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Solutions of the Generalized Heat Equation and its Integral Representations

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ABSTRACT

In this paper we have explored the problem for generalized temperature functions considered over positive and negative time. We have established representation theorems and their applications.

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1. Introduction

In recent past, characterizations for generalized temperature functions defined for positive time which may be represented by Poisson-Hankel-Stieltjesintegral transforms were derived in the papers of Cholewinski and Haimo [1], Haimo [3].Our aim in the present paper is to explore the problem for generalized temperature functions considered over negative time. Although the representation theorems obtained can be proved by techniques analogous to those of the previous results, we use the more elegant approach of appealing to the Appell transform to reduce these cases to those dealt with earlier. In addition, we investigate some other generalized temperature functions which have integral representations.

2.Notations and terminology:

The generalized heat equation is

$$\Delta_x u(x,t) = \frac{\partial}{\partial t} u(x,t), \text{ where} \tag{2.1}$$

$\Delta_x f(x) = f''(x) + \frac{2(\alpha-\beta)+1}{x} f'(x)$, where $\left(\alpha-\beta+\frac{1}{2}\right)$ is a fixed positive number.

A generalized temperature function is a function of class C^2which satisfies the generalized heat equation. We denote the class of such functions by H.

The fundamental solution of the generalized heat equation is the function

$$G(x,y;t) = (1/2t)^{\alpha-\beta+1} e^{-(x^2+y^2)/4t}\, g(xy/2t) \tag{2.2}$$

where$g(z) = 2^{\alpha-\beta}\,\Gamma(\alpha-\beta+1) z^{-(\alpha-\beta)} I_{\alpha-\beta}(z)$, $I_\lambda(z)$is the Bessel function of imaginary argument of order λ.We write $G(x;t)$ for $G(x, 0; t)$.

If $V(x,t)$ is an arbitrary function of two variables, then its Appell type transform$V^A(x,t)$ is given by

$$V^A(x,t) = V^A_{x,t}(x,t) = G(x;t)\, V(x/t, -1/t). \tag{2.3}$$

Now we define a subclass H^* of H which plays an important role in our theory:

A generalized temperature function $u(x,t)$ is a member of H^* for $a < t < b$, if and only if, for every $t, t', a < t' < t < b$,

$$u(x,t) = \int_0^\infty G(x,y;t-t')\, u(y,t')\, d\mu(y), \tag{2.4}$$

where

$$d\mu(x) = \frac{2^{-(\alpha-\beta)}}{\Gamma(\alpha-\beta+1)} x^{2(\alpha-\beta)+1}\, dx,$$

The integral converging absolutely. Functions in $H^*$are said to have the Huygens property.

By Theorem 6.4 of [3], functions in H^*have a complex integral representation as well. Infact, we have the following result.

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Lemma 2.1

If $u(x,t) \in H^*$, $a < t < b$, then

$$u(x,t) = \int_0^\infty G(ix,y;t'-t)\, u(iy,t')\, d\mu(y), \quad a < t < t' < b. \tag{2.5}$$

A fundamental result is the invariance of membership in H^* under an Appell transformation which is stated in the following lemma.

Lemma 2.2

If $u(x,t) \in H^*$for $a < t < b$, then $u^A(x,t) \in H^*$ for $-1/a < t < -1/b$.

Proof

$u^A(x,t) \in H$ for $-1/a < t < -1/b$ may be verified by making use of the fact that

$\Delta_x[f(x)g(x)] = f(x)\Delta_x g(x) + g(x)\Delta_x f(x) + 2f'(x)g'(x).$

Next, consider

$$\int_0^\infty G(x,y;t-t')\, u^A(y,t')\, d\mu(y)$$

$$= \int_0^\infty G(x,y;t-t')\, G(y,t')\, u(y/t',-1/t')\, d\mu(y)$$

$$= G(x;t) \int_0^\infty G(x/t,y;\ 1/t' - 1/t)\ u(y,-1/t')\, d\mu(y).$$

As $u(x,t) \in H^*$ for $a < t < b$, the integral on the right reduces to $u(x/t,-1/t)$for $a < -1/t' < -1/t < b$. Thus

$$\int_0^\infty G(x,y;t-t')u^A(y,t')\, d\mu(y)$$

$= G(x;t)\ u(x/t,-1/t)\ a < -1/t' < -1/t < 1/b,$

$= u^A(x,t), \quad {}^{-1}/_a < t' < t < {}^{-1}/_b$

and this proves the lemma.

