



Inversion Theorem for Distributional Fourier-Finite Mellin Transform

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ARTICLE INFO

Article history:

Received: 29 October 2015;

Received in revised form:

30 November 2015;

Accepted: 4 December 2015;

Keywords

Fourier Transform,
Finite Mellin Transform,
Fourier-Finite Mellin
Transform,
Generalized Function,
Testing Function Space.

ABSTRACT

The Fourier-Finite Mellin transform is a variant of the Fourier transform; however, it certainly does not have as glorious or as predominant a history as the Fourier transform. This Fourier-Finite Mellin transform is used to correct various optical distortions, including noise in lenses, it is also used in radar classification of ships. Theoretically, the Fourier-Finite Mellin transform should provide a truly translation, rotation and scale invariant measure of an image. Fourier-Finite Mellin transform is frequently used in content-based image retrieval and digital image watermarking.

The Object of the present paper is to prove an Inversion theorem for distributional Fourier-Finite Mellin transform with the help of two lemmas which are also given in this paper.

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Introduction

Integral transformation is one of the well known techniques used for function transformation. And the transform analysis of generalized functions concentrates on finite parts of integrals, generalized functions and distributions. In this paper we stressed on Fourier-Finite Mellin transform. The Pioneer of this Fourier-Finite Mellin Transform is Robbin's and Huang [1972]. The Fourier-Finite Mellin Transform is used in signal processing, image processing and many more.

Human face recognition is, indeed, a challenging task, especially under illumination and pose variations. So the effectiveness of a simple face recognition algorithm is based on Fourier-Finite Mellin Transform. [1]. Fourier-Finite Mellin Transform has found application in optical pattern recognition, Ship classification by sonar and radar and image processing. It is also used to correct various optical distortions, including noise, in lenses. Fourier-Finite Mellin Transform is frequently used in content-based image retrieval and digital image watermarking. The Fourier-Finite Mellin Transform is used to help identify the leaf of a plant, regardless of the leaf's scale or rotation, or location in the image [2].

There are various theorems for the above said Fourier-Finite Mellin Transform and Fourier-Laplace transform like Analyticity Theorem, Abelian theorem, Representation Theorem which are already discussed in our previous papers. The main aim of this paper is to generalized the Fourier-Finite Mellin Transform in the Distributional Sense and to presents the Inversion Theorem for Distributional Fourier-Finite Mellin Transform.

Above work requires testing function space and definition of distributional generalized Fourier-Finite Mellin transform which are given as:

The space $FM_{f,b,c,\alpha}$

This space is given by

$$FM_{f,b,c,\alpha} = \left\{ \phi : \phi \in E_+ / \xi_{b,c,k,q,l} \phi(t,x) = 0 < t < \infty \left| t^k \lambda_{b,c}(x) x^{q+1} D_t^l D_x^q \phi(t,x) \right| \leq C_{lq} A^k k^{k\alpha} \right\} \quad (1.1.1)$$

$$0 < x < a$$

for each $k, l, q = 0, 1, 2, 3, \dots$

where,

$$\lambda_{b,c}(x) = \begin{cases} x^{+b} & 0 < x < 1 \\ x^{+c} & 1 < x < a \end{cases}$$

Where the constants A and C_{lq} depend on the testing function ϕ .

Distributional Generalized Fourier-Finite Mellin transforms ($FM_f T$)

For $f(t,x) \in FM_{f,b,c,\alpha}^{*\beta}$, where $FM_{f,b,c,\alpha}^{*\beta}$ is the dual space of $FM_{f,b,c,\alpha}^\beta$. It contains all distributions of compact

support. The Distributional Fourier-Finite Mellin transform is a function of $f(t, x)$ and is defined as

$$FM_f \{ f(t, x) \} = F(s, p) = \left\langle f(t, x), e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right\rangle \quad (1.2.1)$$

where, for each fixed t ($0 < t < \infty$), x ($0 < x < \infty$), $s > 0$ and $p > 0$, the right hand side of (1.2.1) has a sense as an application of $f(t, x) \in FM_{f,b,c,\alpha}^{*\beta}$ to $e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \in FM_{f,b,c,\alpha}^\beta$.

