Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 86 (2015) 35377-35382

ISSN: 2229-712X

Mathematical study of the effect of industrialization on the resource biomass under going harvesting and diffusion in heterogeneous habitat

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ARTICLE INFO

Article history:
Received: 13 May 2012;
Received in revised form: 25 September 2015;
Accepted: 30 September 2015;

Keywords
Biomass,
Diffusion,
Haditat.

ABSTRACT

In this chapter, a Mathematical model is proposed to the study of depletion of a uniformly distributed forest biomass caused by different levels of industrialization and population in two adjoining regions of the habitat. Industrialization dependent, constant, instantaneous, and periodic emissions of pollutant into the environment are taken into consideration. Criteria for local stability, instability, and global stability of non-negative equilibrium are obtained in the absence of diffusion and in presence of diffusion. A model of a single species population living in two patch habitats with migration between them across a barrier was proposed by Freedman and Waltman [5]. The model was extended in [17,19] to include the case where animal species leaving one habitat does not necessarily reach the other habitat, the existence of a positive equilibrium as a function of barrier strengths was examined. Also Freedman [7] studied a single species diffusion model by assuming that the habitat consists of two patches and has shown that there exists a positive, monotonic, continuous non uniform steady state solution that is linearly asymptotically stable under both reservoir and no-flux boundary conditions.

Introduction

Depletion of forest resource by industrialization and rapid growth of population, particularly in the third world country is of grave concern. A typical example in this regard is the degradation of the forestry resources in the Doon Valley located in the foothills of Himalayas, Uttaranchal, India. Here the degradation of forest has been caused mainly by limestone quarries, paper, other wood-based industries and associated population growth [14]. [21] have proposed a Mathematical model for forest degradation caused by resource independent industrialization population by considering the spatial distribution of both the forest biomass as well as the density of industrialization by studying the behavior of uniform steady state solution.

It may be pointed out here that in real ecological situations when forest is degraded by industrialization distributed spatially, patchiness is caused in the forest habitat. It is worth nothing that little efforts have been made to study such systems using mathematical models [6, 16, 21, and 23]. How ever in [21], has not considered the effect of patchiness caused by industrialization. Further [2-4,7] studied a single species diffusion model by assuming that the habitat consists of two adjoining patches and studied the behavior of steady state distribution and local asymptotically stable conditions.

Biotic populations are usually distributed non-uniformly in their habitat, and the distribution is often patchy, due to patchiness of the habitat which arises from a variety of mechanisms and processes under various conditions including deforestation in the case of a forest habitat. It would, thus, seem natural to study the population dynamics of a single species by including diffusion effects, in a patchy habitat. Many investigators [1,5,6,8-13] have shown that in a homogeneous habitat, the diffusion , increases the stability of the system, but this may not always be true, if the habitat is patchy [7,15,17,18].

A model of a single species population living in two patch habitats with migration between them across a barrier was proposed by Freedman and Waltman [5]. The model was extended in [17,19] to include the case where animal species leaving one habitat does not necessarily reach the other habitat, the existence of a positive equilibrium as a function of barrier strengths was examined. Also Freedman [7] studied a single species diffusion model by assuming that the habitat consists of two patches and has shown that there exists a positive, monotonic, continuous non uniform steady state solution that is linearly asymptotically stable under both reservoir and no-flux boundary conditions.

Mathematical Model

We consider a forest habitat $0 \le s \le L_2$ linearly distributed, where forest resources are depleted by different levels of industrialization in the above adjoining regions $0 \le s \le L_1$ and $L_1 \le s \le L_2$ where $L_0 = 0$ and L_1 is the interface of two regions. Let $R_i(s,t)$ *and* $I_i(s,t)$, (i=1,2) be respectively the densities of resource biomass and industrialization (population) pressure at location s and time t in the above mentioned ith regions [see figure 1]. It is assumed that $R_i(s,t)$ grows logistically in both the region with the same intrinsic growth rate and carrying capacity, i.e. in absence of industrialization leading to a uniform spatial distribution. Since the levels of industrialization are assumed to be different in these two regions, the growth rate α_i and carrying capacity

