Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 89 (2015) 36639-36641

Elixir ISSN: 2229-712X

Eigenvalue of Sturm-Liouville problem in Neumann conditions with turning points

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ARTICLE INFO

Article history:
Received: 08 December 2012;
Received in revised form:
22 November 2015;
Accepted: 27 November 2015;

Keywords
Turning points,
Numann conditions,
Asymptotic form.

ABSTRACT

The present paper is concerned the second-order differential equation $-W''+\left(u^2(\zeta^2-1)+q(\zeta)\right)W=0$,(*) with Neumann boundary conditions. By using the asymptotic solutions we find the distribution of eigenvalues of (*) in two turning points.

Introduction

Let us consider second-order differential equation

$$-W''+\left(u^2(\zeta^2-1)+q(\zeta)\right)W=0 \qquad (1)$$

$$\zeta\in[a,b] \quad W'(a)=W'(b)=0 \quad -a \phi b$$

$W'(a)=W'(b)=0$, are called Neumann boundary conditions. The function $f(\zeta)=\zeta^2-1$, is weight function and the zeros 1and -1 are turning points.

Differential equations with turning points plays important role in branch of sciences.

2. Approximation of the solutions and **Derivative of solutions**

In [1] the asymptotic solutions of differential equation

$$\omega''=u^2\zeta\omega(\zeta), \qquad (2)$$

are the Airy functions $A_i(u^{\frac{2}{3}}\eta_\zeta)$, $B_i(u^{\frac{2}{3}}\eta_\zeta)$ in form of

$$\begin{cases} A_i(u^{\frac{2}{3}}\eta_\zeta)\cong \dfrac{e^{-\frac{2}{3}u\eta_\zeta^{\frac{3}{2}}}}{2\pi^{\frac{1}{2}}u^{\frac{1}{6}}\eta_\zeta^{\frac{1}{4}}}\displaystyle\sum_{s=0}^{\infty}(-1)^s u_s\left(\frac{2}{3}u\eta_\zeta^{\frac{3}{2}}\right)^{-s} & \zeta\ge 1 \\ B_i(u^{\frac{2}{3}}\eta_\zeta)\cong \dfrac{e^{\frac{2}{3}u\eta_\zeta^{\frac{3}{2}}}}{2\pi^{\frac{1}{2}}u^{\frac{1}{6}}\eta_\zeta^{\frac{1}{4}}}\displaystyle\sum_{s=0}^{\infty}(-1)^s u_s\left(\frac{2}{3}u\eta_\zeta^{\frac{3}{2}}\right)^{-s} & \zeta\ge 0 \end{cases}$$

$$\begin{cases} A_i(0)=\dfrac{1}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})}=\dfrac{B_i(0)}{\sqrt{3}}, \\ u_s=\dfrac{(2s+1)(2s+3)(2s+5)\dots(6s+1)}{(216)^s s!} \quad , \qquad v_s=-\dfrac{6s+1}{6s-1} \qquad s\ge 1 \\ u_0=v_0=1 \end{cases}$$

$$\begin{cases} A_i(u^{\frac{2}{3}}\eta)\sim \dfrac{1}{2\pi^{\frac{1}{2}}u^{\frac{1}{6}}(-\eta)^{\frac{1}{4}}}\{\cos(\frac{2}{3}u(-\eta)^{\frac{3}{2}}-\frac{\pi}{4})\displaystyle\sum_{s=0}^{\infty}(-1)^s\dfrac{u_{2s}}{(\frac{2}{3}u(-\eta)^{\frac{3}{2}})^{2s}}+ \\ \quad \sin(\frac{2}{3}u(-\eta)^{\frac{3}{2}}-\frac{\pi}{4})\displaystyle\sum_{s=0}^{\infty}(-1)^s\dfrac{u_{2s+1}}{(\frac{2}{3}u(-\eta)^{\frac{3}{2}})^{2s+1}}\}, \qquad for \quad \eta<0 \\ B_i(u^{\frac{2}{3}}\eta)\sim \dfrac{1}{2\pi^{\frac{1}{2}}u^{\frac{1}{6}}(-\eta)^{\frac{1}{4}}}\{-\sin(\frac{2}{3}u(-\eta)^{\frac{3}{2}}-\frac{\pi}{4})\displaystyle\sum_{s=0}^{\infty}(-1)^s\dfrac{u_{2s}}{(\frac{2}{3}u(-\eta)^{\frac{3}{2}})^{2s}}+ \\ \quad \cos(\frac{2}{3}u(-\eta)^{\frac{3}{2}}-\frac{\pi}{4})\displaystyle\sum_{s=0}^{\infty}(-1)^s\dfrac{u_{2s+1}}{(\frac{2}{3}u(-\eta)^{\frac{3}{2}})^{2s+1}}\}, \end{cases}$$

