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Analysis of Batch Arrival Retrial G-Queue with Multi-Types of Heterogeneous Service, Feedback, Randomized J Vacations and Orbital Search

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ABSTRACT

This paper is concerned with the analysis of a single server batch arrival retrial G-queue with feedback, multi-types of service and orbital search under a randomized vacation policy. Positive customers arrive in batches according to Poisson processes. Server provides M types of heterogeneous service. If the server is idle upon the arrival of a batch, one of the customers in the batch receives any one of the types immediately and others join the orbit. Otherwise, all the customers in the batch join the orbit. After completion of service, the unsatisfied customer joins the orbit as a feedback customer. A breakdown at the busy server is represented by the arrival of a negative customer which causes the customer being in service to be lost. The repair of the failed server starts immediately. Whenever the orbit becomes empty the server takes a vacation of random length. At a vacation completion epoch, if the system is still empty, the server leaves for another vacation of same length or remains idle in the system. This pattern continues until the server finds at least one customer in the orbit or the number of vacations reaches J. At the end of J^{th} vacation, even if the orbit is empty the server remains in the system for new arrival. If the orbit is non-empty during the idle period, the server may search customers from the orbit. Using supplementary variable technique, various performance measures are derived. Stochastic decomposition property is established.

Introduction

Queue with negative arrivals called G-queue was introduced by Gelenbe [3] with a view to modeling neural networks. Here the arrival of negative customer brings the server down and makes the interrupted customer to leave the system. For a comprehensive analysis of queueing systems with negative arrivals, reader may refer to Gelenbe[4-6] and Artalejo[1]. Liu et al. [7] analyzed an M/G/1 retrial G-queue with preemptive resume and feedback under N-policy vacation. Wu and Yin [10] considered an unreliable M/G/1 retrial G-queue with non-exhaustive random vacations and derived steady state results. Gao and Wang [2] considered a repairable M/G/1 queue with negative customers and orbital search. Yang et al. [11] investigated an $M^{[X]}$/G/1 retrial G-queue with single vacation subject to the server breakdown and repair. Wu and Lian [9] discussed an M/G/1 retrial G queue with priority resume, Bernoulli vacation and server breakdown.

This paper is concerned with the analysis of retrial queueing system with negative customers, multi-types of service, feedback, randomized J vacation and orbital search. Under this policy, at the end of each vacation if no customers are waiting for service, the server leaves for another vacation with certain probability or remains in the system with complementary probability.

Model Description

Consider a single server retrial queueing system with two types of customers, positive and negative. Positive customers arrive in batches according to Poisson process with rate λ^+. At every arrival epoch, a batch of k customers arrives with probability C_k. The generating function of the sequence $\{C_k\}$ is C(z) with first two moments m_1 and m_2. The server provides M types of services. Customers opt the i^{th} type of service with probability p_i ($1 \le i \le M$). If the arriving batch of positive customers finds the server idle, then one of the customers receives the service and others join the orbit. Otherwise, all the customers in the batch join the orbit. The retrial time is generally distributed with distribution function A(x), density function a(x), Laplace Stieltje's transform $A^*(\theta)$ and conditional completion rate $\eta(x) = a(x)/[1-A(x)]$.

The service time of type i follows a general distribution with distribution function $B_i(x)$, density function $b_i(x)$, Laplace Stieltje's transform $B_i^*(\theta)$, n^{th} factorial moments $\mu_i^{(n)}$ and conditional completion rate $\mu_i(x) = b_i(x)/[1-B_i(x)]$, for (i=1,2,…,M). After receiving service, the customer may again join the orbit as a feedback customer with probability δ or depart the system with its complementary probability $\bar{\delta}$ (=1- δ).

Negative customers arrive singly according to Poisson process with rate λ^-. The arrival of a negative customer removes the positive customer being in service from the system and makes the server down. When the server fails during i^{th} type service, it stops providing service and is sent for repair immediately. The repair time follows general distribution with distribution function $R_i(x)$, density function $r_i(x)$, Laplace Stieltje's transform $R_i^*(\theta)$, n^{th} factorial moments $\beta_i^{(n)}$ and conditional completion rate $\beta_i(x) = r_i(x)/[1-R_i(x)]$, for (i=1,2,…,M).

If the orbit is empty, the server leaves for a vacation of random length V. At a vacation completion epoch, if the system is still empty the server either remains idle in the system with probability q or takes another vacation with probability $\bar{q}$ (=1-q) of same

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length. This pattern continues until the server finds at least one customer in the orbit or number of vacations reaches J. At the end of J^{th} vacation, even if the orbit is empty the server remains in the system. The vacation time also follows general distribution with distribution function V(x), density function v(x), Laplace Stieltje's transform $V^*(\theta)$, n^{th} factorial moments $v^{(n)}$ and conditional completion rate $\gamma(x) = v(x)/[1-V(x)]$.

