Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 86 (2015) 34932-34942

ISSN: 2229-712X

Inf – J and Sup - U_j Compositions Between Fuzzy Relations

Thangaraj Beaula[1] and K. Saraswathi[2]

[1]Department of Mathematics, T.B.M.L. College PorayarTamilnadu, India – 609 307.

[2]Department of Mathematics, Bharathidasan University Constituent College of Arts and Science, Nagapattinam.

ARTICLE INFO

Article history:
Received: 28 July 2015;
Received in revised form:
30 August 2015;
Accepted: 5 September 2015;

Keywords
Inf- j,
Sup- u_j composition.

ABSTRACT

In this paper inf – j composition (where j refers to a t- conorm) and sup- u_j composition are defined. Relation between inf- j and sup- $u_j$compositions are established. Theorems are proved that express the basic properties of inf- j and sup- $u_j$composition.

Introduction

Usually fuzzy relation equations are dealt using sup- i composition where i is a continuous t- norm. this is done by Zadeh [6,7] where he introduce compositional rule of interference with the help of sup- min composition. L.A. Zadeh [8] in 1965 introduced fuzzy set in his seminal paper. Goguen [4] in 1967 generalizes the concept of fuzzy sets defining them in terms of maps from a non empty set to a partially ordered set. Brown [2] in 1971 shows that Zadehs basics results carry over to the maps from a non empty set to a lattice. Sanchez [5] suggested the composition inf- j operation. The properties of this composition have been investigated in [1,2]. However in our paper we have defined inf- j along with sup- u_j composition some of its properties are characterized.

Preliminaries

Fuzzy Set

Let X be a non empty set. A fuzzy set A in X is characterized by its membership function $\mu_A: X \to [0,1]$ and $\mu_A(x)$ is interpreted as the degree of membership of element x in fuzzy set A for each $x \in X$.

t- Norm

A fuzzy intersection/t- norm i is a binary operation on the unit interval that satisfies atleast the following four axioms for all $a, b, d \in [0,1]$

(i) $i(a,1) = a$ (boundary condition)

(ii) $b \leq d$ implies $i(a,b) \leq i(a,d)$ (monotonicity)

(iii) $i(a,b) = i(b,a)$ (commutativity)

(iv) $i[a, i(b,d)] = i[i(a,b), d]$ (associativity)

Three of the most important requirements are expressed by the following axioms

(v) i is a continuous function (continuity)

(vi) $i(a,a) \leq a$ (sub idempotency)

(vii) $a_1 < a_2$ and $b_1 < b_2$ implies $i(a_1,b_2) < i(a_2,b_2)$ (strict monotonicity)

t- Conorm

A fuzzy union/t-conorm j is a binary operation on the unite interval that satisfies atleast the following four axioms for all $a, b, d \in [0,1]$

(i) $j(a,0) = a$ (boundary condition)

(ii) $b \leq d$ implies $j(a,b) \leq j(a,d)$ (monotonicity)

(iii) $j(a,b) = j(b,a)$ (commutativity)

(iv) $j[a, j(b,d)] = j[j(a,b), d]$ (associativity)

The most important additional requirements for fuzzy unions are expressed by the following axioms

(v) j is a continuous function (continuity)

(vi) $j(a,a) > a$ (super idempotency)

(vii) $a_1 < a_2$ and $b_1 < b_2$ implies $u(a_1,b_1) < u(a_2,b_2)$ (strict monotonicity)

Tele:
E-mail addresses: edwinbeaula@yahoo.co.in

Sup- i Composition

Sup- i composition of binary fuzzy relations, where i refers to a t- norm, generalize the standard max- min composition. Given a particular t- norm i and two fuzzy relations $P(X,Y)$ and $Q(Y,Z)$, the sup-i composition of P and Q is a fuzzy relation $P \overset{i}{\circ} Q$ on $X \times Z$ defined by

$$[P \overset{i}{\circ} Q](x,z) = \sup_{y \in Y} i[P(x,y), Q(y,z)] \ldots\ldots\ldots\ldots \quad (1) \text{ for all } x \in X, z \in Z.$$

Definition

Given a continuous t- norm i, let $w_i(a,b) = \sup\{x \in [0,1] \backslash i(a,x) \le b\}$ ....(2)

For every $a, b \in [0,1]$. This operation referred to as operation w_i. While t- norm i may be interpreted as logical conjunction, the corresponding operation w_i may be interpreted as logical implication.

Definition

Given a t- norm i and the associated operation w_i, the inf- w_i composition, $P \overset{w_i}{\circ} Q$, of fuzzy relation $P(x,y)$ and $Q(y,z)$ is defined by the equation

$$[P \overset{w_i}{\circ} Q](x,z) = \inf_{y \in Y} w_i[P(x,y), Q(y,z)] \quad \ldots\ldots\ldots \quad (3)$$

for all $x \in X, z \in Z$.