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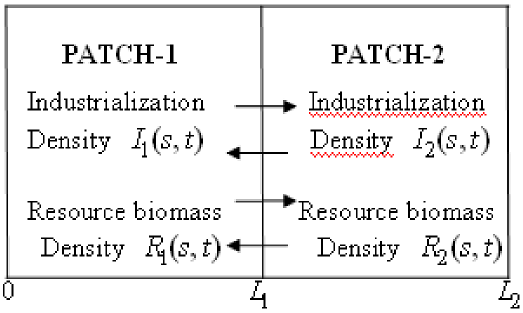
M_i of $I_i(s,t)$ for $i=1$, 2 will be different. Since the industrialization is partially dependent on resource biomass, it is assumed that the rate of depletion of biomass density is equal to biomass density is equal to $\beta_i R_i I_i$. The harvesting in resource biomass and its diffusion in both the patches is also considered here.



Keeping all these in view, our proposed mathematical model is given by the following system of partial differential equations:

$$\frac{\partial R_i}{\partial t} = rR_i - \frac{r_0 R_i^2}{c} - \beta_i R_i I_i - R_i P_i(R_i(s,t)) + d_i \frac{\partial^2 R_i}{\partial s^2} \tag{1.1}$$

$$\frac{\partial I_i}{\partial t} = \alpha_i I_i - \alpha_{i0} I_i^2 + D_i \frac{\partial^2 I_i}{\partial s^2} \tag{1.2}$$

Where

$r =$ Intrinsic growth rate of resource biomass in both regions

$c =$ Carrying capacity of resource biomass in both regions

$\beta_i =$ Depletion coefficient in the i[th] region; i=1, 2

$d_i =$ Diffusion coefficient of $R_i(s,t)$ in the i[th] region; i=1, 2

$D_i =$ Diffusion coefficient of $I_i(s,t)$ in the i[th] region; i=1, 2

$\alpha_i =$ Growth rate of $I_i(s,t)$ in the i[th] region; i=1, 2

$P_i(R_i(s,t)) =$ Harvesting functional response function such that $P_i(0)>0$, for $R_i \ge 0$; $P_i'(R_i)<0$ and when the habitat has carrying capacity C in the i[th] patch , then $P_i(C)=0, \forall i$.

Uniform steady states

To analyse the model (1.1) and (1.2) , we have taken the harvesting function in the following form:

$P_i(R_i(s,t)) = a_i R_i$

Using this value in (1.1), the positive uniform equilibrium point $E_i\ (R_i^*, I_i^*)$ is given by following equations:

$$R_i^* = \frac{[r\alpha_{i0} - \alpha_i\beta_i]}{W_i\alpha_{i0}} > 0 \quad \text{If } r\alpha_{i0} > \alpha_i\beta_i \tag{2.1}$$

$$I_i^* = \frac{\alpha_i}{\alpha_{i0}} > 0 \tag{2.2}$$

Where $W_i = \frac{r_0}{C} + a_i > 0 \quad \forall i = 1,2$

From equation (2.1) it is clear that

$$R_i^* = \frac{r - \beta_i I_i^*}{W_i} \le \frac{r}{W_i} \tag{3.1}$$

We also observe that in absence of industrialization ($i.e.\ \beta_i = 0$, i=1,2)

$$R_1^* = R_2^* = \frac{r}{W_i} \tag{3.2}$$

This shows that the biomass is uniformly distributed in entire habitat.

The model (1.1) and (1.2) is studied by assuming the following initial, boundary and flux-matching conditions. The model is completed by assuming some positive initial distribution for forest resource biomass and industrialization, that is,

$$R_i(s,0) = \chi_i(s) > 0 \quad , L_{i-1} < s < L_i \quad i = 1,2 \tag{4.1}$$

$$I_i(s,0) = \delta_i(s) > 0 \quad , L_{i-1} < s < L_i \quad i = 1,2 \tag{4.2}$$

If the region is closed then there is no diffusion of industrialization and resource biomass across the boundary, no-flux boundary condition for forest resource biomass and industrialization are

$$\frac{\partial R_1(0,t)}{\partial s} = 0 = \frac{\partial R_2(L_2,t)}{\partial s} \quad , \quad \frac{\partial I_1(0,t)}{\partial s} = 0 = \frac{\partial I_2(L_2,t)}{\partial s} \tag{4.3}$$