This paper is summarized as follows:

In Section 2, Lemma 1 and Lemma 2 are given. In section 3, Inversion theorem for Fourier-Finite Mellin Transform is proved. Uniqueness theorem is proved in section 4. Lastly conclusions are given in Section 5.

Notations and terminology are as per Zemanian [3], [4].

Inversion Theorem for Distributional Fourier-Finite Mellin Transform

Lemma 1

Statement

Let

$$FM_f \{ f(t, x) \} = F(s, p)$$

And

$$\text{supp } f \subset S_A \cap S_B,$$

where

$$S_A = \{ t : t \in \mathbb{R}^n, |t| \leq A, A > 0 \}$$

and $S_B = \{ x : x \in \mathbb{R}^n, |x| \leq B, B > 0 \}$, for $s > 0$ and $\rho_1 < \text{Re } p < \rho_2$. Let $\phi \in D$ and

$$\psi(s, p) = \int_{-\infty}^{\infty} \int_0^a \phi(t, x) e^{ist} x^{-p} dt dx$$

then for any fixed real number τ and r with $-\infty < r < \infty$, $0 < \tau < a$,

$$\int_{-r-\tau}^r \int_{-\tau}^{\tau} \left\langle f(t, x), e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right\rangle \psi(s, p) ds dw = \left\langle f(t, x), \int_{-r-\tau}^r \int_{-\tau}^{\tau} e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \psi(s, p) ds dw \right\rangle,$$

where

$p = \rho + iw$, also s and ρ are fixed with $\sigma_1 < s < \sigma_2$ and $\rho_1 < p < \rho_2$.

Proof

For $\phi(t, x) \equiv 0$, the result is trivial, so assume that $\phi(t, x) \neq 0$. If $FM_f \{ f(t, x) \} = F(s, p)$, then

$$F(s, p)$$

is analytic for $s > 0$, $\rho_1 < \text{Re } p < \rho_2$ and $\psi(s, p)$ is an entire function. Therefore above integrals certainly exist.

In order that right hand side is meaningful, we show that

$$\int_{-r-\tau}^r \int_{-\tau}^{\tau} \psi(s, p) e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) ds dw \in FM_{f,b,c,\alpha}.$$

Consider,

$$\begin{aligned} &\leq \int_{-r-\tau}^r \int_{-\tau}^{\tau} \left| t^k \lambda_{b,c}(x) x^{q+1} D_t^l D_x^q e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \psi(s, p) \right| ds dw \\ &\leq \int_{-r-\tau}^r \int_{-\tau}^{\tau} \left| t^k \lambda_{b,c}(x) x^{q+1} (-i)^l s^l e^{-ist} \{ a^{2p} P(-p-q) x^{-p-q-1} - P(p-q) x^{p-q-1} \} \psi(s, p) \right| ds dw \\ &\leq \int_{-r-\tau}^r \int_{-\tau}^{\tau} \left| t^k \lambda_{b,c}(x) (s)^l e^{-ist} \{ a^{2p} P(-p) x^{-p} - P(p) x^p \} \psi(s, p) \right| ds dw \end{aligned}$$

Where $P(p)$ is a polynomial in p

$$\begin{aligned}
&\leq \int_{-r-\tau}^r \int_{-\tau}^{\tau} \left| t^k \lambda_{b,c}(x)(s)^l e^{-ist} \left\{ \left(\frac{a}{x} \right)^{2p} P(-p)x^p - P(p)x^p \right\} \psi(s, p) \right| ds dw \\
&\leq \int_{-r-\tau}^r \int_{-\tau}^{\tau} \left| t^k \lambda_{a,b}(x)(s)^l \left\{ \left(\frac{a}{x} \right)^{2(\rho+iw)} P(-p) - P(p) \right\} e^{-ist} x^{\rho+iw} \psi(s, p) \right| ds dw < \infty \\
&\Rightarrow \int_{-r-\tau}^r \int_{-\tau}^{\tau} e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \psi(s, p) ds dw \in FM_{f,b,c,\alpha}
\end{aligned} \tag{2.1.1}$$

Partition the path of integration on the straight line from $s = -r$ to $s = r$ into m -intervals, each of length

$$\frac{2r}{m}$$

and from $p = \rho - i\tau$ to $p = \rho + i\tau$ into n -intervals, each of length $\frac{2\tau}{n}$.