Definition (ELIE SANCHEZ[4])

Let $P(x,y)$ be a fuzzy relation, the fuzzy relation $P^{-1}(y,x)$, the inverse or transpose of P, is defined by

$P^{-1}(y,x) = P(x,y)$ for all $(y,x) \in Y \times X$ ....(4)

3. Inf- j Compositions of Fuzzy Relations

Inf- j composition of fuzzy relations, where j refers to a t- conorm, generalize the standard min- max composition.

Definition

Given a particular t- conorm j and two fuzzy relations $P(X,Y)$ and $Q(Y,Z)$, the inf- j composition of P and Q is a fuzzy relation $P \overset{j}{\circ} Q$ on $X \times Z$ defined by

$$[P \overset{j}{\circ} Q](x,z) = \inf_{y \in Y} j[P(x,y), Q(y,z)] \quad \ldots\ldots\ldots (5) \text{ for all } x \in X, z \in Z.$$

Basic properties of t- conorm are expressed by the following theorem.

Theorem

For any $a, a_i, b, d \in [0,1]$, where i takes values from an index set I, operation u_j has the following properties.

(i) $b \le d$ implies $j(a,b) \le j(a,d)$ and $j(b,a) \le j(d,a)$

(ii) $j[a, j(b,d)] = j[j(a,b), d]$

(iii) $j\left(b, \sup_{i \in I} a_i\right) \ge \sup_{i \in I} j(b, a_i)$

(iv) $j\left(b, \inf_{i \in I} a_i\right) = \inf_{i \in I} j(b, a_i)$

(v) $j\left(\sup_{i \in I} a_i, b\right) \ge \sup_{i \in I} j(a_i, b)$

(vi) $j\left(\inf_{i \in I} a_i, b\right) = \inf_{i \in I} j(a_i, b)$

Proof

(i) By definition, t- conorm j is monotonic increasing function i.e., if $b \le d$ then $j(a,b) \le j(a,d)$, also j is commutative, i.e., $j(b,a) \le j(d,a)$

(ii) By definition, t-conorm j is associative i.e., $j(j(a,b),d) = j(a, j(b,d))$

(iii) Now we have to prove that $j\left(b, \sup_{i \in I} a_i\right) \ge \sup_{i \in I} j(b, a_i)$

Let $s = \sup_{i \in I} a_i$

$a_i \le s$ for any $i \in I$

$j(b, a_i) \le j(b, s)$ for any $i \in I$

$$\Rightarrow \sup_{i \in I} j(b, a_i) \le j\left(b, \sup_{i \in I} a_i\right)$$

(iv) Now we have to prove that $j\left(b, \inf_{i\in I} a_i\right) = \inf_{i\in I} j(b, a_i)$

Let $l = \inf_{i\in I} a_i \quad \Rightarrow l \le a_i$

$j(b,l) \le j(b,a_i)$

$j(b,l) \le \inf_{i\in I} j(b,a_i)$ ……………………………(i)

But $\inf_{i\in I} j(b,a_i) \le j(b,a_i)$

$\Rightarrow j(b, \inf_{i\in I} j(b,a_i) \le j(b, j(b,a_i))$

$\Rightarrow j(b, \inf_{i\in I} j(b,a_i) \le j(j(b,b),a_i))$

$\Rightarrow \inf_{i\in I} j(b,a_i) \le a_i$ for all $i \in I$

$\Rightarrow \inf_{i\in I} j(b,a_i) \le \inf_{i\in I} a_i$

$\Rightarrow j(b, \inf_{i\in I} j(b,a_i)) \le j(j(b,b), \inf_{i\in I} a_i)$

$\Rightarrow j(b, \inf_{i\in I} j(b,a_i)) \le j(b, j(b, \inf_{i\in I} a_i))$

$\Rightarrow \inf_{i\in I} j(b,a_i) \le j(b, \inf_{i\in I} a_i)$ ………………………………………….(ii)

From equation (i) and (ii) we get

$j(b, \inf_{i\in I} a_i) = \inf_{i\in I} j(b,a_i)$

(v) Now let us prove $j\left(\sup_{i\in I} a_i, b\right) \ge \sup_{i\in I} j(a_i, b)$

By property (iii) $j(b, \sup_{i\in I} a_i) \ge \sup_{i\in I} j(b,a_i)$

Since t-conorm j is commulative therefore $j(\sup_{i\in I} a_i, b) \ge \sup_{i\in I} j(a_i, b)$

(vi) Now we have to prove that $j(\inf_{i\in I} a_i, b) = \inf_{i\in I} j(a_i, b)$

By property (iv) we have $j(b, \inf_{i\in I} a_i) = \inf_{i\in I} j(b,a_i)$

Since t-conorm j is commulative therefore $j(\inf_{i\in I} a_i, b) = \inf_{i\in I} j(a_i, b)$

Theorem

Let $P(X,Y), P_j(X,Y), Q(Y,Z)$ and $Q_j(Y,Z)$ be fuzzy relations. Then

(i) $Q_1 \subseteq Q_2$ then $P \overset{j}{\circ} Q_1 \subseteq P \overset{j}{\circ} Q_2$ and $Q_1 \overset{j}{\circ} R \supseteq Q_2 \overset{j}{\circ} R$