And finally considering the continuity and the flux-matching conditions at the interface $s = L_1$ for $R_i(s,t)$ and $I_i(s,t)$

$$R_1(L_1,t) = R_2(L_1,t)\ , \quad d_1 \frac{\partial R_1}{\partial s}(L_1,t) = d_2 \frac{\partial R_2}{\partial s}(L_1,t) \quad \forall t \ge 0 \tag{4.4}$$

$$I_1(L_1,t) = I_2(L_1,t)\ , \quad D_1 \frac{\partial I_1}{\partial s}(L_1,t) = D_2 \frac{\partial I_2}{\partial s}(L_1,t) \quad \forall t \ge 0 \tag{4.5}$$

Linear and Non-linear stability analysis

Theorem-1 The steady state solution of the system (1.1) and (1.2) with conditions (4.1-4.5) is locally asymptotically stable if the following conditions holds

(i) $A_i = r - 2W_i R_i^* - \beta_i I_i^* \le 0$

(ii) $B_i = \alpha_i - 2\alpha_{i0} I_i^* \le 0$

(iii) $\beta_i^2 R_i^{*2} \le 4A_i B_i$

Proof:

Let $\quad R_i(s,t) = R_i^* + m_i(s,t) \quad (5.1)$

$$I_i(s,t) = I_i^* + n_i(s,t) \tag{5.2}$$

Where $m_i(s,t)$ *and* $n_i(s,t)$ are the small perturbations around equilibrium states.

Using (5.1) and (5.2), the linearised system of differential equations for the equilibrium point E_i is given by (6.1) and (6.2) as follows:

$$\frac{\partial m_i}{\partial t} = m_i\left[r - 2W_i R_i^* - \beta_i I_i^*\right] - n_i(\beta_i R_i^*) + d_i \frac{\partial^2 m_i}{\partial s^2} \tag{6.1}$$

$$\frac{\partial n_i}{\partial t} = n_i\left[\alpha_i - 2\alpha_{i0} I_i^*\right] + D_i \frac{\partial^2 n_i}{\partial s^2} \tag{6.2}$$

Now consider the following Liapunov function

$$V(s,t) = \frac{1}{2}\sum_{i=1}^{2} \int_{L_{i-1}}^{L_i} (m_i^2 + n_i^2)\,ds$$

Its time derivative is given by

$$\frac{\partial V}{\partial t} = \sum_{i=1}^{2} \int_{L_{i-1}}^{L_i} \left(m_i \frac{\partial m_i}{\partial t} + n_i \frac{\partial n_i}{\partial t}\right) ds \tag{7.1}$$

Now using (7.1) and (7.2) in (8.1) we get:

$$\frac{\partial V}{\partial t} = \sum_{i=1}^{2} \int_{L_{i-1}}^{L_i} \left\{ m_i^2 \left[r - 2W_i R_i^* - \beta_i I_i^* \right] + n_i^2 \left[\alpha_i - 2\alpha_{i0} I_i^* \right] - (m_i n_i)(\beta_i R_i^*) \right\} ds$$
$$+ \sum_{i=1}^{2} d_i \int_{L_{i-1}}^{L_i} m_i \frac{\partial^2 m_i}{\partial s^2} ds + \sum_{i=1}^{2} D_i \int_{L_{i-1}}^{L_i} n_i \frac{\partial^2 n_i}{\partial s^2} ds \quad (7.2)$$

Using boundary and flux-matching conditions for forest resource and industrialization density (4.3)-(4.5), we get

$$\sum_{i=1}^{2} d_i \int_{L_{i-1}}^{L_i} m_i \frac{\partial^2 m_i}{\partial s^2} ds = - \sum_{i=1}^{2} d_i \int_{L_{i-1}}^{L_i} \left(\frac{\partial m_i}{\partial s} \right)^2 ds \quad (8.1)$$

$$\sum_{i=1}^{2} D_i \int_{L_{i-1}}^{L_i} n_i \frac{\partial^2 n_i}{\partial s^2} ds = - \sum_{i=1}^{2} D_i \int_{L_{i-1}}^{L_i} \left(\frac{\partial n_i}{\partial s} \right)^2 ds \quad (8.2)$$