If the orbit is non-empty during the idle period, the server searches for the customers in the orbit with probability θ or remains idle with probability $\bar{\theta}$ (=1- θ). Various stochastic process involved in the system are independent of each other.

Definitions and Notations

Assuming the existence of steady state, define the following probabilities

$I_n(x)$ is the probability that there are n customers in the system, the server is idle and the elapsed retrial time is x.

$P_n^{(i)}(x)$ is the probability that there are n customers in the system, the server is busy in i^{th} type service and the elapsed service time is x, $1 \le i \le M$.

$R_n^{(i)}(x)$ is the probability that there are n customers in the system, the server failed during i^{th} type service is under repair and the elapsed repair time is x, $1 \le i \le M$.

$V_n^{(j)}(x)$ is the probability that there are n customers in the system, the server is in j^{th} vacation and the elapsed vacation time is x ,$1 \le j \le J$.

I_0 is the steady state probability that the server is idle in the empty system.

Define the following probability generating functions

$I(x,z) = \sum_{n=1}^{\infty} I_n(x)z^n$, $P^{(i)}(x,z) = \sum_{n=0}^{\infty} P_n^{(i)}(x)z^n$, $R^{(i)}(x,z) = \sum_{n=0}^{\infty} R_n^{(i)}(x)z^n$,$1 \le i \le M$ and

$V^{(j)}(x,z) = \sum_{n=0}^{\infty} V_n^{(j)}(x)z^n, 1 \le j \le J$

Steady State Distributions

The system of equations that governs the model under steady state are given below

$$\lambda^+ I_0 = \int_0^{\infty} V_0^{(J)}(x)\gamma(x)dx + q\sum_{j=1}^{J-1}\int_0^{\infty} V_0^{(j)}(x)\gamma(x)dx \tag{1}$$

$$\frac{d}{dx}I_n(x) = -(\lambda^+ + \eta(x))I_n(x), n \ge 1 \tag{2}$$

$$\frac{d}{dx}P_n^{(i)}(x) = -(\lambda^+ + \lambda^- + \mu_i(x))P_n^{(i)}(x) + (1-\delta_{n0})\lambda^+ \sum_{k=1}^{n} C_k P_{n-k}^{(i)}(x), n \ge 0, i = 1,2,\ldots,M \tag{3}$$

$$\frac{d}{dx}R_n^{(i)}(x) = -(\lambda^+ + \beta_i(x))R_n^{(i)}(x) + (1-\delta_{n0})\lambda^+ \sum_{k=1}^{n} C_k R_{n-k}^{(i)}(x), n \ge 0, i = 1,2,\ldots,M \tag{4}$$

$$\frac{d}{dx}V_n^{(j)}(x) = -(\lambda^+ + \gamma(x))V_n^{(j)}(x) + (1-\delta_{n0})\lambda^+ \sum_{k=1}^{n} C_k V_{n-k}^{(j)}(x), n \ge 0, j = 1,2,\ldots,J \tag{5}$$

with boundary conditions

$$I_n(0) = \sum_{i=1}^{M}\left[\delta\int_0^{\infty} P_{n-1}^{(i)}(x)\mu_i(x)dx + \bar{\delta}\int_0^{\infty} P_n^{(i)}(x)\mu_i(x)dx\right] + \bar{\theta}\left[\sum_{i=1}^{M}\int_0^{\infty} R_n^{(i)}(x)\beta_i(x)dx + \sum_{j=1}^{J}\int_0^{\infty} V_n^{(j)}(x)\gamma(x)dx\right], n \ge 1 \tag{6}$$

$$P_0^{(i)}(0) = p_i\left[\lambda^+ c_1 I_0 + \int_0^{\infty} I_1(x)\eta(x)dx + \theta\left(\sum_{i=1}^{M}\int_0^{\infty} R_1^{(i)}(x)\beta_i(x)dx + \sum_{j=1}^{J}\int_0^{\infty} V_1^{(j)}(x)\gamma(x)dx\right)\right] \tag{7}$$

$$P_n^{(i)}(0) = p_i\left[\lambda^+ c_{n+1} I_0 + \int_0^{\infty} I_{n+1}(x)\eta(x)dx + \lambda^+\sum_{k=1}^{n} c_k \int_0^{\infty} I_{n-k+1}(x)dx + \theta\left(\sum_{i=1}^{M}\int_0^{\infty} R_{n+1}^{(i)}(x)\beta_i(x)dx + \sum_{j=1}^{J}\int_0^{\infty} V_{n+1}^{(j)}(x)\gamma(x)dx\right)\right], n \ge 1, i = 1,2,\ldots,M \tag{8}$$