The asymptotic solution of the standard equation of the form

$$U''=u^2(\zeta^2-1)U \qquad (3)$$

For large value of the real parameter u is obtained by Olver [1]. In fact let $U_1(u,\zeta)$ and $U_2(u,\zeta)$ be two independent solutions of the equation (3).For ζ >0 are of the form

$$\begin{cases} U_1(-\frac{u}{2} \ , \ \zeta\sqrt{2u})\sim 2^{\frac{1}{2}}\pi^{\frac{1}{4}}\{\Gamma(\frac{1}{2}+\frac{u}{2})\}^{\frac{1}{2}}u^{\frac{-1}{12}}(\frac{\eta_\zeta}{\zeta^2-1})^{\frac{1}{4}}A_i(u^{\frac{2}{3}}\eta_\zeta) \ (1+O(u^{-2})) \\ U_2(-\frac{u}{2} \ , \ \zeta\sqrt{2u})\sim 2^{\frac{1}{2}}\pi^{\frac{1}{4}}\{\Gamma(\frac{1}{2}+\frac{u}{2})\}^{\frac{1}{2}}u^{\frac{=1}{12}}(\frac{\eta_\zeta}{\zeta^2-1})^{\frac{1}{4}}B_i(u^{\frac{2}{3}}\eta_\zeta) \ (1+O(u^{-2})) \end{cases} \qquad (4)$$

where,

$$\eta_\zeta=\begin{cases} \{\frac{3}{2}\int_1^\zeta(\tau^2-1)^{\frac{1}{2}}d\tau\}^{\frac{2}{3}} & \zeta\ge 1 \\ -\{\frac{3}{2}\int_\zeta^1(1-\tau^2)^{\frac{1}{2}}d\tau\}^{\frac{2}{3}} & 0\le\zeta\le 1 \end{cases}$$

For every value of u, equation (1) has two solutions $W_1(u\ ,\ \zeta)$ and $W_2(u\ ,\ \zeta)$,

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$$\begin{cases} W_1(u\,,\zeta)=U_1(-\frac{u}{2}\,,\zeta\sqrt{2u})\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{u^{2s}}+\frac{1}{u^2}\frac{\partial U_1(u\,,\zeta)}{\partial\zeta}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{u^{2s}} \\ W_2(u\,,\zeta)=U_2(-\frac{u}{2}\,,\zeta\sqrt{2u})\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{u^{2s}}+\frac{1}{u^2}\frac{\partial U_2(u\,,\zeta)}{\partial\zeta}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{u^{2s}} \end{cases},\qquad (5)$$

$$\begin{cases} A_0(\zeta)=1\,,\;\; A_s(\zeta)=\frac{-1}{2}B'_{s-1}(\zeta)+\frac{1}{2}\int^{\zeta}q(t)B_{s-1}(t)\,dt, \\ B_s(\zeta)=\frac{1}{2}(1-\zeta^2)^{-\frac{1}{2}}\frac{1}{2}\int_{\pm}^{\zeta}(1-t^2)^{-\frac{1}{2}}\{A''_s(t)-q(t)A_s(t)\}\,dt \end{cases}$$

Let us define

$$\begin{cases} K_1(n\,,\zeta)=\sum_{s=0}^{n}A_s(\zeta)u^{-2s}+u^{-2}(u\zeta-\sqrt{2u})\sum_{s=0}^{n-1}B_s(\zeta)u^{-2s} \\ K_2(n\,,\zeta)=\sum_{s=0}^{n}A_s(\zeta)u^{-2s}+u^{-2}(\sqrt{2u}-u\zeta)\sum_{s=0}^{n-1}B_s(\zeta)u^{-2s} \end{cases}.$$