Using Sylvester's criteria for (8.2) and choosing

(i) $A_i = r - 2W_i R_i^* - \beta_i I_i^* \le 0$

(ii) $B_i = \alpha_i - 2\alpha_{i0} I_i^* \le 0$

(iii) $\beta_i^2 R_i^{*2} \le 4 A_i B_i$

It is shown that $\frac{\partial V}{\partial t}$ is negative definite and hence it is proved that E_i is locally asymptotically stable.

Theorem-2 The steady state solution of the system (1.1) and (1.2) with conditions (4.1-4.5) is global asymptotically stable if the following condition $\beta_i^2 < 4\alpha_{i0} W_i$ holds.

Proof: Using the same transformation as taken in (5.1) and (5.2), the nonlinearised system of differential equations for the equilibrium point E_i is given below by (9.1) and (9.2)

$$\frac{\partial m_i}{\partial t} = (R_i^* + m_i)\left[-W_i m_i - \beta_i n_i \right] + d_i \frac{\partial^2 m_i}{\partial s^2} \quad (9.1)$$

$$\frac{\partial n_i}{\partial t} = (I_i^* + n_i)\left[-\alpha_{i0} n_i \right] + D_i \frac{\partial^2 n_i}{\partial s^2} \quad (9.2)$$

Now consider the following Liapunov function

$$V(s,t) = \sum_{i=1}^{2} \int_{L_{i-1}}^{L_i} \left[\left\{ m_i - R_i^* \ln\left(1 + \frac{m_i}{R_i^*}\right) \right\} + \left\{ n_i - I_i^* \ln\left(1 + \frac{n_i}{I_i^*}\right) \right\} \right] ds$$

Its time derivative is given by

$$\frac{\partial V}{\partial t} = \sum_{i=1}^{2} \int_{L_{i-1}}^{L_i} \left[\left(\frac{m_i}{R_i^* + m_i} \right) \frac{\partial m_i}{\partial t} + \left(\frac{n_i}{I_i^* + n_i} \right) \frac{\partial n_i}{\partial t} \right] ds \quad (10)$$

Now using (9.1) and (9.2) in (10) we get:

$$\frac{\partial V}{\partial t} = -\sum_{i=1}^{2} \int_{L_{i-1}}^{L_i} \left\{ W_i m_i^2 + \beta_i m_i n_i + \alpha_{i0} n_i^2 \right\} ds + \sum_{i=1}^{2} d_i \int_{L_{i-1}}^{L_i} \left(\frac{m_i}{R_i^* + m_i} \right) \frac{\partial^2 m_i}{\partial s^2} ds + \sum_{i=1}^{2} D_i \int_{L_{i-1}}^{L_i} \left(\frac{n_i}{I_i^* + n_i} \right) \frac{\partial^2 n_i}{\partial s^2} ds$$

Using boundary and flux-matching conditions for forest resource and industrialization density (4.3)-(4.5), we get

$$\sum_{i=1}^{2} d_i \int_{L_{i-1}}^{L_i} \left(\frac{m_i}{R_i^* + m_i} \right) \frac{\partial^2 m_i}{\partial s^2} ds = - \sum_{i=1}^{2} d_i \int_{L_{i-1}}^{L_i} \frac{R_i^*}{\left(R_i^* + m_i\right)^2} \left(\frac{\partial m_i}{\partial s} \right)^2 ds \quad (11.1)$$

And

$$\sum_{i=1}^{2} D_i \int_{L_{i-1}}^{L_i} \left(\frac{n_i}{I_i^* + n_i} \right) \frac{\partial^2 n_i}{\partial s^2} ds = - \sum_{i=1}^{2} D_i \int_{L_{i-1}}^{L_i} \frac{I_i^*}{\left(I_i^* + n_i\right)^2} \left(\frac{\partial n_i}{\partial s} \right)^2 ds \quad (11.2)$$

Using Sylvester's criteria that $\frac{\partial V}{\partial t}$ is negative definite provided that the following condition is satisfied,

$$\beta_i^2 < 4\alpha_{i0} W_i$$

Hence E_i is globally asymptotically stable if $\beta_i^2 < 4\alpha_{i0} W_i$.