(ii) $(P \overset{j}{\circ} Q) \overset{j}{\circ} R = P \overset{j}{\circ} (Q \overset{j}{\circ} R)$

(iii) $P \overset{j}{\circ} \left(\bigcup_{i\in I} Q_i\right) \supseteq \bigcup_{i\in I} (P \overset{j}{\circ} Q_i)$

(iv) $P \overset{j}{\circ} \left(\bigcap_{i\in I} Q_i\right) = \bigcap_{i\in I} (P \overset{j}{\circ} Q_i)$

(v) $\left(\bigcup_{i\in I} P_i\right) \overset{j}{\circ} Q \supseteq \bigcup_{i\in I} (P_i \overset{j}{\circ} Q)$

(vi) $\left(\bigcap_{i\in I} P_i\right) \overset{j}{\circ} Q = \bigcap_{i\in I} (P_i \overset{j}{\circ} Q)$

(vii) $(P \overset{j}{\circ} Q)^{-1} = Q^{-1} \overset{j}{\circ} P^{-1}$

(i) Since $Q_1 \subseteq Q_2$

Let $[P\overset{j}{\circ}Q_1](x,z)=\inf_{y\in Y} j[P(x,y).Q_1(y,z)]\subseteq \inf_{y\in Y} j[P(x,y),Q_2(y,z)]=[P\overset{j}{\circ}Q_2](x,z)$

Therefore $P\overset{j}{\circ}Q_1\subseteq P\overset{j}{\circ}Q_2$

Similarly $Q_1\overset{j}{\circ}R\supseteq Q_2\overset{j}{\circ}R$

(ii) By definition of inf-j composition,

$$[(P\overset{j}{\circ}Q)\overset{j}{\circ}R](x,v)=\inf_{z\in Z} j[(P\overset{j}{\circ}Q)(x,z),R(z,v)]$$

$$=\inf_{z\in Z}[\inf_{y\in Y} j[P(x,y),Q(y,z)],R(z,v)]$$

$$=\inf_{z\in Z}\inf_{y\in Y} j[j[P(x,y),Q(y,z)],R(z,v)]]$$

$$=\inf_{z\in Z}\inf_{y\in Y} j[P(x,y),j[Q(y,z)],R(z,v)]]$$

$$=\inf_{z\in Z} j[P(x,y),\inf_{y\in Y} j[Q(y,z)],R(z,v)]$$

$$=[P\overset{j}{\circ}(Q\overset{j}{\circ}R)](x,z)$$

(iii) We know that $Q_i\subseteq\left(\bigcup_{i\in I}Q_i\right)$ for all $i\in I$

Then by property (i) have $P\overset{j}{\circ}Q_i\subseteq P\overset{j}{\circ}\left(\bigcup_{i\in I}Q_i\right)$ for all $i\in I$

$$\Rightarrow\bigcup_{i\in I}(P\overset{j}{\circ}Q_i)\subseteq P\overset{j}{\circ}\left(\bigcup_{i\in I}Q_i\right)$$

therefore $P\overset{j}{\circ}\left(\bigcup_{i\in I}Q_i\right)\supseteq\bigcup_{i\in I}(P\overset{j}{\circ}Q_i)$

(iv) We know that $\left(\bigcap_{i\in I}Q_i\right)\subseteq Q_i$ for any $i\in I$

Then by property (i) have

$P\overset{j}{\circ}\left(\bigcap_{i\in I}Q_i\right)\subseteq P\overset{j}{\circ}Q_i$ for any $i\in I$

$$\Rightarrow P\overset{j}{\circ}\left(\bigcap_{i\in I}Q_i\right)\subseteq\bigcap(P\overset{j}{\circ}Q_i)\qquad\text{…………………..(i)}$$

But $\bigcap(P\overset{j}{\circ}Q_i)\subseteq P\overset{j}{\circ}Q_i$ for any $i\in I$

Then by property (i) we have

$$P^{-1}\overset{j}{\circ}\left(\bigcap_{i\in I}(P\overset{j}{\circ}Q_i)\right)\subseteq P^{-1}\overset{j}{\circ}(P\overset{j}{\circ}Q_i)$$

By property (ii) we have

$$P^{-1}\overset{j}{\circ}\left(\bigcap_{i\in I}(P\overset{j}{\circ}Q_i)\right)\subseteq(P^{-1}\overset{j}{\circ}P)\overset{j}{\circ}Q_i$$