$$R_n^{(i)}(0) = \lambda^- \int_0^{\infty} P_n^{(i)}(x)dx, n \ge 0, i = 1,2,\ldots,M \tag{9}$$

$$V_n^{(j)}(0) = \begin{cases} \bar{q}\int_0^{\infty} V_n^{(j-1)}(x)\gamma(x)dx, & n = 0, j = 2,3,\ldots,J \\ 0 & ,n \ne 0, j = 2,3,\ldots,J \end{cases} \tag{10}$$

$$V_n^{(1)}(0)=\begin{cases}\sum_{i=1}^{M}\left(\bar{\delta}\int_0^{\infty}P_n^{(i)}(x)\mu_i(x)dx+\bar{\theta}\int_0^{\infty}R_n^{(i)}(x)\beta_i(x)dx\right), & n=0\\ 0, & n\neq 0\end{cases} \quad (11)$$

The normalizing conditions is

$$I_0+\sum_{n=1}^{\infty}\int_0^{\infty}I_n(x)dx+\sum_{n=0}^{\infty}\sum_{i=1}^{M}\left[\int_0^{\infty}P_n^{(i)}(x)dx+\int_0^{\infty}R_n^{(i)}(x)dx\right]+\sum_{n=0}^{\infty}\sum_{j=1}^{J}\int_0^{\infty}V_n^{(j)}(x)dx=1 \quad (12)$$

Multiplying equations (2) to (11) by z^n and summing over all possible values of n, we get the following partial differential equations

$$\left[\frac{\partial}{\partial x}+(\lambda^{+}+\eta(x))\right]I(x,z)=0 \quad (13)$$

$$\left[\frac{\partial}{\partial x}+(\lambda^{+}+\lambda^{-}-\lambda^{+}C(z)+\mu_i(x))\right]P^{(i)}(x,z)=0,\ i=1,2,\ldots,M \quad (14)$$

$$\left[\frac{\partial}{\partial x}+(\lambda^{+}-\lambda^{+}C(z)+\beta_i(x))\right]R^{(i)}(x,z)=0,\ i=1,2,\ldots,M \quad (15)$$

$$\left[\frac{\partial}{\partial x}+(\lambda^{+}-\lambda^{+}C(z)+\gamma(x))\right]V_j(x,z)=0,\ j=1,2,\ldots,J \quad (16)$$

$$I(0,z)=\bar{\theta}\sum_{j=1}^{J}\int_0^{\infty}V^{(j)}(x,z)\gamma(x)dx+\sum_{i=1}^{M}\left[(\delta(z-1)+1)\int_0^{\infty}P^{(i)}(x,z)\mu_i(x)dx+\bar{\theta}\int_0^{\infty}R^{(i)}(x,z)\beta_i(x)dx\right]-\lambda^{+}I_0-\sum_{j=1}^{J}V_0^{(j)}(0) \quad (17)$$

$$P^{(i)}(0,z)=\frac{p_i}{z}\left[\lambda^{+}C(z)I_0+\int_0^{\infty}I(x,z)\eta(x)dx+\lambda^{+}C(z)\int_0^{\infty}I(x,z)dx+\theta\Big(\sum_{j=1}^{J}\int_0^{\infty}V^{(j)}(x,z)dx+\sum_{j=1}^{M}\int_0^{\infty}R^{(i)}(x,z)\beta_i(x)dx\Big)\right],i=1,2,\ldots,M \quad (18)$$

$$R^{(i)}(0,z)=\lambda^{-}\int_0^{\infty}P^{(i)}(x,z)dx,\quad i=1,2,\ldots,M \quad (19)$$

Solving equation (5) at n=0, we have

$$V_0^{(j)}(x)=V_0^{(j)}(0)e^{-\lambda^{+}x}[1-V(x)],\ j=1,2,\ldots,J \quad (20)$$

Multiplying equation (20) by $\gamma(x)$ and integrating with respect to x from 0 to ∞, we have

$$\begin{aligned}\int_0^{\infty}V_0^{(j)}(x)\gamma(x)dx&=\int_0^{\infty}V_0^{(j)}(0)e^{-\lambda^{+}x}(1-V(x))\gamma(x)dx\\&=V_0^{(j)}(0)V^{*}(\lambda^{+}),\ j=1,2,\ldots,J\end{aligned} \quad (21)$$

Equation (10) gives

$$V_0^{(j)}(0)=\bar{q}V_0^{(j-1)}(0)V^{*}(\lambda^{+}),\ j=2,3,\ldots,J \quad (22)$$

From equations (10) and (11) it is clear that $V^{(j)}(0,z)=V^{(j)}(0)$.