Table-1

Parameters	Patch-1	Figure	Patch-2	Figure
	r_0		r_0	
Intrinsic growth rate of Resource biomass	r	0.06	r	0.08
Carrying capacity of Resource biomass	C	400	C	800
Depletion rate of forest resource	β_1	0.00004	β_2	0.00002
Growth rate of the resource	a_1	6	a_2	7
Growth rate of the Industrialization	α_1	8.759	α_2	74.97
	α_{10}	0.00003	α_{20}	0.00001
Diffusion coefficient of Resource biomass	d_1	0.5	d_2	0.6
Diffusion coefficient of Industrialization	D_1	0.7	D_2	0.8

Table-2

Parameters (Patch-1)	Figure	Parameters (Patch-2)	Figure
R_1^*	0.94492378	R_2^*	0.967
I_1^*	400	I_2^*	800

MODEL IN HOMOGENEOUS HABITAT

MODEL WITHOUT DIFFUSION

Here the forest resource biomass and industrialization (population) are uniformly distributed throughout the habitat ($0 \le s \le L_2$) with no diffusion. Let R and I denote the forest resource and Industrialization densities respectively at the location s and at any time t. Then our model takes the following form

$$\frac{\partial R}{\partial t} = rR - \frac{r_0 R^2}{c} - \beta RI - aR^2 \quad (12.1)$$

$$\frac{\partial I}{\partial t} = \alpha I - \alpha_0 I^2 \quad (12.2)$$

Uniform equilibrium states

The uniform positive equilibrium point $E^*(R^*, I^*)$ becomes

$$R^* = \frac{\left[r\alpha_0 - \alpha\beta\right]}{W\alpha_0} > 0 \quad if \quad r\alpha_0 > \alpha\beta \quad (13.1)$$

$$I^* = \frac{\alpha}{\alpha_0} > 0 \quad (13.2)$$

Where $W = \frac{r_0}{c} + a$

In this case the initial and boundary conditions of the forestry resource biomass and Industrialization become

$$R(s,0) = \chi(s) > 0 \quad , L_0 < s < L_2 \quad (14.1)$$

$$I(s,0) = \delta(s) > 0 \quad , L_0 < s < L_2 \quad (14.2)$$

And $$\frac{\partial R(0,t)}{\partial s} = 0 = \frac{\partial R(L_2,t)}{\partial s} \quad , \frac{\partial I(0,t)}{\partial s} = 0 = \frac{\partial I(L_2,t)}{\partial s} \quad (14.3)$$

Linear and Non-linear stability analysis

Theorem 3 The steady state solution of the system (12.1) and (12.2) with conditions (14.1-14.3) is locally asymptotically stable if the following conditions holds

(i) $A = r - 2WR^* - \beta I^* \leq 0$

(ii) $B = \alpha - 2\alpha_0 I^* \leq 0$

(iii) $\beta^2 R^{*2} \leq 4AB$

Proof

Let $R = R^* + m(s,t)$ (15.1)

$I = I^* + n(s,t)$ (15.2)

Where $m(s,t)$ *and* $n(s,t)$ are the small perturbations around equilibrium states. Using these, the linearised system of differential equations for the equilibrium point E^* is given by (16.1) and (16.2) as follows:

$$\frac{\partial m}{\partial t} = m\left[r - 2WR^* - \beta I^*\right] - n(\beta R^*) \quad (16.1)$$

$$\frac{\partial n}{\partial t} = n\left[\alpha - 2\alpha_0 I^*\right] \quad (16.2)$$

Now consider the following Liapunov function

$$V(s,t) = \frac{1}{2}\int_{L_0}^{L_2}(m^2 + n^2)ds$$

Its time derivative is given by

$$\frac{\partial V}{\partial t} = \int_{L_0}^{L_2}\left(m\frac{\partial m}{\partial t} + n\frac{\partial n}{\partial t}\right)ds \quad (17)$$