Since $\boldsymbol{P^{-1}\subseteq P^{-1}\overset{j}{\circ}P}$ ($\boldsymbol{j}$ is super idempotent)

$$\Rightarrow\bigcap_{i\in I}(P\overset{j}{\circ}Q_i)\subseteq Q_i \text{ for all } i\in I$$

$$\Rightarrow\bigcap_{i\in I}(P\overset{j}{\circ}Q_i)\subseteq\bigcap_{i\in I}Q_i$$

$$P^{-1}\overset{j}{\circ}\left(\bigcap_{i\in I}(P\overset{j}{\circ}Q_i)\right)\subseteq(P^{-1}\overset{j}{\circ}P)\overset{j}{\circ}\left(\bigcap_{i\in I}Q_i\right)$$

$$P^{-1}\overset{j}{\circ}\left(\bigcap_{i\in I}(P\overset{j}{\circ}Q_i)\right)\subseteq P^{-1}\overset{j}{\circ}\left(P\overset{j}{\circ}\left(\bigcap_{i\in I}Q_i\right)\right)$$

$$\bigcap_{i\in I}(P\overset{j}{\circ}Q_i)\subseteq P\overset{j}{\circ}\left(\bigcap_{i\in I}Q_i\right)$$ ……………………………(ii)

From equations (i) and (ii)

$$P\overset{j}{\circ}\left(\bigcap_{i\in I}Q_i\right)=\bigcap_{i\in I}(P\overset{j}{\circ}Q_i)$$

(v) $$\left(\bigcup_{i\in I}P_i\right)\overset{j}{\circ}Q\supseteq\bigcup_{i\in I}(P_i\overset{j}{\circ}Q)$$

We know that

$P_i\subseteq\bigcup_{i\in I}P_i$ for all $i\in I$

Then by property (i)

$P_i\overset{j}{\circ}Q\subseteq\left(\bigcup_{i\in I}P_i\right)\overset{j}{\circ}Q$ for all $i\in I$

$$\bigcup_{i\in I}(P_i\overset{j}{\circ}Q)\subseteq\left(\bigcup_{i\in I}P_i\right)\overset{j}{\circ}Q$$

(vi) $$\left(\bigcap_{i\in I}P_i\right)\overset{j}{\circ}Q=\bigcap_{i\in I}(P_i\overset{j}{\circ}Q)$$

But we know that

$\bigcap_{i\in I}P_i\subseteq P_i$ for all $i\in I$

Then by property (i)

$\left(\bigcap_{i\in I}P_i\right)\overset{j}{\circ}Q\subseteq P_i\overset{j}{\circ}Q$ for all $i\in I$

$$\left(\bigcap_{i\in I}P_i\right)\overset{j}{\circ}Q\subseteq\bigcap_{i\in I}(P_i\overset{j}{\circ}Q)$$ ……………………………(1)

But $\bigcap(P_i\overset{j}{\circ}Q)\subseteq P_i\overset{j}{\circ}Q$ for all $i\in I$

Then by property (i) we have

$$\left(\bigcap_{i\in I}(P_i\overset{j}{\circ}Q)\right)\overset{j}{\circ}Q^{-1}\subseteq(P_i\overset{j}{\circ}Q)\overset{j}{\circ}Q^{-1}$$

$$\Rightarrow\left(\bigcap_{i\in I}(P_i\overset{j}{\circ}Q)\right)\overset{j}{\circ}Q^{-1}\subseteq P_i\overset{j}{\circ}(Q\overset{j}{\circ}Q^{-1})$$

Because $Q^{-1}\subseteq(Q\overset{j}{\circ}Q^{-1})$ j is a super idempotent

$\Rightarrow\bigcap_{i\in I}(P_i\overset{j}{\circ}Q)\subseteq P_i$ for all $i\in I$

$$\Rightarrow\left(\bigcap_{i\in I}(P_i\overset{j}{\circ}Q)\right)\subseteq\bigcap_{i\in I}P_i$$

$$\Rightarrow\left(\bigcap_{i\in I}(P_i\overset{j}{\circ}Q)\right)\overset{j}{\circ}Q^{-1}\subseteq\left(\bigcap_{i\in I}P_i\right)\overset{j}{\circ}(Q\overset{j}{\circ}Q^{-1})$$

$$\Rightarrow\left(\bigcap_{i\in I}(P_i\overset{j}{\circ}Q)\right)\overset{j}{\circ}Q^{-1}\subseteq\left(\left(\bigcap_{i\in I}P_i\right)\overset{j}{\circ}Q\right)\overset{j}{\circ}Q^{-1}$$

$$\Rightarrow\bigcap_{i\in I}(P_i\overset{j}{\circ}Q)\subseteq\left(\bigcap_{i\in I}P_i\right)\overset{j}{\circ}Q$$ ……………………………(II)

From equations (I) and (II) we get

$$\bigcap_{i\in I}(P_i \overset{j}{\circ} Q) = \left(\bigcap_{i\in I} P_i\right)\overset{j}{\circ} Q$$

(vii) $(P \overset{j}{\circ} Q)^{-1} = Q^{-1} \overset{j}{\circ} P^{-1}$

Here $(P \overset{j}{\circ} Q)^{-1}(z,x) = (P \overset{j}{\circ} Q)(x,z)$

$$= \inf_{y\in Y} j(P(x,y), Q(y,z))$$

$$= \inf_{y\in Y} j(P(z,y), P^{-1}(y,x))$$

$$= \left(Q^{-1} \overset{j}{\circ} P^{-1}\right)(z,x)$$

$$(P \overset{j}{\circ} Q)^{-1} = Q^{-1} \overset{j}{\circ} P^{-1}$$

Sup-u$_j$ Composition of Fuzzy Relations

Definition

Given a t-conorm j, let $u_j\ (a,b) = \inf\{X \in [0,1] j(a,x) \geq b\}$ ………………..(6)

For every a,b∈ [0,1]

Theorem

The operation u_j satisfies the following properties.