We shall be concerned with the following integral transforms: (see [1],[2]):

The Hankel type transform $\hat{\psi}(x)$ of a function ψ defined on $(0,\infty)$is given by

$$\hat{\psi}(x) = \int_0^\infty h(xy)\, \psi(y)\, d\mu(y), \quad 0 \le x < \infty, \tag{2.6}$$

where

$h(x) = 2^{\alpha-\beta}\, \Gamma(\alpha-\beta+1) x^{-(\alpha-\beta)} J_{\alpha-\beta}(x),$

And $J_v(x)$ is the ordinary Bessel type function of order v, whenever the integral converges. We write

$$\hat{\psi}_i(x) = \int_0^\infty g(xy)\, \psi(y)\, d\mu(y), \quad 0 \le x < \infty, \tag{2.7}$$

whenever the integral (2.7) converges.

The Hankel-Stieltjes type transform $\hat{\psi}^s(x)$ of a function λ of bounded variation in every finite interval is given by

$$\hat{\psi}^s(x) = \int_0^\infty h(xy)\, d\lambda(y), \quad 0 \le x < \infty, \tag{2.8}$$

whenever the integral converges.

We also write

$$\hat{\psi}_i^s(x) = \int_0^\infty g(xy)\, d\lambda(y), \quad 0 \le x < \infty \tag{2.9}$$

whenever the integral converges.

The Poisson-Hankel type transform of a function ψ integrable in every finite interval is the function $\psi^P(x,t)$ given by

$$\psi^P(x,t) = \int_0^\infty G(x,y;t)\, \psi(y)\, d\mu(y), \quad 0 \le x < \infty, \tag{2.10}$$

wheneverthe integral converges.

The Poisson-Hankel-Stieltjes type transform of a function $\lambda$of bounded variation in every finite interval is the function $\psi^{P_s}(x,t)$given by

$$\psi^{P_s}(x,t) = \int_0^\infty G(x,y;t)\, d\lambda(y), \quad 0 \le x < \infty, \tag{2.11}$$

whenever the integral converges.

By Theorem 6.2 of [3], we know that within the interval of absolute convergence of the integral (2.11), $\psi^{P_s}(x,t) \in H^*$. Hence we may readily establish the following result.

Lemma 2.3

If $\psi(x) \in L$, $0 \le x < \infty$, then $\psi^P(x,t) \in H^*$for $t > 0$, and

$$\psi^P(x,t) = \left[e^{-tx^2}\hat{\psi}(x)\right]^{\Lambda}. \tag{2.12}$$

Proof

As $\psi(x) \in L$ for $0 \le x < \infty$,

$$\int_0^\infty |G(x,y;t)\, \psi(y)|\, d\mu(y) \le (1/2t)^{\alpha-\beta+1} \int_0^\infty |\psi(y)|\, d\mu(y) < \infty, \quad t > 0,$$

so that the integral defining $\psi^P(x,t)$converges absolutely. Hence by theorem 2.6 of [3] $\psi^P(x,t) \in H^*$ for $t > 0$and the firstpart of the lemma is proved.

Next, we make use of Fubini's theorem to obtain

$$\psi^P(x,t) = \int_0^\infty G(x,y;t)\,\psi(y)\,d\mu(y)$$

$$= \int_0^\infty \psi(y)\,d\mu(y)\int_0^\infty e^{-tu^2} h(xu)\,h(yu)\,d\mu(u)$$

$$= \int_0^\infty e^{-tu^2} h(xu)\hat{\psi}(u)\,d\mu(u).$$

But the final integral is the right hand side of (2.12) and thus proof is completed.Now we have following companion result:

Lemma 2.4

If $\psi(x) \in L,\ 0 \le x < \infty,$ then$\psi^P(ix,-t) \in H^*$ for $t < 0,$

and

$$\psi^P(ix,-t) = \left[e^{tx^2}\hat{\psi}(x)\right]\hat{\imath}. \tag{2.13}$$

Proof

As the integral defining $\hat{\psi}(x,t)$ converges absolutely for $t > 0$, we know that $\hat{\psi}(x,t) \in H^*$ for $t > 0$.Hence by Lemma 2.1, for $0 < t < t' < \infty$,

$$\hat{\psi}(x,t) = \int_0^\infty G\,(ix,y;\ t'-t)\hat{\psi}\,(iy,t')\,d\mu(y),$$