Now using (16.1) and (16.2) in (17) we get :

$$\frac{\partial V}{\partial t} = \int_{L_0}^{L_2}\left\{m^2\left[r - 2WR^* - \beta I^*\right] + n^2\left[\alpha - 2\alpha_0 I^*\right] - (mn)(\beta R^*)\right\}ds$$

Using Sylvester's criteria and choosing

(i) $A = r - 2WR^* - \beta I^* \leq 0$

(ii) $B = \alpha - 2\alpha_0 I^* \leq 0$

(iii) $\beta^2 R^{*2} \leq 4AB$

It is shown that $\frac{\partial V}{\partial t}$ is negative definite and hence it is proved that E^* is locally asymptotically stable.

Theorm 4

The steady state solution of the system (12.1) and (12.2) with conditions (14.1-14.3) is global asymptotically stable if the following condition $\beta_i^2 < 4\alpha_{i0}W_i$ holds.

Proof: Using the same transformation as taken in (15.1) and (15.2), the non linearised system of differential equations for the equilibrium point E^* is given below by (18.1) and (18.2)

$$\frac{\partial m}{\partial t} = (R^* + m)\left[-Wm - \beta n\right] \quad (18.1)$$

$$\frac{\partial n}{\partial t} = (I^* + n)\left[-\alpha_0 n\right] \quad (18.2)$$

Now consider the following Liapunov function

$$V(s,t) = \int_{L_0}^{L_2}\left[\left\{m - R^*\ln\left(1 + \frac{m}{R^*}\right)\right\} + \left\{n - I^*\ln\left(1 + \frac{n}{I^*}\right)\right\}\right]ds$$

Its time derivative is given by

$$\frac{\partial V}{\partial t} = \int_{L_0}^{L_2}\left[\left(\frac{m}{R^* + m}\right)\frac{\partial m}{\partial t} + \left(\frac{n}{I^* + n}\right)\frac{\partial n}{\partial t}\right]ds \quad (19)$$

Now using (20.2) and (20.2) in (21) we get:

$$\frac{\partial V}{\partial t} = -\int_{L_0}^{L_2}\left\{Wm^2 + \beta mn + \alpha_0 n^2\right\}ds$$

It is then found using Sylvester's criteria that $\frac{\partial V}{\partial t}$ is negative definite proving E^* is global asymptotically stable in the region $(0, L_2)$ provided $\beta_i^2 < 4\alpha_{i0}W_i$.

Table 1

Parameters	Patch-1	Figure	Patch-2	Figure
	r_0		r_0	
Intrinsic growth rate of Resource biomass	r	0.06	r	0.08
Carrying capacity of Resource biomass	C	400	C	800
Depletion rate of forest resource	β_1	0.00004	β_2	0.00002
Growth rate of the resource	a_1	6	a_2	7
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Diffusion coefficient of Industrialization	D_1	0.7	D_2	0.8

Table-2

Parameters (Patch-1)	Figure	Parameters (Patch-2)	Figure
R_1^*	0.94492378	R_2^*	0.967
I_1^*	400	I_2^*	800

Model with diffusion

Now we study of the behavior of the uniform steady state solution of the model (1.1) and (1.2) with diffusion in a single homogeneous habitat. In this case we consider that both the Industrialization (population) and forest resource is not spatially uniformly distributed. Here $R_i(s,t), I_i(s,t), d_i$, D_i, β_i becomes $R(s,t), I(s,t), d, D$ and β respectively ; $\forall s \in [0, L_2]$. Then the model in this case can be written as

$$\frac{\partial R}{\partial t} = rR - \frac{r_0 R^2}{c} - \beta RI - RP(R(s,t)) + d\frac{\partial^2 R}{\partial s^2} \quad (19.1)$$

$$\frac{\partial I}{\partial t} = \alpha I - \alpha_0 I^2 + D\frac{\partial^2 I}{\partial s^2} \quad (19.2)$$

Uniform equilibrium states

To analysis the model (19.1) and (19.2), we have taken the harvesting function in the following form:

Using this value in (19.1), we get positive equilibrium points

$$R^* = \frac{\left[r\alpha_0 - \alpha\beta\right]}{W\alpha_0} > 0 \quad r\alpha_0 > \alpha\beta \quad (20.1)$$

$$I^* = \frac{\alpha}{\alpha_0} > 0 \tag{20.2}$$

Where $W = \frac{r_0}{C} + a \ > 0$

In this case the initial and boundary conditions of the forestry resource biomass and industrialization become

$$R\,(s,0) = \chi\,(s) > 0 \qquad , L_0 < s < L_2 \tag{21.1}$$

$$I\,(s,0) = \delta\,(s) > 0 \qquad , L_0 < s < L_2 \tag{21.2}$$

And $\frac{\partial R(0,t)}{\partial s} = 0 = \frac{\partial R(L_2,t)}{\partial s}$, $\frac{\partial I\,(0,t)}{\partial s} = 0 = \frac{\partial I\,(L_2,t)}{\partial s}$ (21.3)

Linear and Non-linear stability analysis

Theorem 5 The steady state solution of the system (19.1) and (19.2) with conditions (21.1-21.3) is locally asymptotically stable if it satisfies the same conditions as in theorem-3 .

Proof: Using the same transformation as taken in (15.1) and (15.2), the linearised system of differential equations for the equilibrium point E^* is given below by (22.1) and (22.2)

$$\frac{\partial m}{\partial t} = m\left[r - 2WR^* - \beta I^*\right] - n(\beta R^*) + d\,\frac{\partial^2 m}{\partial s^2} \tag{22.1}$$

$$\frac{\partial n}{\partial t} = n\left[\alpha - 2\alpha_0 I^*\right] + D\,\frac{\partial^2 n}{\partial s^2} \tag{22.2}$$

Now consider the following Liapunov function

$$V(s,t) = \frac{1}{2}\int_{L_0}^{L_2} (m^2 + n^2)ds$$

Its time derivative is given by

$$\frac{\partial V}{\partial t} = \int_{L_0}^{L_2} \left(m\,\frac{\partial m}{\partial t} + n\,\frac{\partial n}{\partial t}\right)ds \tag{23.1}$$

Now using (22.1) and (22.2) in (23.1) we get :

$$\frac{\partial V}{\partial t} = \int_{L_0}^{L_2} \left\{m^2\left[r - 2WR^* - \beta I^*\right] + n^2\left[\alpha - 2\alpha_0 I^*\right] - (mn)(\beta R^*)\right\}ds$$

$$+\, d\int_{L_0}^{L_2} m\,\frac{\partial^2 m}{\partial s^2}\,ds + D\int_{L_0}^{L_2} n\,\frac{\partial^2 n}{\partial s^2}\,ds \tag{23.2}$$

Using boundary and flux-matching conditions for forest resource and industrialization density (21.1)-(21.3) , we get

$$\int_{L_0}^{L_2} m\,\frac{\partial^2 m}{\partial s^2}\,ds = -\int_{L_0}^{L_2}\left(\frac{\partial m}{\partial s}\right)^2 ds \tag{24.1}$$

$$\int_{L_0}^{L_2} n\,\frac{\partial^2 n}{\partial s^2}\,ds = -\int_{L_0}^{L_2}\left(\frac{\partial n}{\partial s}\right)^2 ds \tag{24.2}$$

Using Sylvester's criteria for (23.2) and choosing

(i) $A = r - 2WR^* - \beta I^* \le 0$

(ii) $B = \alpha - 2\alpha_0 I^* \le 0$

(iii) $\beta^2 R^{*2} \le 4AB$

It is shown that $\frac{\partial V}{\partial t}$ is negative definite and hence it is proved that E^* is locally asymptotically stable.

Theorm 6 The steady state solution of the system (12.1) and (12.2) with conditions (14.1-14.3) is global asymptotically stable if it satisfies the same condition an in theorem-4.