(i) $j(a,b) \geq d$ iff $u_j\ (a,d) \leq b$

(ii) $u_j(u_j(a,b),b) \leq a$

(iii) $u_j(j(a,b),d) = u_j(a,u_j(b,d))$

(iv) $a \leq b$ implies $u_j(a,d) \geq u_j(b,d)$ and $u_j(d,a) \leq u_j(d,b)$

(v) $u_j(\sup_{i\in I} a_i, b) \leq \inf_{i\in I} u_j(a_i,b)$

(vi) $u_j(\inf_{i\in I} a_i, b) = \sup_{i\in I} u_j(a_i,b)$

(vii) $u_j(b, \inf_{i\in I} a_i) \leq \inf_{i\in I} u_j(b,a_i)$

(viii) $u_j(b, \sup_{i\in I} a_i) \leq \sup_{i\in I} u_j(b,a)$

(ix) $j[a,u_j(a,b)] \geq b$ for any $a, b, d \in [0,1]$. where $i \in I$

Proof

(i) Now let us prove $j(a,b) \geq d$ iff $u_j(a,d) \leq b$

If $j(a,b) \geq d$

Then $b \in \{x/j(a,x) \geq d$

$$b \geq \inf\{x/j(a,x) \geq d\} = u_j(a,d)$$

$$\Rightarrow u_j(a,d) \leq b$$

if $u_j(a,d) \leq b$

then $j(a,b) \geq j[a,u_j(a,d)]$

$$= j[a, \inf\{x/j(a,x) \geq d\}]$$
$$= \inf[j(a,x)/j(a,x) \geq d]$$
$$\geq d$$

$$\Rightarrow j(a,d) \geq d$$

(ii) We have to prove $u_j(u_j(a,b),b) \leq a$

i.e., $j(u_j(a,b),a) \geq b$ by (i)

Let $j\big(u_j(a,b),a\big) = j(\inf\{x/j(a,x) \geq b\},a)$

$$= \inf\{j(x,a)/j(a,x) \geq b\}$$
$$= \inf\{j(a,x)/j(a,x) \geq b\}$$

$$\geq b$$

$$\Rightarrow j(u_j(a,b),a) \geq b$$

$\therefore u_j(u_j(a,b),b) \leq a$

(iii) Now we have to prove that $u_j[j(a,b),d] = u_j[a,u_j(b,d)]$

Let $u_j[a,u_j(b,d)] = \inf\{x/j(a,x) \geq u_j(b,d)\}$

$=\inf\{x/x \geq u_j[(a,b),d]\}$

$= u_j[j(a,b),d]$

Since by property 1

$j(a,x) \geq u_j(b,d)$

$\Leftrightarrow j[b,j(a,x)] \geq d$

$\Leftrightarrow j[j(a,b),x] \geq d$

$\Leftrightarrow x \geq u_j[j(a,b),d]$

(iv) $a \leq b$ implies $u_j(a,d) \geq u_j(b,d)$ and $u_j(d,a) \leq u_j(d,b)$

Let $u_j(d,a) = \inf\{x/j(d,x) \geq a\}$

$\leq \inf\{x/j(d,x) \geq b\}$

$= u_j(d,b)$

$\therefore u_j(d,a) \leq u_j(d,b)$

Next we have to prove that $u_j(a,d) \geq u_j(b,d)$

i.e.,$u_j(b,d) \leq u_j(a,d)$

i.e.,$j(b,u_j(a,d)) \geq d$

let $j(b,u_j(a,d)) = j(b,\inf\{x/j(a,x) \geq d\})$

$= j(\inf\{x/j(a,x) \geq d\},b)$

$\geq j(\inf\{x/j(a,x) \geq d\},a)$

$= \inf\{j(x,a)/j(a,x) \geq d\}$

$= \inf\{j(a,x)/j(a,x) \geq d\}$s

$\geq d.$

$\therefore j(b,u_j(a,d)) \geq d$

$u_j(b,d) \leq u_j(a,d)$

i.e.,$u_j(a,d) \geq u_j(b,d)$

(v) now let us prove that $u_j\left[\sup_{i\in I} a_i, b\right] \leq \inf_{i\in I} u_j(a_i,b)$

Let $s = \sup_{i\in I} a_i$ then $s \geq a_i$

$\Rightarrow \quad a_i \leq s$

$\Rightarrow \quad u_j(a_i,b) \geq u_j(s,b)$ for any $i \in I$ (by property(iv))

$\Rightarrow \inf_{i\in I} u_j(a_i,b) \geq u_j(s,b)$ for any $i \in I$

i.e., $\inf_{i\in I} u_j(a_i,b) \geq u_j(\sup_{i\in I} a_i, b)$

(vi) we have to prove that $u_j\left[\inf_{i\in I} a_i, b\right] = \sup_{i\in I} u_j[a_i,b]$