The integral converging absolutely and we have

$$\psi^P(ix,t) = \int_0^\infty G\,(x,y;t'-t)\psi^P(iy,t')\,d\mu(y),\quad 0 < t < t' < \infty,$$

So that $\psi^P(ix,-t) \in H^*$ for $t < 0$. Further

$$\psi^P(ix,-t) = \int_0^\infty G(ix,y;-t)\,\psi(y)\,d\mu(y)$$

$$= \int_0^\infty \psi(y)\,d\mu\,(y)\int_0^\infty h(yu)\ g(xu)e^{tu^2}\,d\mu(u)$$

$$= \int_0^\infty g(xu)e^{tu^2}\hat{\psi}(u)\,d\mu(u),$$

This completes the proof.

Finally we complete this section with a formula giving the value of an integral transform of an Appell transform of a function of H^*.

Theorem 2.5

If $u(x,t) \in H^*$for $|t| < \sigma$, then for any $t > 1/\sigma$,

$u\,(2x,0)e^{tx^2} = [u^A(x,t)]_i^\Lambda.$

Proof

We have

$$e^{-tx^2}\int_0^\infty g(xy)\,u^A(y,t)\,d\mu(y) = e^{-tx^2}\int_0^\infty g(xy)\ G(y;t)\ u(y/t,-1/t)\,d\mu(y)$$

$$= e^{-tx^2}\int_0^\infty g(xyt)\ G\,(y;\ 1/t)\ u\,(y,-1/t)\,d\mu(y).$$

$$= \int_0^\infty G(2x,y;1/t)\ u(y,-1/t)\,d\mu(y).$$

But, since $u(x,t) \in H^*$ for $|t| < \sigma$, the last integral is simply $u(2x,0)$, for $t > 1/\sigma$, and the proof is complete.

3. Integral representation of Appell type transforms

In this section we obtain main representation theorems but before this we need following definitions and lemmas:

Definition 3.1

An even entire function

$$f(x) = \sum_{n=0}^\infty a_n x^{2n} \tag{3.1}$$

belongs to the class $(1,\sigma)$, or has growth $(1,\sigma)$, if and only if

$$\lim\ \mathrm{Sup}_{n\to\infty}\, n\,|a_n|^{1/n} \le e\sigma\,. \tag{3.2}$$

Lemma 3.2

If $u(x,t) \in H^*$ for $|t| < \sigma$, then $u(-2ix,0) \in (1,1/\sigma)$.

Proof

By Theorem 5.1 of [5], we have that

$$u(x,t) = \sum_{n=0}^{\infty} a_n P_{n,\alpha,\beta}(x,t) \quad , \quad |t| < \sigma,$$

where $P_{n,\alpha,\beta}(x,t)$ is the generalized heat polynomial (See [5]).

Further by Theorem 3.8 of [5], it follows that $u(x,0)$ is an even function of growth $(1,1/4\sigma)$. Since

$$u(x,0) = \sum_{n=0}^{\infty} a_n x^{2n},$$

by (3.2), we find that

$\lim \lim_{n\to\infty} n\,|2^{2n} a_n|^{1/n} \le e/\sigma$.

Thus it easily follows that $u(-2ix,0) \in (1,1/\sigma)$. This completes the proof.

Lemma 3.3

For $s,t > 0$,

$$G^{A}_{x,t}(x,y;s+t) = G^{A}_{y,s}(x,y;s+t). \tag{3.3}$$

Proof

The result is immediate from the definitions as each side is equal to

$$\left[\frac{1}{4(st-1)}\right]^{\alpha-\beta+1} e^{-\left(\frac{x^2 s+y^2 t}{4(st-1)}\right)} g\left(\frac{xy}{2(st-1)}\right). \tag{3.4}$$

Now we are ready to establish a principal result.

Theorem 3.4

A necessary and sufficient condition that

$$u(x,t) = \left[e^{-tx^2}\psi(x)\right]^{\Lambda},\ t > 1/\sigma, \tag{3.5}$$

where $\psi(x) \in (1,1/\sigma)$ is an even, is that there exist a function $v(x,t) \in H^*$ for $|t| < \sigma$, and such that $u(x,t) = v^{A}(x,t)$.