On the other hand, for $\xi>0$, by using the derivative of $U_1(u\,,\xi)\,,U_2(u\,,\xi)$ and inserting them in the derivative of $W_1(u\,,\xi)$ and $W_2(u\,,\xi)$ we have,

$$W_1'(u\,,\;\xi)=\sqrt{2u}\,2^{\frac{1}{2}}\pi^{\frac{1}{4}}\{\Gamma(\frac{1}{2}+\frac{u}{2})\}^{\frac{1}{2}}u^{\frac{-1}{12}}(\frac{\eta_\xi}{\xi^2-1})^{\frac{1}{4}}A_i(u^{\frac{2}{3}}\;\eta_\xi)(\frac{\xi\sqrt{2u}}{2}-e^{-\frac{4}{3}\eta_\xi^{\frac{2}{3}}})(1+O(u^{-1}))$$

$$=W_1(u\,,\;\xi)(\xi u-\sqrt{2u}\,e^{-\frac{4}{3}\eta_\xi^{\frac{2}{3}}}),$$

Similarly for $W_2(u\,,\zeta)$ we get

$$W_2'\;(u\,,\;\xi)=W_2(u\,,\;\xi)(\xi u-\sqrt{2u}\,e^{-\frac{4}{3}\eta_\xi^{\frac{2}{3}}})$$

The asymptotic behavior of $W_1(u\,,\zeta)\,,W_2(u\,,\zeta)$ for large u and $\zeta\le 0$ can be determined by use of connection formula, on replacing ζ by $-\zeta$

$$\begin{cases} W_1(u\,,\zeta)=[\sin(\frac{\pi u}{2})+O(u^{-1})][U_1(-\frac{u}{2}\,,-\zeta\sqrt{2u})]+[\cos(\frac{\pi u}{2})+O(u^{-1})][U_2(-\frac{u}{2}\,,-\zeta\sqrt{2u})]K_1(\infty,\xi) \\ W_2(u\,,\zeta)=[\cos(\frac{\pi u}{2})+O(u^{-1})][U_1(-\frac{u}{2}\,,-\zeta\sqrt{2u})+[-\sin(\frac{\pi u}{2})+O(u^{-1})][U_2(-\frac{u}{2}\,,-\zeta\sqrt{2u})]K_2(\infty,\xi) \end{cases}$$

Not that for $\xi<0$, the derivative of $W_1(u\,,\xi)\,,W_2(u\,,\xi)$ with respect to ξ are

$$W_1'(u,\xi)=[\sin(\frac{\pi u}{2})+O(u^{-1})]W_1(u,-\xi)(\xi u+\sqrt{2u}\,e^{\frac{-4}{3}\eta_{-\xi}^{\frac{3}{2}}})+[\cos(\frac{\pi u}{2})+O(u^{-1})]W_2(u,-\xi)(\xi u+\sqrt{2u}\,e^{\frac{4}{3}\eta_{-\xi}^{\frac{3}{2}}})$$

$$W_2'(u,\xi)=[\cos(\frac{\pi\,u}{2})+O(u^{-1})]\;W_1(u,-\xi)\;(-\xi\;u-\sqrt{2u}\;e^{\frac{-4}{3}\eta_{-\xi}^{\frac{3}{2}}})\;+[-\sin\;(\frac{\pi\;u}{2})+O(u^{-1})]W_2(u,-\xi)\;(-\xi\;u-\sqrt{2u}\;e^{\frac{4}{3}\eta_{-\xi}^{\frac{3}{2}}})$$

3.Asymptotic of eigenvalue for case $-a\,\phi\,b,$ with Neumann conditions

In this section we will study distributions of the eigenvalues of equation (1) with boundary conditions $W'(a)=W'(b)=0$.

If $W_1(u,\zeta)\,and\,W_2(u,\zeta)$ be two independent solutions of equation (1) then

$$W(u,\zeta)=c_1W_1(u,\zeta)+c_2W_2(u,\zeta)\quad c_1\neq 0,c_2\neq 0\,,$$

also is a solutions equation (1) therefore

$$W'(u,\zeta)=c_1W_1'(u,\zeta)+c_2W_2'(u,\zeta).$$

By attention to Neumann boundary conditions

$$\begin{cases} W'(a)=c_1W_1'(a)+c_2W_2'(a)=0 \\ W'(b)=c_1W_1'(b)+c_2W_2'(b)=0 \end{cases},$$

because above system having non trivial solution then the determinant coefficients must be zero therefore

$$\begin{vmatrix} W_1'(a) & W_2'(a) \\ W_1'(b) & W_2'(b) \end{vmatrix}=0.$$

In this case the eigenvalues of equation (1) are the zeros of $\Delta(u)=0$

where,

$$\Delta(u)=\begin{vmatrix} W_1'(u\,,a) & W_2'(u\,,a) \\ W_1'(u\,,b) & W_2'(u\,,b) \end{vmatrix}.\qquad (6)$$