Let $l = \inf_{i\in I} a_i$ then $a_i \geq l \Rightarrow l \leq a_i$

$u_j(l,b) \geq u_j(a_i,b)$ for any $i \in I$ (by property 4)

Hence $u_j(l,b) \geq \sup_{i\in I} u_j(a,b)$ ……………………………(i)

Since $\sup_{i\in I} u_j(a_i,b) \geq u_j(a_{i_o},b)$ for all $i_o \in I$

$j(a_{i_o}, \sup_{i\in I} u_j(a_i,b)) \geq b$ for all $i_o \in I$

$j(l, \sup_{i\in I} u_j(a_i,b)) = \inf_{i\in I} j(a_{i_0}, \sup_{i\in I} u_j(a_i,b)) \geq b$

Again by property (i)

$u_j(l,b) \leq \sup_{i\in I} u_j(a_i,b)$ ……………………………………..(ii)

From equations (i) and (ii)

$u_j\left(\inf_{i\in I} a_i, b\right) = \sup_{i\in I} u_j(a_i,b)$

(vii) $u_j\left(b, \inf_{i\in I} a_i\right) \le \inf_{i\in I} u_j(b, a_i)$

Let $l = \inf_{i\in I} a_i \Rightarrow l \le a_i$

$u_j(b,l) \le u_j(b,a_i)$ for any $i \in I$ (by prop.4)

Hence $u_j(b,l) \le \inf_{i\in I} u_j(b,a_i)$

$u_j\left(b, \inf_{i\in I} a_i\right) \le \inf_{i\in I} u_j(b, a_i)$

(viii)

$$u_j\left(b, \sup_{i\in I} a_i\right) = \sup_{i\in I} u_j(b, a_i)$$

Let $s = \sup_{i\in I} a_i \Rightarrow s \ge a_i \Rightarrow a_i \le s$

$u_j(b,a_i) \le u_j(b,s)$ for any $i \in I$ (by prop. 4)

$\sup_{i\in I} u_j(b,a_i) \le u_j(b,s)$ for any $i \in I$ ………………………..(i)

On the other hand

Since $\sup_{i\in I} u_j(b,a_i) \ge u_j(b,a_{i_o})$ for all $i_o \in I$

By property (i) we have

$$j\left(b, \sup_{i\in I} u_j(b,a_i)\right) \ge \sup_{i\in I} a_{i_o}$$

$$j\left(b, \sup_{i\in I} u_j(b,a_i)\right) \ge s$$

$$u_j(b,s) \le \sup_{i\in I} u_j(b,a_i)$$

$$u_j\left(b, \sup_{i\in I} a_i\right) \le \sup_{i\in I} u_j(b,a_i)$$ ……………………………………..(ii)

From equations (i) and (ii)we get

$$u_j\left(b, \sup_{i\in I} a_i\right) = \sup_{i\in I} u_j(b,a_i)$$

(xi) $j[a, u_j(a,b)] \ge b$

Let $j[a,u_j(a,b)] = j[a, \inf(x / j(a,x) \ge b)]$

$= \inf\{ j(a,x) / j(a,x) \ge b\}$

$\ge b$

$\therefore j[a,u_j(a,b)] \ge b$

Hence the theorem.

Definition

Given a t- conorm j and the associated operation u_j, the sup-u_j composition, $P \overset{u_j}{\circ} Q$ of fuzzy relation $P(X,Y)$ and $Q(Y,Z)$ is defined by the equation $(P \overset{u_j}{\circ} Q)(x,z) = \sup_{y\in Y} u_j[P(x,y), Q(y,z)]$ ………..(7)

For all $x \in X, z \in Z.$

Basic properties of the sup-u_j composition are expressed by the following theorems.

Theorem

Let $P(X,Y), Q(Y,Z), R(X,Z)$ and $S(Z,V)$ be fuzzy relations. Then

(1) The following properties are equivalent

(i) $P \overset{j}{\circ} Q \supseteq R$ ……………………..(8)

(ii) $Q \supseteq P^{-1} \overset{u_j}{\circ} R$ ……………………..(9)

(iii) $P \supseteq (Q \overset{u_j}{\circ} R^{-1})^{-1}$ ……………………(10)

(2) $P \overset{u_j}{\circ}(Q \overset{u_j}{\circ} S) = (P \overset{j}{\circ} Q) \overset{u_j}{\circ} S$

Proof

First let us prove (i) $\Rightarrow$(ii)