Proof

To prove sufficiency, we note that since $v(x,t) \in H^*$ for $|t| < \sigma$, then for any $\sigma',\ 0 < \sigma' < \sigma$,

$$v(x,t) = \int_0^{\infty} G(x,y;t+\sigma')\, v(y,-\sigma')\, d\mu(y), \tag{3.6}$$

The integral converging absolutely for $-\sigma' < t < \sigma$. Next,

$u(x,t) = v^{A}(x,t) = G(x,t)\, v(x/t - 1/t)$,

so that by (3.6) and (3.3), we have

$$u(x,t) = \int_0^{\infty} G(y;\sigma')\, G\left(x, y/\sigma'; t - {}^{1}/_{\sigma'}\right) v(y,-\sigma')\, d\mu(y)$$

$$= \int_0^{\infty} G(y;\sigma') v(y,-\sigma')\, d\mu(y) \int_0^{\infty} e^{-(t-1/\sigma')s^2} h(xs)\, h(ys/\sigma')\, d\mu(s),$$

or if the order of integration may be reversed.

$$u(x,t) = \int_0^{\infty} h(xs)\, e^{-ts^2}\, d\mu(s) \int_0^{\infty} G(-2is, \quad y;\ \sigma')\ v(y,-\sigma')\, d\mu(y)$$

$$= \int_0^{\infty} h(xs) e^{-ts^2}\ v(-2is, \quad 0)\, d\mu(s), \qquad t > 1/\sigma'.$$

If we set $\psi(s) = v(-2is,0)$, then by Lemma 3.2, $\psi(x) \in \left(1, {}^{1}/_{\sigma}\right)$.

It is clearly an even function and we then have, for all $t > 1/\sigma$,

$u(x,t) = v^{A}(x,t) = \left[e^{-tx^2}\psi(x)\right]^{\Lambda}$.

To justify the interchange in order of integration, we observe that

$$\int_0^{\infty} |h(xs)|\, e^{-ts^2}\, d\mu(s) \int_0^{\infty} \left|G(-2is,y;\sigma')\right|\left|v(y,-\sigma')\right| \Big|\, d\mu(y)$$

$$\le \int_0^{\infty} e^{-ts^2+s^2/\sigma'}\, d\mu(s) \int_0^{\infty} G(y;\sigma')\, |v(y,-\sigma')|\, d\mu(y).$$

The inner integral converges by (3.6) with $x = t = o$. Further, the outer integral converges for $t > 1/\sigma'$, and the proof of sufficiency of the condition is complete.

To prove the necessity of the condition, assume (3.5) with $\psi(x)$ an even function of growth $(1,1/\sigma)$. Then by Theorem 6.1 of [5], we have

$u(x,t) = \sum_{n=0}^{\infty} b_n W_{n,\alpha,\beta}(x,t)$, $0 \leq 1/\sigma < t$, (3.7)

with

$b_n = (-1)^n \frac{\psi^{(2n)}(0)}{2^{2n}(2n)!}$, (3.8)

Where $W_{n,\alpha,\beta}$ is the Appell type transform of the generalized heat polynomial $P_{n,\alpha,\beta}(x,t)$. Thus we have

$v(x,t) = \sum_{n=0}^{\infty} b_n P_{n,\alpha,\beta}(x,t)$, $-\sigma < t < 0$, (3.9)

Where $u(x,t) = v^A(x,t)$. Since the series (3.9) always converges in a strip symmetric about the x –axis of xt –plane, its convergence for $-\sigma < t < 0$ implies its convergence for $|t| < \sigma$. Theorem 5.1 of [5] yields the fact that $v(x,t) \in H^*$for $|t| < \sigma$. Further by (3.8) and (3.9), we have

$$v(x,0) = \sum_{n=0}^{\infty} \psi^{(2n)}(0)/(2n)!\,(ix/2)^{2n}$$

$$= \psi(ix/2),$$

or

$$v(-2is,0) = \psi(s), \tag{3.10}$$

which is the relation between v and ψ established in the sufficiency proof. This completes the proof.

Corollary 3.5

If $v(x,t) \in H^*$ for $a < t < b$, and if $a < t_0 < b$, then

$v^A_{x,t}(x,t+t_0) = \left[e^{-tx^2}\psi(x)\right]^{\Lambda}$, $t > 1/\sigma$, (3.11)

Where $\sigma = \mathrm{Min}(t_0 - a, b - t_0)$, and $\psi(x) = v(-2ix,\ t_0)$is an even function of growth $(1,1/\sigma)$.