Applying equations (22) repeatedly for j=J,J-1,… we get

$$V^{(j)}(0,z)=\frac{V_0^{(J)}(0)}{\left[\bar{p}V^{*}(\lambda^{+})\right]^{J-1}},\ j=1,2,\ldots,J-1 \quad (23)$$

Substituting equation (22) and (23) in equation (1) and after some algebraic manipulations, we get

$$V_0^{(J)}(0)=\frac{\lambda^{+}I_0}{V^{*}(\lambda^{+})\left[1+\frac{q(1-(\bar{q}V^{*}(\lambda^{+}))^{J-1})}{(\bar{q}V^{*}(\lambda^{+}))^{J-1}(1-(\bar{q}V^{*}(\lambda^{+}))}\right]} \quad (24)$$

Solving the partial differential equations (13),(14),(15) and (16) we get respectively

$I(x,z)= I(0,z)\exp[-\lambda^{+}x][1-A(x)]$ (25)

$P^{(i)}(x,z)=P^{(i)}(0,z)\exp[-(\lambda^{-}+\lambda^{+}(1-C(z)))x][1-B_i(x)]$, i=1,2,…,M (26)

$R^{(i)}(x,z)= R^{(i)}(0,z)\exp[-(\lambda^{+}-\lambda^{+}C(z))][1-R_i(x)]$, i=1,2,…,M (27)

$V^{(j)}(x,z)=V^{(j)}(0,z)\exp[-(\lambda^{+}-\lambda^{+}C(z))x][1-V(x)]$, j=1,2,…,J (28)

Substituting the expression of $P^{(i)}(x,z)$ in equation (19), we obtain

$$R^{(i)}(0,z)=\lambda^{-}P^{(i)}(0,z)(1-B_i^{*}(g(z))/g(z),\ i=1,2,\ldots,M \quad (29)$$

where, $g(z)=\lambda^{+}+\lambda^{-}-\lambda^{+}C(z)$

Using equations (25), (26) ,(27)and (28) in equation (17) and (18), we get

$$I(0,z)=\bar{\theta}\sum_{j=1}^{J}V^{(j)}(0,z)V^{*}(h(z))+\sum_{i=1}^{M}\left[(\delta(z-1)+1)P^{(i)}(0,z)B_i^{*}(g(z))+\bar{\theta}R^{(i)}(0,z)R_i^{*}(h(z))\right]-\lambda^{+}I_0-\sum_{j=1}^{J}V_0^{(j)}(0) \tag{30}$$

$$P^{(i)}(0,z)=\frac{p_i}{z}\left[\lambda^{+}I_0C(z)+I(0,z)\left[A^{*}(\lambda^{+})+C(z)(1-A^{*}(\lambda^{+}))\right]+\theta\left[\sum_{j=1}^{J}V^{(j)}(0,z)V^{*}(h(z))+\sum_{j=1}^{M}R^{(i)}(0,z)R^{(i)}(h(z))\right]\right],i=1,2,\ldots,M \tag{31}$$

where , $h(z)=\lambda^{+}-\lambda^{+}C(z)$

Using equations (23),(24),(29) and (31) in equation (30) and simplifying we obtain

$$I(0,z)=I_0\lambda^{+}\begin{bmatrix}[C(z)+\theta QV^{*}(h(z))]\sum_{i=1}^{M}p_i[(\delta(z-1)+1)g(z)B_i^{*}(g(z))+\lambda^{-}\bar{\theta}(1-B_i^{*}(g(z))R_i^{*}(h(z))]+\\ [zg(z)-\lambda^{-}\theta\sum_{i=1}^{M}p_i(1-B_i^{*}(g(z))R_i^{*}(h(z))][((1-\theta)V^{*}(h(z))-1)Q-1]\end{bmatrix}/D(z) \tag{32}$$

$$\text{where, } D(z)=zg(z)-\lambda^{-}\theta\sum_{i=1}^{M}p_i(1-B_i^{*}(g(z))R_i^{*}(h(z))-[A^{*}(\lambda^{+})+C(z))(1-A^{*}(\lambda^{+}))](\sum_{i=1}^{M}p_i[(\delta(z-1)+1)g(z)B_i^{*}(g(z))+\lambda^{-}\bar{\theta}(1-B_i^{*}(g(z))R_i^{*}(h(z))])$$

$$\text{and}\quad Q=\frac{1-\left(\bar{q}V^{*}(\lambda^{+})\right)^{J}}{V^{*}(\lambda^{+})[(\bar{q}V^{*}(\lambda^{+}))^{J-1}(\bar{q}V^{*}(\lambda^{+})-1)+q((\bar{q}V^{*}(\lambda^{+}))^{J-1}-1)]}$$