Assume that $P \overset{j}{\circ} Q \supseteq R$

Then by definition of inf- j composition we have

$$\inf_{y \in Y} j(P(x,y),Q(y,z)) \supseteq R(x,z)$$
$$\Rightarrow j(P(x,y),Q(y,z)) \supseteq R(x,z) \quad \forall x \in X, y \in Y, z \in Z$$
$$\Rightarrow u_j(P^{-1}(y,x),R(x,z)) \subseteq Q(y,z) \quad \forall x \in X, y \in Y, z \in Z$$
$$\Rightarrow \sup_{y \in Y} u_j(P^{-1}(y,x),R(x,z)) \subseteq Q(y,z)$$
$$\Rightarrow (P^{-1} \overset{u_j}{\circ} R)(y,z) \subseteq Q(y,z)$$
$$\Rightarrow Q \supseteq P^{-1} \overset{u_j}{\circ} R$$

This proves (i) implies (ii)

Now we have to prove (ii) imlies (iii)

Let $Q \supseteq P^{-1} \overset{u_j}{\circ} R$

Then by definition of the sup- u_j composition we have

$$\sup_{x \in X} u_j(P^{-1}(y,x),R(x,z)) \subseteq Q(y,z)$$
$$\Rightarrow u_j(P^{-1}(y,x),R(x,z)) \subseteq Q(y,z) \quad \forall x \in X, y \in Y, z \in Z$$
$$\Rightarrow j(P(x,y),Q(y,z)) \supseteq R(x,z) \quad \forall x \in X, y \in Y, z \in Z$$
$$\Rightarrow j(Q^{-1}(z,y),P^{-1}(y,x) \supseteq R^{-1}(z,x)$$
$$\Rightarrow u_j(Q(y,z),R^{-1}(z,x)) \subseteq p^{-1}(y,x)$$
$$\Rightarrow \sup_{z \in Z} u_j(Q(y,z),R^{-1}(z,x)) \subseteq p^{-1}(y,x)$$
$$\Rightarrow \left(Q \overset{u_j}{\circ} R^{-1}\right)(y,x) \subseteq P^{-1}(y,x)$$
$$\Rightarrow \left(Q \overset{u_j}{\circ} R^{-1}\right)^{-1}(x,y) \subseteq P(x,y)$$

This proves (ii) implies (iii)

Now we have to prove (iii) implies (i)

Let us assume that

$$P \supseteq (Q \overset{u_j}{\circ} R^{-1})^{-1}$$
$$\Rightarrow P^{-1} \supseteq Q \overset{u_j}{\circ} R^{-1}$$

Then by equivalent preposition (8) & (9) we have

$$Q^{-1} \overset{j}{\circ} P^{-1} \supseteq R^{-1}$$

$\Rightarrow P \overset{j}{\circ} Q \supseteq R$ This proves (iii) implies (i)

2. Now we have to prove that

$$P \overset{u_j}{\circ}\left(Q \overset{u_j}{\circ} S\right) = \left(P \overset{j}{\circ} Q\right) \overset{u_j}{\circ} S$$
$$\left[P \overset{u_j}{\circ}\left(Q \overset{u_j}{\circ} S\right)\right](x,v) = \sup_{y \in Y} u_j\left[P(x,y),\left(Q \overset{u_j}{\circ} S\right)(y,v)\right]$$

By equivalent preposition in (1) we have

$$j\left[P^{-1}(y,x),\left(P\overset{u_j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right)(x,v)\right]\supseteq\left(Q\overset{u_j}{\circ}S\right)(y,v)$$

$$\supseteq\sup_{z\in Z}u_j\left(Q(y,z),S(z,v)\right)$$

$$\Rightarrow j\left[P^{-1}(y,x),\left(P\overset{u_j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right)(x,v)\right]\supseteq u_j\left(Q(y,z),S(z,v)\right)\ \ \forall z\in Z$$

$$\Rightarrow u_j\left[P(x,y),u_j\left(Q(y,z),S(z,v)\right)\right]\subseteq\left[P\overset{u_j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right](x,v)$$

$$\Rightarrow u_j\left[j\left(P(x,y),Q(y,z)\right),S(z,v)\right]\subseteq\left[P\overset{u_j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right](x,v)$$

$$\Rightarrow j\left[\left[j\left(P(x,y),Q(y,z)\right)\right]^{-1},\left[P\overset{u_j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right](x,v)\right]\supseteq S(z,v)$$

$$\Rightarrow u_j\left[\left[P\overset{u_j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right](x,v),S^{-1}(v,z)\right]\subseteq j(P(x,y),Q(y,z))\quad\forall x\in X,y\in Y,z\in Z,v\in V$$

$$\subseteq\inf_{y\in Y}j(P(x,y),Q(y,z))$$

$$\Rightarrow u_j\left[\left[P\overset{u_j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right](x,v),S^{-1}(v,z)\right]\subseteq\left(P\overset{j}{\circ}Q\right)(x,z)$$

$$\Rightarrow j\left[\left[P\overset{u_j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right]^{-1}(v,x),\left(P\overset{j}{\circ}Q\right)(x,z)\right]\supseteq S^{-1}(v,z)$$

$$\Rightarrow j\left[\left(P\overset{j}{\circ}Q\right)^{-1}(z,x),\left[P\overset{u_j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right](x,v)\right]\supseteq S(z,v)$$

$$\Rightarrow u_j\left[\left(P\overset{j}{\circ}Q\right)(x,z),S(z,v)\right]\subseteq\left[P\overset{u_j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right](x,v)$$

$$\Rightarrow\sup_{z\in Z}u_j\left[\left(P\overset{j}{\circ}Q\right)(x,z),S(z,v)\right]\subseteq\left[P\overset{u_j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right](x,v)$$

$$\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right](x,v)\subseteq\left[P\overset{u_j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right](x,v)\qquad\text{………………………………………(i)}$$