Proof

The hypothesis clearly implies that $v(x,t+t_0) \in H^*$ for $|t| < \sigma$.We may thus apply the theorem and the result is immediate.

Example

An example illustrating the corollary is given by

$v(x,t) = G(x;t)$ (3.12)

which is in H^* for $0 < t < \infty$. Thus we have

$G^A_{x,t}(x,t+a) = \left[e^{-tx^2} G(-2ix,a)\right]^{\Lambda}$, $t > 1/a$, (3.13)

$$= (1/2a)^{\alpha-\beta+1}\int_0^{\infty} h(xy)e^{-y^2(t-1/a)}\,d\mu(y), \qquad t > 1/a.$$

Note that $\psi(x) = G(-2ix,a)$ is an even function of growth $(1,1/a)$, and that the integral (3.13) converges in no larger regionthan that predicted in (3.11).

Actually, the region of convergence of the integral (3.11) as given by the corollary is not the largest possible in every instance as the following example illustrate consider, for $\sigma,\delta > 0$,

$v(x,t) = G(ix;\ \sigma+\delta-t) + G(x;t)$, (3.14)

which is in H^* for $0 < t < \sigma+\delta$. Hence by the corollary,

$v^A_{x,t}(x,t+\sigma) = [G(ix;\ \delta-t) + G(x;t+\sigma)]^A$ (3.15)

$$= \int_0^{\infty} h(xy)e^{-ty^2}[G(2y;\delta) + G(2iy;\sigma)]\,d\mu(y)$$

with the integral (3.15) converging for $t > [1/Min(\sigma,\delta)]$. If $\sigma > \delta$, corollary thus predicts the convergence of the integral (3.15) for $t > 1/\delta$,whereas actually, it is clear that the integral converges in the larger region $t > 1/\sigma$.

To take care of such cases, we introduce a theorem to indicate the conditions under which Corollary 3.5 may be strengthened to give the larger region of convergence.

Theorem 3.6

If $v(x,t) \in H^*$ for $-\sigma < t < \delta, \sigma, \delta > 0$, then

$v^A(x,t) = \left[e^{-ty^2}\psi(x)\right]^{\Lambda}$, $t > 1/\sigma$, (3.16)

Where $\psi(x) = v(2ix,0)$ is an even entire function and

$\lim Sup_{x\to\pm\infty} \frac{\log|\psi(x)|}{x^2} \leq 1/\sigma$. (3.17)

Proof

Note that if $\sigma > \delta$ then Corollary 3.5, with $t_0 = 0$ asserts that $\psi(x)$ is an even function of growth $(1,1/\delta)$ and that (3.16) holds with integral converging for $t > 1/\delta$ rather than the larger region $t > 1/\sigma$ indicated in the present theorem.

To prove the theorem, we note that since $v(x,t) \in H^*\ for - \sigma < t < \delta$, we have, for any $\sigma', 0 < \sigma' < \sigma$,

$v(x,t) = \int_0^{\infty} G(x,y;t+\sigma')\,v(y,-\sigma')\,d\mu(y)$, (3.18)

the integral converging absolutely for $-\sigma' < t < \delta$. Then, as in the proof of Theorem3.4

$$v^A(x,t) = \int_0^{\infty} e^{-ts^2}\,h(xs)\,\psi(s)\,d\mu(s), \qquad t > 1/\sigma',$$

or since σ' may be taken arbitrarily close to σ,

$v^A(x,t) = \left[e^{-tx^2}\psi(x)\right]^{\Lambda}, \qquad t > 1/\sigma,$

Where $\psi(x) = v(-2ix, 0)$, but we no longer can conclude that $\psi(x) \in (1, 1/\sigma)$. Instead, we have, since $v(x,t) \in H^*$ for $-\sigma < t < \delta$,

$$\psi(x) = \int_0^\infty G(2ix, y; \sigma')\, v(y, -\sigma')\, d\mu(y),$$

so that

$$|\psi(x)| \le e^{x^2/\sigma'} \int_0^\infty G(y; \sigma')|v(y, -\sigma')|\, d\mu(y).$$

Thus it is immediate that

$$\lim Sup_{x\to\pm\infty} \frac{\log|\psi(x)|}{x^2} \le \frac{1}{\sigma'}\,, \tag{3.19}$$

or since σ' may be taken arbitrarily close to σ, (3.17) is established and thus proof is complete.