Let $s_\nu = \sigma$ be any point in ν^{th} interval and $p_\mu = \rho + iw$ be any point in μ^{th} interval.

$$\text{Suppose,} \quad \phi_{m,n}(t, x) = \sum_{\nu=1}^m \sum_{\mu=1}^n e^{-is_\nu t} \left(\frac{a^{2p_\mu}}{x^{p_\mu+1}} - x^{p_\mu-1} \right) \psi(s_\nu, p_\mu) \frac{2r}{m} \frac{2\tau}{n} \tag{2.1.2}$$

To show that

$$\phi_{m,n}(t, x)$$

converges in $FM_{f,b,c,\alpha}$ to

$$\int_{-r-\tau}^r \int_{-\tau}^{\tau} e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \psi(s, p) ds dw$$

we have that

$$\phi_{m,n}(t, x) - \int_{-r-\tau}^r \int_{-\tau}^{\tau} e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \psi(s, p) ds dw$$

converges to zero in $FM_{f,b,c,\alpha}$ as $m, n \rightarrow \infty$

We write,

$$\begin{aligned}
&\left| t^k \lambda_{b,c}(x) x^{q+1} D_t^l D_x^q \left[\phi_{m,n}(t, x) - \int_{-r-\tau}^r \int_{-\tau}^{\tau} e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \psi(s, p) ds dw \right] \right| \\
&= \left| t^k \lambda_{b,c}(x) x^{q+1} D_t^l D_x^q \left[\sum_{\nu=1}^m \sum_{\mu=1}^n e^{-is_\nu t} \left(\frac{a^{2p_\mu}}{x^{p_\mu+1}} - x^{p_\mu-1} \right) \psi(s_\nu, p_\mu) \frac{2r}{m} \frac{2\tau}{n} \right. \right. \\
&\quad \left. \left. - \int_{-r-\tau}^r \int_{-\tau}^{\tau} e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \psi(s, p) ds dw \right] \right| \\
&= \left| t^k \lambda_{b,c}(x) \sum_{\nu=1}^m \sum_{\mu=1}^n s_\nu^l e^{-is_\nu t} \left\{ a^{2p_\mu} P(-p_\mu) x^{-p_\mu} - P(p_\mu) x^{p_\mu} \right\} \psi(s_\nu, p_\mu) \frac{2r}{m} \frac{2\tau}{n} \right. \\
&\quad \left. - t^k \lambda_{b,c}(x) \int_{-r-\tau}^r \int_{-\tau}^{\tau} s^l e^{-ist} \left\{ a^{2p} P(-p) x^{-p} - P(p) x^p \right\} \psi(s, p) ds dw \right|
\end{aligned}$$

Where $P(-p_\mu - q)$ is a polynomial in p_μ etc.

$$= \left| t^k \lambda_{b,c}(x) \sum_{\nu=1}^m \sum_{\mu=1}^n s_{\nu}^l e^{-is_{\nu}t} x^{p_{\mu}} \left\{ \left(\frac{a}{x} \right)^{2p_{\mu}} P(-p_{\mu}) - P(p_{\mu}) \right\} \psi(s_{\nu}, p_{\mu}) \frac{2r}{m} \frac{2\tau}{n} \right. \\ \left. - t^k \lambda_{b,c}(x) \int_{-r-\tau}^r \int_{-\tau}^{\tau} s^l e^{-ist} \left\{ \left(\frac{a}{x} \right)^{2p} P(-p) - P(p) \right\} x^p \psi(s, p) ds dw \right| \quad (2.1.3)$$

Since

$$\int_{-r-\tau}^r \int_{-\tau}^{\tau} s^l \left\{ \left(\frac{a}{x} \right)^{2p} P(-p) - P(p) \right\} \psi(s, p) ds dw$$

is finite by (4.1.1) and

$$|t^k \lambda_{b,c}(x) x^p e^{-ist}| \rightarrow 0,$$

for sufficiently large values of x and t .

