for variable viscosity, variable thermal conductivity and their combined effect where as porous parameter suppresses the flow in all the three cases.
3. The solutions obtained by Runge-Kutta shooting method and perturbation method are exact in the absence of buoyancy parameter and the error increases as the buoyancy parameter increases.

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