Using equations (32) in equation (31) and on solving we have

$$P^{(i)}(0,z)=I_0\lambda^{+}p_ig(z)[A^{*}(\lambda^{+})(C(z)-1)+\theta QV^{*}(h(z))+[A^{*}(\lambda^{+})+C(z))(1-A^{*}(\lambda^{+}))][((1-\theta)V^{*}(h(z))-1)Q]]/D(z) \tag{33}$$

Using equations (33) in equation (29) and simplifying we get

$$R^{(i)}(0,z)=I_0\lambda^{+}\lambda^{-}p_i([1-B_i^{*}(g(z))][A^{*}(\lambda^{+})(C(z)-1)+\theta QV^{*}(h(z))+[A^{*}(\lambda^{+})+C(z))(1-A^{*}(\lambda^{+}))][((1-\theta)V^{*}(h(z))-1)Q]])/D(z) \tag{34}$$

By defining the partial probability generating functions $\chi(z)=\int_0^{\infty}\chi(x,z)dx$, we can find the orbit size probability generating functions as follows

The partial probability generating function of the orbit size when the server is idle is

$$I(z)=I_0(1-A^{*}(\lambda^{+}))\begin{bmatrix}[C(z)+\theta QV^{*}(h(z))]\sum_{i=1}^{M}p_i[(\delta(z-1)+1)g(z)B_i^{*}(g(z))+\lambda^{-}\bar{\theta}(1-B_i^{*}(g(z))R_i^{*}(h(z))]+\\ [zg(z)-\lambda^{-}\theta\sum_{i=1}^{M}p_i(1-B_i^{*}(g(z))R_i^{*}(h(z))][((1-\theta)V^{*}(h(z))-1)Q-1]\end{bmatrix}/D(z) \tag{35}$$

The partial probability generating function of the orbit size when the server is busy in type i service is

$$P^{(i)}(z)=I_0\lambda^{+}p_i[1-B_i^{*}(g(z))][A^{*}(\lambda^{+})(C(z)-1)+\theta QV^{*}(h(z))+[A^{*}(\lambda^{+})+C(z))(1-A^{*}(\lambda^{+}))][((1-\theta)V^{*}(h(z))-1)Q]]/D(z) \tag{36}$$

$,1\le i\le M$

The partial probability generating function of the orbit size when the server failed during type i service is under repair is given by

$$R^{(i)}(z)=I_0\lambda^{-}p_i([1-B_i^{*}(g(z))][1-R_i^{*}(h(z))]\bullet$$
$$[A^{*}(\lambda^{+})(C(z)-1)+\theta QV^{*}(h(z))+[A^{*}(\lambda^{+})+C(z))(1-A^{*}(\lambda^{+}))][((1-\theta)V^{*}(h(z))-1)Q]])/[(1-C(z))D(z)] \tag{37}$$

$,1\le i\le M$

The partial probability generating function of the orbit size when the server is on vacation is

$$V^{(j)}(z)=\frac{I_0\,Q\,[1-V^{*}(h(z))]}{(1-C(z))},\quad 1\le j\le J \tag{38}$$

Using the normalizing condition (12), I_0 can be obtained as \

$$I_0=\frac{\lambda^{-}[1-m_1(1-A^{*}(\lambda^{+}))\sum_{i=1}^{M}p_i[1-\theta(1-B_i^{*}(\lambda^{-}))]-\lambda^{+}m_1[1-\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]-\sum_{i=1}^{M}p_i[\lambda^{-}\delta B_i^{*}(\lambda^{-})+\lambda^{+}\lambda^{-}m_1\beta_i^{(1)}(1-B_i^{*}(\lambda^{-}))]}{(\lambda^{+}\lambda^{-}v_1Q+\lambda^{-}A^{*}(\lambda^{+})-\theta Q(1-A^{*}(\lambda^{+}))[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]} \tag{39}$$

Performance Measures

Probability that the server is idle in the non-empty system is given by

$$I=\lim_{z\to1}I(z)=\frac{\begin{array}{c}(1-A^{*}(\lambda^{+}))\{(\lambda^{-}m_1+\lambda^{+}\lambda^{-}m_1v_1Q)(1-\theta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-}))+(1+\theta Q)[\lambda^{+}m_1-\lambda^{-}+\lambda^{-}\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})\\ -\lambda^{+}m_1\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})+\lambda^{+}\lambda^{-}m_1\sum_{i=1}^{M}p_i\beta_i^{(1)}(1-B_i^{*}(\lambda^{-}))\}\end{array}}{(\lambda^{+}\lambda^{-}v_1Q+\lambda^{-}A^{*}(\lambda^{+})-\theta Q(1-A^{*}(\lambda^{+}))[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]} \tag{40}$$