Given any $\varepsilon > 0$, we can choose x_0 and t_0 so large that for $x > x_0$ and $t > t_0$,

$$\left| t^k \lambda_{b,c}(x) \int_{-r-\tau}^r \int_{-\tau}^{\tau} \left\{ \left(\frac{a}{x} \right)^{2p} P(-p) - P(p) \right\} s^l e^{-ist} x^p \psi(s, p) ds dw \right| < \frac{\varepsilon}{3} \quad (2.1.4)$$

Now consider the first term of (4.1.3), choosing m_0 and n_0 so large that, for $m > m_0$ and $n > n_0$,

$$\left| t^k \lambda_{b,c}(x) \sum_{\nu=1}^m \sum_{\mu=1}^n \left\{ \left(\frac{a}{x} \right)^{2p_{\mu}} P(-p_{\mu}) - P(p_{\mu}) \right\} s_{\nu}^l e^{-is_{\nu}t} x^{p_{\mu}} \psi(s_{\nu}, p_{\mu}) \frac{2r}{m} \frac{2\tau}{n} \right| < \frac{2\varepsilon}{3} \quad (2.1.5)$$

for all $x > x_0$ and $t > t_0$.

In view of above in equations (2.1.4), (2.1.5) and (2.1.3),

$$\left| t^k \lambda_{b,c}(x) x^{q+1} D_t^l D_x^q \left[\phi_{m,n}(t, x) - \int_{-r-\tau}^r \int_{-\tau}^{\tau} e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \psi(s, p) ds dw \right] \right| < \varepsilon \\ \Rightarrow \phi_{m,n}(t, x) \text{ converges to } \int_{-r-\tau}^r \int_{-\tau}^{\tau} e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \psi(s, p) ds dw \text{ in } FM_{f,b,c,\alpha}.$$

$$\text{Hence} \quad \langle f(t, x), \phi_{m,n}(t, x) \rangle = \left\langle f(t, x), \int_{-r-\tau}^r \int_{-\tau}^{\tau} e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \psi(s, p) ds dw \right\rangle \quad (2.1.6)$$

Further left hand side of (2.1.6)

$$= \left\langle f(t, x), \sum_{\nu=1}^m \sum_{\mu=1}^n e^{-is_{\nu}t} \left(\frac{a^{2p_{\mu}}}{x^{p_{\mu}+1}} - x^{p_{\mu}-1} \right) \psi(s_{\nu}, p_{\mu}) \frac{2r}{m} \frac{2\tau}{n} \right\rangle \\ = \sum_{\nu=1}^m \sum_{\mu=1}^n \left\langle f(t, x), e^{-is_{\nu}t} \left(\frac{a^{2p_{\mu}}}{x^{p_{\mu}+1}} - x^{p_{\mu}-1} \right) \psi(s_{\nu}, p_{\mu}) \frac{2r}{m} \frac{2\tau}{n} \right\rangle \\ = \sum_{\nu=1}^m \sum_{\mu=1}^n \left\langle f(t, x), e^{-is_{\nu}t} \left(\frac{a^{2p_{\mu}}}{x^{p_{\mu}+1}} - x^{p_{\mu}-1} \right) \right\rangle \psi(s_{\nu}, p_{\mu}) \frac{2r}{m} \frac{2\tau}{n} \\ = \int_{-r-\tau}^r \int_{-\tau}^{\tau} \left\langle f(t, x), e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right\rangle \psi(s, p) ds dw.$$