Since $a\,\pi\,-1,$ $W_1'(u\,,a)$ is trigonometry and $b\,\phi\,1,$ $W_2'(u\,,b)$ is algebraic.

$$W_1'(u\,,a)W_2'(u\,,b)=\{[\sin(\frac{\pi u}{2})+O(u^{-1})]W_1(u\,,-a)(au+\sqrt{2u}\,e^{\frac{-4}{3}\eta_{-a}^{\frac{3}{2}}})+$$

$$[\cos(\frac{\pi u}{2})+O(u^{-1})]W_2(u\,,-a)(au+\sqrt{2u}\,e^{\frac{4}{3}\eta_{-a}^{\frac{3}{2}}})\}\times W_2(u\,,b)(bu-\sqrt{2u}e^{\frac{-4}{3}\eta_b^{\frac{3}{2}}})\qquad (7)$$

$$W_1'(u\,,b)W_2'(u\,,a)=\{[\cos(\frac{\pi u}{2})+O(u^{-1})]W_1(u\,,-a)(-au-\sqrt{2u}\,e^{\frac{-4}{3}\eta_{-a}^{\frac{3}{2}}})+$$

$$[\sin(\frac{\pi u}{2})+O(u^{-1})]W_2(u\,,-a)(au+\sqrt{2u}\,e^{\frac{4}{3}\eta_{-a}^{\frac{3}{2}}})\}\times W_1(u\,,b)(bu-\sqrt{2u}\,e^{-\frac{4}{3}\eta_b^{\frac{3}{2}}})\qquad (8)$$

$$\begin{cases} \Gamma_1=abu^2-au\sqrt{2u}\,e^{\frac{4}{3}\eta_b^{\frac{3}{2}}}+bu\sqrt{2u}\,e^{-\frac{4}{3}\eta_{-a}^{\frac{3}{2}}}-2ue^{\frac{4}{3}(\eta_b^{\frac{3}{2}}-\eta_{-a}^{\frac{3}{2}})} \\ \Gamma_1=abu^2-au\sqrt{2u}\,e^{\frac{4}{3}\eta_b^{\frac{3}{2}}}+bu\sqrt{2u}\,e^{-\frac{4}{3}\eta_{-a}^{\frac{3}{2}}}-2ue^{\frac{4}{3}(\eta_b^{\frac{3}{2}}-\eta_{-a}^{\frac{3}{2}})} \\ \Gamma_3=abu^2-au\sqrt{2u}\,e^{-\frac{4}{3}\eta_b^{\frac{3}{2}}}+bu\sqrt{2u}\,e^{\frac{4}{3}\eta_{-a}^{\frac{3}{2}}}-2ue^{\frac{4}{3}(\eta_{-a}^{\frac{3}{2}}-\eta_b^{\frac{3}{2}})} \\ \Gamma_4=abu^2-au\sqrt{2u}\,e^{-\frac{4}{3}\eta_b^{\frac{3}{2}}}+bu\sqrt{2u}\,e^{-\frac{4}{3}\eta_{-a}^{\frac{3}{2}}}-2ue^{-\frac{4}{3}(\eta_{-a}^{\frac{3}{2}}+\eta_b^{\frac{3}{2}})} \end{cases}$$

We know from $\Delta(u)=0,$ (7) = (8),

$$[\sin(\frac{\pi u}{2})+O(u^{-1})]W_1(u\,,-a)W_2(u\,,b)\Gamma_1+[\cos(\frac{\pi u}{2})+O(u^{-1})]W_2(u\,,-a)W_2(u\,,b)\Gamma_2$$

$$=[\sin(\frac{\pi u}{2})+O(u^{-1})]W_1(u\,,b)W_2(u\,,-a)\Gamma_3-[\cos(\frac{\pi u}{2})+O(u^{-1})]W_1(u\,,-a)W_1(u\,,b)\Gamma_4$$

we can obtain,

$$[\sin(\frac{\pi u}{2})+O(u^{-1})][W_1(u\,,-a)W_2(u\,,b)\Gamma_1-W_1(u\,,b)W_2(u\,,-a)\Gamma_3]$$

$$=[\cos(\frac{\pi u}{2})+O(u^{-1})][-W_1(u\,,b)W_2(u\,,-a)\Gamma_3-W_2(u\,,-a)W_2(u\,,b)\Gamma_2].\qquad (9)$$