Proof: Using the same transformation as taken in (15.1) and (15.2), the nonlinearised system of differential equations for the equilibrium point E^* is given below by

$$\frac{\partial m}{\partial t} = (R^* + m)\left[-Wm - \beta n\right] + d\,\frac{\partial^2 m}{\partial s^2} \tag{25.1}$$

$$\frac{\partial n}{\partial t} = -\alpha_0 n(I^* + n)\left[-\alpha_0 n\right] + D\,\frac{\partial^2 n}{\partial s^2} \tag{25.2}$$

Now consider the following Liapunov function

$$V(s,t) = \int_{L_0}^{L_2}\left[\left\{m - R^*\ln\left(1 + \frac{m}{R^*}\right)\right\} + \left\{n - I^*\ln\left(1 + \frac{n}{I^*}\right)\right\}\right]ds$$

Its time derivative is given by

$$\frac{\partial V}{\partial t} = \int_{L_0}^{L_2}\left[\left(\frac{m}{R^* + m}\right)\frac{\partial m}{\partial t} + \left(\frac{n}{I^* + n}\right)\frac{\partial n}{\partial t}\right]ds \tag{26}$$

Now using (25.1) and (25.2) in (26) we get:

$$\frac{\partial V}{\partial t} = -\int_{L_0}^{L_2}\left\{Wm^2 + \beta mn + \alpha_0 n^2\right\}ds$$

$$+\, d\int_{L_0}^{L_2}\left(\frac{m}{R^* + m}\right)\frac{\partial^2 m}{\partial s^2}\,ds + D\int_{L_0}^{L_2}\left(\frac{n}{I^* + n}\right)\frac{\partial^2 n}{\partial s^2}\,ds$$

But

$$\int_{L_0}^{L_2}\left(\frac{m}{R^* + m}\right)\frac{\partial^2 m}{\partial s^2}\,ds = -\int_{L_0}^{L_2}\frac{R^*}{(R^* + m)^2}\left(\frac{\partial m}{\partial s}\right)^2 ds \tag{27.1}$$

$$\int_{L_0}^{L_2}\left(\frac{n}{I^* + n}\right)\frac{\partial^2 n}{\partial s^2}\,ds = -\int_{L_0}^{L_2}\frac{I^*}{(I^* + n)^2}\left(\frac{\partial n}{\partial s}\right)^2 ds \tag{27.2}$$

It is then found using Sylvester's criteria that $\frac{\partial V}{\partial t}$ is negative definite proving E^* is global asymptotically stable in the region $(0, L_2)$ provided $\beta^2 < 4\alpha_0 W$.

Table-1

Parameters	Patch-1	Figure	Patch-2	Figure
	r_0		r_0	
Intrinsic growth rate of Resource biomass	r	0.06	r	0.08
Carrying capacity of Resource biomass	C	400	C	800
Depletion rate of forest resource	β_1	0.00004	β_2	0.00002
Growth rate of the resource	a_1	6	a_2	7
Growth rate of the Industrialization	α_1	8.759	α_2	74.97
	α_{10}	0.00003	α_{20}	0.00001
Diffusion coefficient of Resource biomass	d_1	0.5	d_2	0.6
Diffusion coefficient of Industrialization	D_1	0.7	D_2	0.8

Table-2

Parameters (Patch-1)	Figure	Parameters (Patch-2)	Figure
R_1^*	0.94492378	R_2^*	0.967
I_1^*	400	I_2^*	800

Conclusion

In the present model, we have assumed that the density of resource biomass is governed by the logistic function with the same intrinsic growth rate and carrying capacity in the entire habitat. Further the harvesting in resource biomass and distribution of resource biomass in the patchy habitat are assumed. The rate of depletion of forest resource biomass density due to industrialization, harvesting function and diffusion coefficients are considered to be different in each patch. It is further assumed that the density of industrialization is also governed by general logistic function in both the regions but with different growth rates and diffusion coefficients.

The linear and non linear stability analysis is carried out for the positive uniform equilibrium state in homogeneous as well as patchy habitat by using Liapunov direct method. It is found that the positive uniform equilibrium state is both linear (local) and non linear (global) asymptotically stable under some conditions involving parameters in each case. It is shown that the equilibrium level of the resource biomass in two patches decreases as the density of industrialization or the rate of depletion due to industrialization increases.

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