Since $\left\langle f(t, x), e^{-is_v t} \left(\frac{a^{2p_\mu}}{x^{p_\mu+1}} - x^{p_\mu-1} \right) \right\rangle \psi(s_v, p_\mu)$ is a continuous function of s and w from (2.1.6). We have

$$\int_{-r}^r \int_{-\tau}^{\tau} \left\langle f(t, x), e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \right\rangle \psi(s, p) ds dw$$

$$= \left\langle f(t, x), \int_{-r}^r \int_{-\tau}^{\tau} e^{-ist} \left(\frac{a^{2p}}{x^{p+1}} - x^{p-1} \right) \psi(s, p) ds dw \right\rangle$$
Lemma 2

If $\phi \in D(I)$ and

$$\frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_0^a \frac{\sin(t-v)r}{t-v} \left[\frac{a^{2p} x^{-\sigma+1}}{u^{\sigma+1}} \frac{\sin w(\log xu)}{x \log xu} - \left(\frac{u}{x} \right)^{\sigma-1} \frac{\sin w \log \left(\frac{u}{x} \right)}{x \log \left(\frac{u}{x} \right)} \right] \phi(t, x) dt dx = A(v, u),$$

Then $A(v, u)$ converges in $FM_{f, b, c, \alpha}$ to

$$\phi(v, u)$$

as $W \rightarrow \infty$, where b, c and W are real numbers.

Proof: For this we have to show that

$$\sup_{\substack{-\infty < t < \infty \\ 0 < x < a}} \gamma_{l,q} \left[A(v, u) - \phi(v, u) \right] = \left| v^k \lambda_{b,c}(u) u^{q+1} D_v^l D_u^q \left\{ \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_0^a \frac{\sin(t-v)r}{(t-v)} \left[\frac{a^{2p} x^{-\sigma+1}}{u^{\sigma+1}} \frac{\sin w \log xu}{x \log xu} - \left(\frac{u}{x} \right)^{\sigma-1} \frac{\sin w \log \left(\frac{u}{x} \right)}{x \log \left(\frac{u}{x} \right)} \right] \phi(t, x) dt dx - \phi(v, u) \right\} \right| \rightarrow 0$$

As $W \rightarrow \infty$,

$$\text{Now } \int_{-\infty}^{\infty} \frac{\sin(t-v)r}{(t-v)} dt = \int_{-\infty}^{\infty} \frac{\sin(rt_1)}{t_1} dt_1 = \pi,$$

$$\int_0^a \frac{\sin w \log xu}{x \log xu} dx = \int_{\infty}^d \frac{\sin(wx_1)}{x_1} dx_1 = \frac{\pi}{2}, \text{ as } W \rightarrow \infty,$$

$$\int_0^a \frac{\sin w \log \left(\frac{u}{x} \right)}{x \log \left(\frac{u}{x} \right)} dx = \int_{\infty}^d \frac{\sin(wx_1)}{x_1} dx_1 = \frac{\pi}{2}, \text{ as } W \rightarrow \infty,$$

$$\sup_{\substack{-\infty < t < \infty \\ 0 < x < a}} \gamma_{l,q} \left[A(v, u) - \phi(v, u) \right] = \left| v^k \lambda_{b,c}(u) u^{q+1} D_v^l D_u^q \left\{ \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_0^a \frac{\sin(t-v)r}{(t-v)} \right. \right.$$