In fact, form (9) we have obtained

$$\tan(\frac{\pi u}{2})=\frac{-W_2(u\,,-a)W_2(u\,,b)\,\Gamma_2-W_1(u\,,-a)W_1(u\,,b)\Gamma_4}{W_1(u\,,-a)W_2(u\,,b)\Gamma_1-W_2(u\,,-a)W_1(u\,,b)\Gamma_3}(1+O(u^{-1})).\qquad (10)$$

By attention that,

$$\begin{cases} W_1(u\,,\xi)\cong U_1(u\,,\xi)K_1(n\,,\xi) \\ W_2(u\,,\xi)\cong U_2(u\,,\xi)K_2(n\,,\xi) \end{cases}$$

$$\tan(\frac{\pi u}{2})=\frac{U_1(u\,,-a)U_1(u\,,b)K_1(n\,,-a)K_1(n\,,b)\Gamma_4+U_2(u\,,-a)U_2(u\,,b)K_2(n\,,-a)K_2(n\,,b)\Gamma_2}{U_1(u\,,b)U_2(u\,,-a)K_1(n\,,b)K_2(n\,,-a)\Gamma_3-U_1(u\,,-a)U_2(u\,,b)K_1(n\,,-a)K_2(n\,,b)\Gamma_1}$$

$$U_1(u\,,-a)U_1(u\,,b)=$$

$$2^{\frac{1}{2}}\pi^{\frac{1}{4}}\{\Gamma(\frac{1}{2}+\frac{u}{2})\}^{\frac{1}{2}}u^{\frac{-1}{12}}(\frac{\eta_{-a}}{a^2-1})^{\frac{1}{4}}A_i(u^{\frac{2}{3}}\;\eta_{-a})\times 2^{\frac{1}{2}}\pi^{\frac{1}{4}}\{\Gamma(\frac{1}{2}+\frac{u}{2})\}^{\frac{1}{2}}u^{\frac{-1}{12}}(\frac{\eta_b}{b^2-1})^{\frac{1}{4}}A_i(u^{\frac{2}{3}}\;\eta_b)\qquad (11)$$

$$=N_1{}^2N_2A_i(u^{\frac{2}{3}}\;\eta_{-a})A_i(u^{\frac{2}{3}}\;\eta_b)$$

In calculating and summarizing equation (20) we define the notations,

$$N_1=\sqrt[4]{4\pi}\{\Gamma(\frac{1}{2}+\frac{u}{2})\}^{\frac{1}{2}}u^{-\frac{1}{12}}\quad,\qquad N_2=(\frac{\eta_{-a}\eta_b}{(a^2-1)(b^2-1)})^{\frac{1}{4}}$$

$$\begin{cases} U_1(u,-a)U_1(u,b) = N_1^2 N_2 A_i(u^{\frac{2}{3}}\eta_{-a})A_i(u^{\frac{2}{3}}\eta_b), \\ U_2(u,-a)U_2(u,b) = N_1^2 N_2 B_i(u^{\frac{2}{3}}\eta_{-a})B_i(u^{\frac{2}{3}}\eta_b), \\ U_1(u,-a)U_2(u,b) = N_1^2 N_2 A_i(u^{\frac{2}{3}}\eta_{-a})B_i(u^{\frac{2}{3}}\eta_b), \\ U_1(u,b)U_2(u,-a) = N_1^2 N_2 A_i(u^{\frac{2}{3}}\eta_b)B_i(u^{\frac{2}{3}}\eta_{-a}). \end{cases} \quad (12)$$

$$\begin{cases} K_1(n,-a) = K_{11}, K_1(n,b) = K_{12} \\ K_2(n,-a) = K_{21}, K_2(n,b) = K_{22}, \end{cases}$$