Probability that the server is busy is given by

$$P=\lim_{z\to 1}\sum_{i=1}^{M}P^{(i)}(z)=\frac{\sum_{i=1}^{M}\lambda^{+}m_1p_i(1-B_i^{*}(\lambda^{-}))}{\lambda^{-}[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]} \qquad (41)$$

Probability that the server is under repair is given by

$$R=\lim_{z\to 1}\sum_{i=1}^{M}R^{(i)}(z)=\frac{\lambda^{+}m_1\sum_{i=1}^{M}\beta_i^{(1)}p_i(1-B_i^{*}(\lambda^{-}))}{[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]} \qquad (42)$$

Probability that the server is on vacation is given by

$$V=\lim_{z\to 1}\sum_{j=1}^{J}V^{(j)}(z)=\frac{\lambda^{+}v_1Q[\lambda^{-}[1-m_1(1-A^{*}(\lambda^{+}))\sum_{i=1}^{M}p_i(1-\theta(1-B_i^{*}(\lambda^{-})))]-\lambda^{+}m_1[1-\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]-\sum_{i=1}^{M}p_i[\lambda^{-}\delta B_i^{*}(\lambda^{-})+\lambda^{+}\lambda^{-}m_1\beta_i^{(1)}(1-B_i^{*}(\lambda^{-}))]}{(\lambda^{+}\lambda^{-}v_1Q+\lambda^{-}A^{*}(\lambda^{+})-\theta Q(1-A^{*}(\lambda^{+})))[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]} \qquad (43)$$

Probability generating function of the number of customers in the orbit is given by

$$P_q(z)=\frac{I_0+I(z)+\sum_{i=1}^{M}[P^{(i)}(z)+R^{(i)}(z)]+\sum_{j=1}^{J}V_j(z)}{}$$

$$=I_0(z-1)g(z)[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(g(z))]\,[(A^{*}(\lambda^{+})+C(z)(1-A^{*}(\lambda^{+})))[Q(1-(1-\theta)V^{*}(h(z))]-\theta QV^{*}(h(z))+A^{*}(\lambda^{+})(1-C(z))]/[(1-C(z))D(z)] \qquad (44)$$

Mean number of customers in the orbit L_q is given by

Let Nr(z) and Dr(z) be the numerator and denominator of $P_q(z)$.Then $Nr(1)=Dr(1)=Nr'(1)=Dr'(1)=0$.Applying L'Hospital rule we get

$$L_q=P_q'(1)=\frac{[Dr''(1)Nr'''(1)-Nr''(1)Dr'''(1)]}{3Dr''(1)^2} \qquad (45)$$

where,

$$Nr''(1)=-I_0 2\lambda^{-}m_1[A^{*}(\lambda^{+})-\theta Q(1-A^{*}(\lambda^{+}))+\lambda^{+}v_1Q][1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]$$

$$\begin{aligned}Nr(1)''=&I_0[6((-m_1A^{*}(\lambda^{+})+m_1\theta Q(1-A^{*}(\lambda^{+}))-\lambda^{+}m_1v_1Q)(-\delta\lambda^{+}\lambda^{-}m_1\sum_{i=1}^{M}p_i\mu_i^{(1)}-\lambda^{+}m_1[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]))\\&+3\lambda^{-}[1-\delta\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})](-m_2A^{*}(\lambda^{+})+m_2\theta Q(1-A^{*}(\lambda^{+}))-2\lambda^{+}m_1^{2}v_1Q(1-\theta)(1-A^{*}(\lambda^{+}))\\&-Q(\lambda^{+}m_2v_1+\lambda^{+^{2}}m_1^{2}v_2))]\end{aligned}$$

$$Dr''(1)=-2m_1\{\lambda^{-}[1-m_1(1-A^{*}(\lambda^{+}))\sum_{i=1}^{M}p_i[1-\theta(1-B_i^{*}(\lambda^{-}))]-\lambda^{+}m_1[1-\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]-\sum_{i=1}^{M}p_i[\lambda^{-}\delta B_i^{*}(\lambda^{-})+\lambda^{+}\lambda^{-}m_1\beta_i^{(1)}(1-B_i^{*}(\lambda^{-}))]\}$$