$$\begin{aligned}
& \left[\frac{a^{2p} x^{-\sigma+1}}{u^{\sigma+1}} \frac{\sin w \log xu}{x \log xu} - \left(\frac{u}{x} \right)^{\sigma-1} \frac{\sin w \log \left(\frac{u}{x} \right)}{x \log \left(\frac{u}{x} \right)} \right] \phi(t, x) dt dx - \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin(t-v)r}{(t-v)} dt \\
& \left[\int_0^a \frac{\sin w(\log xu)}{x \log xu} dx + \int_0^a \frac{\sin w \log \left(\frac{u}{x} \right)}{x \log \frac{u}{x}} dx \right] \phi(v, u) \Bigg\} \\
& \sup_{-\infty < t < \infty} \left| v^k \lambda_{b,c}(u) u^{q+1} D_v^l D_u^q \left\{ \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin rt_1}{t_1} dt_1 \left[\int_{-\infty}^d \frac{a^{2p} x^{-\sigma+1}}{u^{\sigma+1}} \frac{\sin wx_1}{x_1} \phi(t, x) dx_1 \right. \right. \right. \\
& \left. \left. \left. - \int_{-\infty}^d \frac{\sin wx_1}{x_1} dx_1 \phi(v, u) + \int_{-\infty}^d \left(\frac{u}{x} \right)^{\sigma-1} \frac{\sin wx_1}{x_1} \phi(t, x) dx_1 - \int_{-\infty}^d \frac{\sin wx_1}{x_1} \phi(v, u) dx_1 \right] \right\} \right| \\
& \sup_{-\infty < t < \infty} \left| v^k \lambda_{b,c}(u) u^{q+1} D_v^l D_u^q \left\{ \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin rt_1}{t_1} dt_1 \left(\int_{-\infty}^d \left[\frac{a^{2p} x^{-\sigma+1}}{u^{\sigma+1}} \phi \left(t_1 + v, \frac{e^{x_1}}{u} \right) - \phi(v, u) \right] \right. \right. \right. \\
& \left. \left. \left. \frac{\sin wx_1}{x_1} dx_1 + \int_{-\infty}^d \left[\left(\frac{u}{x} \right)^{\sigma-1} \phi \left(t_1 + v, \frac{e^{x_1}}{u} \right) - \phi(v, u) \right] \frac{\sin wx_1}{x_1} dx_1 \right) \right\} \right|
\end{aligned}$$

Since ϕ is continuous, as in Zemanian [5] pp.66 it can be shown that

$$\gamma_{l,q} [A(v, u) - \phi(v, u)] \rightarrow 0$$

Hence Proved.

Now the proof of Inversion Theorem can be easily completed.

Inversion Theorem

Statement

Let $FM_f \{f(t, x)\} = F(s, p)$, for $s > 0$ and $\rho_1 < p < \rho_2$. Also, let r and τ be a real variable such that $-\infty < r < \infty$, $0 < \tau < a$. Then in the sense of convergence in D^* ,

$$\lim_{\substack{r \rightarrow \infty \\ \tau \rightarrow \infty}} \frac{1}{4\pi^2 i} \int_{-r}^r \int_{\rho-i\tau}^{\rho+i\tau} F(s, p) e^{ist} x^{-p} ds dp,$$

where $p = \rho + iw$, also s and p are fixed real numbers with $-r < s < r$ and $\rho_1 < p < \rho_2$.

Proof

Let $\phi \in D$. Choose the real numbers c and d such that $c < s < d$ and the real numbers a and b such that $\rho_1 < a < p < b < \rho_2$, we have to show that

$$\lim_{\substack{r \rightarrow \infty \\ \tau \rightarrow \infty}} \left\langle f(t, x), \phi(t, x) \right\rangle = \left\langle \frac{1}{4\pi^2 i} \int_{-r}^r \int_{\rho-i\tau}^{\rho+i\tau} F(s, p) e^{ist} x^{-p} ds dp, \phi(t, x) \right\rangle \quad (3.1)$$

Now, the integral on s and p is a continuous function of t and x and therefore the right hand side of (4.3.1) without the limit notation can be written as

$$\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_0^a \phi(t, x) \int_{-r-\tau}^r \int_{-\tau}^{\tau} F(s, p) e^{ist} x^{-p} ds dw dt dx, \quad p = \rho + iw, \quad r, \tau > 0 \quad (3.2)$$