$$\theta = u^{\frac{2}{3}}\eta_{-a}, \ \theta' = u^{\frac{2}{3}}\eta_b.$$

$$A_i(u^{\frac{2}{3}}\eta_{-a})A_i(u^{\frac{2}{3}}\eta_b) = \frac{e^{-\frac{2}{3}u\eta_{-a}^{\frac{3}{2}}}}{2\pi^{\frac{1}{2}}u^{\frac{1}{6}}\eta_{-a}^{\frac{1}{4}}}\sum_{s=0}^{\infty}(-1)^s u_s(\frac{2}{3}u\eta_{-a}^{\frac{3}{2}})^{-s} \times \frac{e^{-\frac{2}{3}u\eta_b^{\frac{3}{2}}}}{2\pi^{\frac{1}{2}}u^{\frac{1}{6}}\eta_b^{\frac{1}{4}}}\sum_{s=0}^{\infty}(-1)^s u_s(\frac{2}{3}u\eta_b^{\frac{3}{2}})^{-s} =$$

$$= Fe^{-\alpha}M_1M_2, \quad K_2(n,-a) = K_{21}, K_2(n,b) = K_{22}, \quad (13)$$

$$\alpha = \frac{2}{3}u(\eta_{-a}^{\frac{3}{2}} + \eta_b^{\frac{3}{2}}), \ \beta = \frac{2}{3}u(\eta_{-a}^{\frac{3}{2}} - \eta_b^{\frac{3}{2}}), F = \{4\pi u^{\frac{1}{3}}(\eta_{-a}\eta_b)^{\frac{1}{4}}\}^{-1}.$$

Similarly we have following notations,

$$A_i(\theta)A_i(\theta') = M_1M_2F e^{-\alpha}, \quad B_i(\theta)B_i(\theta') = M_3M_4F e^{\alpha},$$

$$A_i(\theta)B_i(\theta') = M_1M_4F e^{-\beta}, \quad A_i(\theta')B_i(\theta) = M_2M_3F e^{\beta}.$$

$$M_1 = \sum_{s=0}^{\infty}(-1)^s u_s(\frac{2}{3}u\eta_{-a}^{\frac{3}{2}})^{-s}, \ M_2 = \sum_{s=0}^{\infty}(-1)^s u_s(\frac{2}{3}u\eta_{-b}^{\frac{3}{2}})^{-s}, \ M_3 = \sum_{s=0}^{\infty}u_s(\frac{2}{3}u\eta_{-a}^{\frac{3}{2}})^{-s}, \ M_4 = \sum_{s=0}^{\infty}u_s(\frac{2}{3}u\eta_{-b}^{\frac{3}{2}})^{-s},$$

Therefore we can write

$$\tan(\frac{\pi u}{2}) = \frac{M_1M_2K_{11}K_{12}\Gamma_4e^{-\alpha} + M_3M_4K_{21}K_{22}\Gamma_2e^{\alpha}}{M_1M_4K_{12}K_{21}\Gamma_3e^{-\beta} - M_2M_3K_{11}K_{22}\Gamma_1e^{\beta}}(1+O(u^{-1})). \quad (14)$$

Theorem: The equation (1) with Neumann conditions has asymptotic eigenvalues in the form of

$$u_n = \frac{n\pi + \frac{\pi}{2}}{\frac{\pi}{2}} - \frac{2}{\pi x_n} + O(\frac{1}{x_n^{\ 2}})$$

Proof: we suppose,

$$\lambda_1 = M_1M_2K_{11}K_{12}, \lambda_2 = M_3M_4K_{21}K_{22}, \lambda_3 = M_1M_4K_{21}K_{12}, \ \lambda_4 = M_3M_2K_{11}K_{22}$$

From (14) we can write

$$\tan(\frac{\pi u}{2}) = \frac{\lambda_1\Gamma_4e^{-\alpha} + \lambda_2\Gamma_2e^{\alpha}}{\lambda_3\Gamma_3e^{-\beta} - \lambda_4\Gamma_1e^{\beta}}(1+O(u^{-1})). \quad (15)$$

Since $u \to \infty$

$$\alpha = \frac{2}{3}u(\eta_{-a}^{\frac{3}{2}} + \eta_b^{\frac{3}{2}}), \alpha \to \infty \ \Rightarrow e^{-\alpha} \to o,$$

and

We know

$$-a > b, \begin{cases} -a > 1 \Rightarrow \eta_{-a} > 0 \\ b > 1 \ \Rightarrow \eta_b > o \end{cases} \Rightarrow \eta_{-a} > \eta_b, \Rightarrow (\eta^{\frac{3}{2}}{}_{-a} - \eta^{\frac{3}{2}}{}_b) > 0$$

$$\beta = \frac{2}{3}u(\eta_{-a}^{\frac{3}{2}} - \eta_b^{\frac{3}{2}}), \quad u \to \infty, \beta \to \infty, \Rightarrow e^{-\beta} \to 0$$