$$\begin{aligned}Dr'''(1)=&-3m_1[-2\lambda^{+}m_1-\lambda^{+}m_2(1-\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-}))-\lambda^{-}m_2(1-A^{*}(\lambda^{+}))\sum_{i=1}^{M}p_i[1-\theta(1-B_i^{*}(\lambda^{-}))]-2m_1(1-A^{*}(\lambda^{+}))\sum_{i=1}^{M}p_i[(\lambda^{-}\delta B_i^{*}(\lambda^{-})\\&-\lambda^{+}m_1B_i^{*}(\lambda^{-})+\lambda^{+}\lambda^{-}m_1(\theta\mu_i^{(1)}+\beta_i^{(1)}(1-\theta)(1-B_i^{*}(\lambda^{-}))]-\sum_{i=1}^{M}p_i\{-2\lambda^{+^{2}}m_1^{2}\lambda^{-}\mu_i^{(1)}\beta_i^{(1)}-2\lambda^{+^{2}}m_1^{2}\mu_i^{(1)}-2\delta\lambda^{+}m_1B_i^{*}(\lambda^{-})\\&+2\lambda^{+}\lambda^{-}m_1\mu_i^{(1)}\delta+\lambda^{-}(1-B_i^{*}(\lambda^{-}))[\lambda^{+^{2}}m_1^{2}\beta_i^{(2)}+\lambda^{+}m_2\beta_i^{(1)}]-3m_2[\lambda^{-}[1-m_1(1-A^{*}(\lambda^{+}))\sum_{i=1}^{M}p_i[1-\theta(1-B_i^{*}(\lambda^{-}))]\\&-\lambda^{+}m_1[1-\sum_{i=1}^{M}p_iB_i^{*}(\lambda^{-})]-\sum_{i=1}^{M}p_i[\lambda^{-}\delta B_i^{*}(\lambda^{-})+\lambda^{+}\lambda^{-}m_1\beta_i^{(1)}(1-B_i^{*}(\lambda^{-}))]]\end{aligned}$$

Probability generating function of the number of customers in the system is given by

$$P_s(z)=\frac{I_0+I(z)+zP(z)+R(z)+\sum_{j=1}^{J}V_j(z)}{}$$

$$=I_0(z-1)[\lambda^{-}+\sum_{i=1}^{M}p_iB_i^{*}(g(z))(h(z)-\delta g(z))]\,[(A^{*}(\lambda^{+})+C(z)(1-A^{*}(\lambda^{+})))[Q(1-(1-\theta)V^{*}(h(z))]-\theta QV^{*}(h(z))+A^{*}(\lambda^{+})(1-C(z))]/[(1-C(z))D(z)] \qquad (46)$$

Mean number of customers in the orbit L_s is given by

Let $Nr_1(z)$ and Dr(z) be the numerator and denominator of $P_s(z)$. Since $Nr_1(1)=Dr(1)=Nr_1'(1)=Dr'(1)=0$, applying L'Hospital rule we get

$$L_S = P_S'(1) = \frac{\lfloor Dr''(1)Nr_1'''(1) - Nr_1''(1)Dr'''(1) \rfloor}{3Dr''(1)^2} \tag{47}$$

where,

$$Nr_1''(1) = -2\lambda^- m_1[A^*(\lambda^+) - \theta Q(1-A^*(\lambda^+)) + \lambda^+ v_1 Q][1-\delta\sum_{i=1}^{M} p_i B_i^*(\lambda^-)]I_0$$

$$\begin{aligned} Nr_1'''(1) = I_0[&6((-m_1 A^*(\lambda^+) + m_1(1-A^*(\lambda^+))\theta Q - \lambda^+ m_1 v_1 Q)(-\delta\lambda^+\lambda^- m_1 \sum_{i=1}^{M} p_i \mu_i^{(1)} + \lambda^+ m_1(\delta-1)\sum_{i=1}^{M} p_i B_i^*(\lambda^-))) \\ &+ 3(\lambda^-[1-\delta\sum_{i=1}^{M} p_i B_i^*(\lambda^-)](-m_2 A^*(\lambda^+) + m_2(1-A^*(\lambda^+))\theta Q - 2\lambda^+ m_1^2 v_1(1-\theta)Q(1-A^*(\lambda^+)) - Q(\lambda^+ m_2 v_1 + \lambda^{+2} m_1^2 v_2))] \end{aligned}$$

Reliability Indices

Let A(t) be the pointwise availability of the server at time t, that is the probability that the server is idle or busy. The steady state availability of the server will be $A = \lim_{t\to\infty} A(t)$.