By the definition of sup- u_j composition we have

$$\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right](x,v)=\sup_{z\in Z}u_j\left[\left(P\overset{j}{\circ}Q\right)(x,z),S(z,v)\right]$$

$$\Rightarrow\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right](x,v)\supseteq u_j\left[\left(P\overset{j}{\circ}Q\right)(x,z),S(z,v)\right]$$

$$\Rightarrow j\left[\left(P\overset{j}{\circ}Q\right)^{-1}(z,x),\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right](x,v)\right]\supseteq S(z,v)\Rightarrow j\left[\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right]^{-1}(v,x),\left(P\overset{j}{\circ}Q\right)(x,z)\right]\supseteq S^{-1}(v,z)$$

$$\Rightarrow u_j\left[\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right](x,v),S^{-1}(v,z)\right]\subseteq P\overset{j}{\circ}Q(x,z)$$

$$\Rightarrow u_j\left[\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right](x,v),S^{-1}(v,z)\right]\subseteq\inf_{y\in Y}j(P(x,y),Q(y,z))$$

$$\Rightarrow u_j\left[\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right](x,v),S^{-1}(v,z)\right]\subseteq j(P(x,y),Q(y,z))$$

$$\Rightarrow j(\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right]^{-1}(v,x),j(P(x,y),Q(y,z))\supseteq S^{-1}(v,z)$$

$$\Rightarrow j\left[\left[j\left(P(x,y),Q(y,z)\right)\right]^{-1},\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right](x,v)\right]\supseteq S(z,v)$$

$$\Rightarrow u_j\left[j\left(P(x,y),Q(y,z)\right),S(z,v)\right]\subseteq\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right](x,v)$$

$$\Rightarrow u_j\left[P(x,y),u_j\left(Q(y,z),S(z,v)\right)\right]\subseteq\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right](x,v)$$

$$\Rightarrow j\left(P^{-1}(y,x),\left[P\overset{j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right](x,v)\right)\supseteq u_j(Q(y,z),S(z,v)) \quad \forall x\in X, y\in Y, z\in Z, v\in V$$

$$\Rightarrow j\left(P^{-1}(y,x),\left[P\overset{j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right](x,v)\right)\supseteq \sup_{z\in Z}u_j(Q(y,z),S(z,v))$$

$$\Rightarrow j\left(P^{-1}(y,x),\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right](x,v)\right)\supseteq\left(Q\overset{u_j}{\circ}S\right)(y,v)$$

$$\Rightarrow u_j\left[P(x,y),\left(Q\overset{u_j}{\circ}S\right)(y,v)\right]\subseteq\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right](x,v)$$

$$\Rightarrow \sup_{y\in Y}u_j\left[P(x,y),\left(Q\overset{u_j}{\circ}S\right)(y,v)\right]\subseteq\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right](x,v)$$

$$\Rightarrow\left[P\overset{u_j}{\circ}\left(Q\overset{u_j}{\circ}S\right)\right](x,v)\subseteq\left[\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S\right](x,v) \qquad \text{……………………………(ii)}$$

From equations (i) and (ii) we get

$$P\overset{u_j}{\circ}\left(Q\overset{u_j}{\circ}S\right)=\left(P\overset{j}{\circ}Q\right)\overset{u_j}{\circ}S$$

This proves (2)

Conclusion

An attempt is made to prove important theorems using the newly defined inf- j and sup- u_j compositions. This can be further developed to solve problems with fuzzy relational equations.

References

[1]. Birkhoff.G (1967) "lattice Theory " 3rd edition, American mathematical society colloquium publications, vol.xxv, Providence, R.I.
[2]. Brown.J.G. (1971), A note on fuzzy sets, information and control 18, 32 – 39.
[3]. George J. klir, Bo Yuan: fuzzy sets and fuzzy logic, theory and applications (1995) 61–64, 76–79.
[4]. Goguen.J.A. (1967), L – fuzzy sets, J.Math.Anal., 18, (145-174).
[5]. Sanchez.E: Resolution of composite fuzzy relation equation and control, 30 (1976) 38 – 48.
[6]. Zadeh.L.A: Outline of a new approach to the analysis of complex systems and decision processes, IEEE Trans. On systems, Man, and cybernetics, SMC – 3 (1973), 28 – 44.
[7]. Zadeh.L.A: The concept of a linguistic variable and its application to approximate reasoning I,II,III. Informations sciences, 8(1975), 199 - 257, 301 – 357; 9(1975), 43 – 80.
[8]. Zadeh.L.A. (1965), fuzzy sets, information and control 8, 338 – 353.