Therefore from (14) we will have,

$$\tan(\frac{\pi u}{2}) = \frac{\lambda_2\Gamma_2e^{\alpha}}{-\lambda_4\Gamma_1e^{\beta}}(1+O(u^{-1})) \quad (16)$$

$$x = \frac{\lambda_2\Gamma_2e^{\alpha}}{-\lambda_4\Gamma_1e^{\beta}} = \frac{\lambda_2}{-\lambda_4}\times\frac{\Gamma_2}{\Gamma_1}\times e^{(\alpha-\beta)} = \frac{\lambda_2}{-\lambda_4}\times\frac{\Gamma_2}{\Gamma_1}e^{\frac{4}{3}u\eta^{\frac{3}{2}}{}_b} \quad (17)$$

$$\frac{1}{x} = \frac{\lambda_4}{-\lambda_2}\times\frac{\Gamma_1}{\Gamma_2}e^{\frac{-4}{3}u\eta^{\frac{3}{2}}{}_b} = \frac{\lambda_4}{-\lambda_2}\times\frac{\Gamma_1}{\Gamma_2}$$

$$\begin{cases} \frac{\lambda_4}{-\lambda_2} = 1-(3u_1\eta_b^{\frac{-3}{2}} + 2aB_0(-a))\frac{1}{u} + O(u^{-2}), \ \frac{\Gamma_1}{\Gamma_2} = 1 - \frac{2\sqrt{2}}{a\sqrt{u}}\sinh(\frac{4}{3}\eta_{-a}^{\frac{3}{2}}), \quad (18) \\ \frac{1}{x} = -[1 - \frac{2\sqrt{2}}{a\sqrt{u}}\sinh(\frac{4}{3}\eta^{\frac{3}{2}}{}_{-a}) - (3u_1\eta_b^{\frac{-3}{2}} + 2aB_0(-a))\frac{1}{u} + \frac{2\sqrt{2}}{a}\left(3u_1\eta_b^{\frac{-3}{2}} + 2aB_0(-a)\right)\frac{1}{u\sqrt{u}}] + O(u^{-2}) \end{cases}.$$

$$\tan(\frac{\pi u}{2}) = x \ \Rightarrow \frac{\pi u}{2} = n\pi + \tan^{-1}x$$

$$\tan^{-1}x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \ldots = \frac{\pi}{2} - \frac{1}{x}[1 - \frac{1}{3x^2} + \frac{1}{5x^4} + \ldots], \quad |x| > 1$$

$$u_n = \frac{n\pi + \frac{\pi}{2}}{\frac{\pi}{2}} - \frac{2}{\pi x_n} + O(\frac{1}{x_n^{\ 2}}).$$

References

[1] F.W.J . Olver, Second linear differential equations with two turning points . Phil. Trans R soc Lon A. 278 (1975) 137 – 174.

[2] Jodayree Akbarfam *A. , Higher-order apprpximation to the eigenvalues of the Sturm-Liouville problems in one turning point case*, Bulletin Iranian Math. Society, vol. 23, No. 2.(1997)37 – 53 .

[3] W. Wosow, Asymptotic expansion for ordinary differential equations, Interscience Publishers, New York 1965.

[4]F.V. Atkinson,and A.B. Mingarelli., Asymptotics of the number of zero and of the eigenvalues of general weighted Sturm-Liouville problems, J.reine ang .math., 375, (1987) 380-393.

[5] A. Neamaty, The canonical product of the solution of the Sturm-Liouville problems, Iran. J. Sci. Technol. Trans. A 2(1999) 61-67.

[6] W. Eberhard, G. Freiling and A. Schneider, Connection formulas for second-order differential equation with a complex parameter and having a arbitrary number of turning points. Math Nachr, 165 (1994) 205-229.

[7] A.Neamaty and A.Jodayree Representation of asymptotic solution of second order differential equation with two transition points in explicit form 28th AIMC, Tabriz University March (1997) 405-412.

[8] G. Freiling and V.Yurko , Inverse problems for differential equations with turning Inverse problems 13 (1997) 1247-1263.

[9] A.Jodayree Akbarfam , Angelo B.Mingarelli : Duality for an indefinite inverse Sturm- Liouville problem. J.Math. Anal.Appl.312 (2005) 435-463.