The availability of the server is given by

$$A = 1-(R+V) = \frac{\begin{aligned}&(\lambda^+\lambda^- v_1 Q + \lambda^- A^*(\lambda^+) - \theta Q(1-A^*(\lambda^+)))\lambda^+ m_1 \sum_{i=1}^{M} \beta_i^{(1)} p_i(1-B_i^*(\lambda^-)) + \lambda^+ v_1 Q[\lambda^-[1-m_1+m_1 A^*(\lambda^+)] \\ &-\lambda^+ m_1[1-\sum_{i=1}^{M} p_i B_i^*(\lambda^-)] - \sum_{i=1}^{M} p_i[\lambda^-\delta B_i^*(\lambda^-) + \lambda^+\lambda^- m_1 \beta_i^{(1)}(1-B_i^*(\lambda^-))]\end{aligned}}{(\lambda^+\lambda^- v_1 Q + \lambda^- A^*(\lambda^+) - \theta Q(1-A^*(\lambda^+)))[1-\delta\sum_{i=1}^{M} p_i B_i^*(\lambda^-)]} \tag{48}$$

Steady state failure frequency of the server is

$$F = \lambda^- P = \frac{\lambda^+ m_1 \sum_{i=1}^{M} p_i(1-B_i^*(\lambda^-))}{[1-\delta\sum_{i=1}^{M} p_i B_i^*(\lambda^-)]} \tag{49}$$

Stochastic Decompostion

Theorem: The number of customers in the system (L_s) can be expressed as the sum of two independent random variables, one of which is the mean number of customers (L) in the unreliable batch arrival G-queue with multi-optional service and feedback and the other is the mean number of customers in the orbit (L_I) given that the server is idle or on vacation.

Proof: The probability generating function $\pi(z)$ of the number of customers in the Classical queueing system with negative customers, feedback, server breakdown and randomized J vacation is given by

$$\pi(z) = \frac{(z-1)[\lambda^- + \sum_{i=1}^{M} p_i B_i^*(g(z))(h(z)-\delta g(z))](1-C(z))(\lambda^- - \lambda^+ m_1[1-\sum_{i=1}^{M} p_i B_i^*(\lambda^-)] - \sum_{i=1}^{M} p_i[\lambda^-\delta B_i^*(\lambda^-) + \lambda^+\lambda^- m_1\beta_i^{(1)}(1-B_i^*(\lambda^-))])}{\lambda^-(1-C(z))[1-\delta\sum_{i=1}^{M} p_i B_i^*(\lambda^-)]\left[zg(z) - \sum_{i=1}^{M} p_i[(\delta(z-1)+1)g(z)B_i^*(g(z)) + \lambda^-(1-B_i^*(g(z))R_i^*(h(z))]\right]} \tag{50}$$

The probability generating function $\chi(z)$ of the number of customers in the orbit given that the server is idle or on vacation is

$$\chi(z) = \frac{I_0 + I(z) + \sum_{j=1}^{J} V_j(z)}{I_0 + I + V} \tag{51}$$

$$= \frac{\left[\begin{array}{l}\left[\begin{array}{l}\left[A^*(\lambda^+)(1-C(z)) - \theta QV^*(h(z)) - [A^*(\lambda^+) + C(z))(1-A^*(\lambda^+))][((1-\theta)V^*(h(z))-1)Q]\right] \\ \left[zg(z) - \sum_{i=1}^{M} p_i[(\delta(z-1)+1)g(z)B_i^*(g(z)) + \lambda^-(1-B_i^*(g(z))R_i^*(h(z))]\right]\end{array}\right] \\ \left[\lambda^-[1-m_1(1-A^*(\lambda^+))\sum_{i=1}^{M} p_i[1-\theta(1-B_i^*(\lambda^-))] - \lambda^+ m_1[1-\sum_{i=1}^{M} p_i B_i^*(\lambda^-)] - \sum_{i=1}^{M} p_i[\lambda^-\delta B_i^*(\lambda^-) + \lambda^+\lambda^- m_1\beta_i^{(1)}(1-B_i^*(\lambda^-))]\right]\end{array}\right]}{(1-C(z))D(z)[(\lambda^+ v_1 Q + A^*(\lambda^+) - \theta Q(1-A^*(\lambda^+))][\lambda^- - \lambda^+ m_1[1-\sum_{i=1}^{M} p_i B_i^*(\lambda^-)] - \lambda^-\delta\sum_{i=1}^{M} p_i B_i^*(\lambda^-) - \lambda^+\lambda^- m_1 \sum_{i=1}^{M} p_i\beta_i^{(1)})(1-B_i^*(\lambda^-))]}$$

From equations (46),(50) and (51), we see that $P_s(z) = \pi(z)\,\chi(z)$. Hence, $L_s = L + L_I$.

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