Since $\phi(t, x)$ is of bounded support and the integrand is a continuous function of t, x, s, w , the order of integration may be changed and we write

$$\begin{aligned} \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_0^a \phi(t, x) \int_{-r-\tau}^r \int_{-\tau}^{\tau} F(s, p) e^{ist} x^{-p} ds dw dt dx &= \frac{1}{4\pi^2} \int_{-r-\tau}^r \int_{-\tau}^{\tau} F(s, p) \int_{-\infty}^{\infty} \int_0^a \phi(t, x) e^{ist} x^{-p} dt dx ds dw \\ &= \frac{1}{4\pi^2} \int_{-r-\tau}^r \int_{-\tau}^{\tau} \left\langle f(v, u), e^{-isv} \left(\frac{a^{2p}}{u^{p+1}} - u^{p-1} \right) \right\rangle \int_{-\infty}^{\infty} \int_0^a \phi(t, x) e^{ist} x^{-p} dt dx ds dw \\ &= \left\langle f(v, u), \frac{1}{4\pi^2} \int_{-r-\tau}^r \int_{-\tau}^{\tau} e^{-isv} \left(\frac{a^{2p}}{u^{p+1}} - u^{p-1} \right) \int_{-\infty}^{\infty} \int_0^a \phi(t, x) e^{ist} x^{-p} dt dx ds dw \right\rangle \end{aligned} \quad (\text{By Lemma 1})$$

The order of integration for the repeated integral herein may be changed because again $\phi(t, x)$ is of bounded support and the integrand is a continuous function of t, x, s, w upon doing this we obtain

$$\begin{aligned} &\left\langle f(v, u), \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_0^a \phi(t, x) \int_{-r-\tau}^r \int_{-\tau}^{\tau} e^{-isv} \left(\frac{a^{2p}}{u^{p+1}} - u^{p-1} \right) e^{ist} x^{-p} dt ds dw dx \right\rangle \\ &= \left\langle f(v, u), \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_0^a \frac{\sin(t-v)r}{t-v} \left[\frac{a^{2p} x^{-\sigma+1}}{u^{\sigma+1}} \frac{\sin w(\log xu)}{x \log xu} - \left(\frac{u}{x} \right)^{\sigma-1} \frac{\sin w \log \left(\frac{u}{x} \right)}{x \log \left(\frac{u}{x} \right)} \right] \phi(t, x) dt dx \right\rangle \end{aligned}$$

The last expression tends to

$\langle f(v, u), \phi(v, u) \rangle$ as $r \rightarrow \infty$ and $\tau \rightarrow \infty$ because $f \in FM_{f, b, c, \alpha}^*$ and according to lemma 2, the testing function in the last expression converges in $FM_{f, b, c, \alpha}$ to $\phi(v, u)$.

This completes the Proof.

Uniqueness Theorem

If $FM_f \{f(t, x)\} = F(s, p)$, for $s, p \in \Omega_f$ and $FM_f \{g(t, x)\} = G(s, p)$, for $s, p \in \Omega_g, s > 0$ and $\rho_1 < \text{Re } p < \rho_2$. If $\Omega_f \cap \Omega_g$ is not empty and if $F(s, p) = G(s, p)$, for $s \in \Omega_f \cap \Omega_g$ and $p \in \Omega_f \cap \Omega_g$ then $f = g$ in the sense of equality $D^*(I)$.

Proof

f and g must assign the same value to each $\phi \in D$. By inversion theorem and equating $F(s, p)$ and $G(s, p)$ in $\langle f - g, \phi(t, x) \rangle$

$$\lim_{\substack{r \rightarrow \infty \\ \tau \rightarrow \infty}} \left\langle \frac{1}{4\pi^2} \int_{-r-\tau}^r \int_{-\tau}^{\tau} (F - G)(s, p) e^{ist} x^{-p} ds dp, \phi(t, x) \right\rangle = 0$$

Thus, $f = g$ in $D^*(I)$.

Conclusion

This paper mainly focused on the proof of Inversion Theorem for Distributional Fourier-Finite Mellin Transform with the help of two lemmas.

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