It is of interest to note that the condition $\delta > 0$ cannot be improved, as indicated by the example

$v(x,t) = G(ix; -t)$,

which is in H^* for $-\infty < t < 0$.Here

$v^A(x,t) = {}^{1}\!/_{2^{2(\alpha-\beta+1)}}$

which cannot have Hankel type representation since such functions must vanish at ∞.

Consider

$$u(x,t) = \int_0^\infty e^{-ty^2} h(xy)\, e^{-y^4}\, d\mu(y)\,, \qquad -\infty < t < \infty.$$

Here $u(x,t) = \left[e^{-tx^2}\psi(x)\right]^{\Lambda}$, where $\psi(x) = e^{-x^4}$is an even function which is not of growth $(1,\sigma)$ for any σ. However there exists a function

$$v(x,t) = \int_0^\infty G(ix, y; -t) e^{-y^4/16}\, d\mu(y)$$

inH^* for $t < 0$ and such that $u(x,t) = v^A(x,t)$.

A modification of the necessity part of Theorem 3.4 to include such an example, is given by the following result.

Theorem 3.7

If

$$u(x,t) = \int_0^\infty e^{-ty^2} h(xy)\, \psi(y)\, d\mu(y), \quad t > 1/\sigma \ge 0, \tag{3.20}$$

where$\psi(x)$ is an even function for which

$$\lim\, \mathrm{Sup}_{x\to\pm\infty} \frac{\log|\psi(x)|}{x^2} \le \frac{1}{\sigma}\,, \tag{3.21}$$

then there exists a function $v(x,t) \in H^*$ for $-\sigma < t < 0$ and such that $u(x,t) = v^A(x,t)$.

Proof

Equation (3.21) implies that for any σ', $0 < \sigma' < \sigma$,

$$\psi(x) = O\left(e^{x^2/\sigma'}\right), \quad x \to \pm\infty\,. \tag{3.22}$$

Now if we set

$$u(x,t) = G(x,t)\, v(x/t, \; -1/t)$$
$$= \int_0^\infty e^{-ty^2}\, h(xy)\, \psi(y)\, d\mu(y)\,,$$

then, formally,

$$v(x,t) = \int_0^\infty G(ix, y; -t)\, \psi(y/2)\, d\mu(y). \tag{3.23}$$

But by (3.22), the integral defining $v(x,t)$ is dominated by

$$(-1/2t)^{\alpha-\beta+1} e^{-x^2/4t} \int_0^\infty e^{y^2/4t}\, O\left(e^{y^2/4\sigma'}\right) d\mu(y)$$

and consequently converges absolutely for $-\sigma' < t < 0$, or since σ' may be taken arbitrarily close to σ, for $-\sigma < t <0$. Now

$$w(x,t) = \int_0^\infty G(x, y; t)\, \psi(y/2)\, d\mu(y) \tag{3.24}$$

may be shown in a similar way to converge absolutely for $0 < t < \sigma$.

It follows by Theorem 6.2 of [3] that $w(x,t) \in H^*$ for $0 < t < \sigma$. Hence by Lemma 2.1

$$w(x,t) = \int_0^\infty G(ix, y; t' - t)\, w(iy, t')\, d\mu(y)\,, \quad 0 < t < t' < \sigma,$$

or by Theorem 5.3 of [1]

$$w(ix,-t) = \int_0^\infty G(x,y;t'+t)w(iy,t')\,d\mu(y), \qquad -\sigma' < -t < t < 0.$$

Thus it follows that $w(ix,-t) \in H^*$for $-\sigma' < t < 0$and consequently for $-\sigma < t < 0$.Since$w(ix,-t) = v(x,t)$, then the theorem is established.

4. Temperatures in positive time:

In [1] and [3] criteria were established for a class of generalized temperature functions defined for positive time to be represented by a Poisson-Hankel-Stieltjes transform as noted in introduction. In this section, we find that in addition, different representation formulas hold as well, if the class of generalized temperature functions considered is further restricted in each case, by an additional condition.

Theorem 4.1

A necessary and sufficient condition that

$$u(x,t) = \left[e^{-tx^2}\psi(x)\right]^{\Lambda}, \quad 0 < t < \infty, \tag{4.1}$$

were $\psi(x) = \hat{\alpha}^s(s)$for a bounded, non-decreasing function λ is that, for $0 < t < \infty$, $u(x,t) \in H$, $u(x,t) \geq 0$, and for some $t_0 > 0$,

$$\int_0^\infty u(x,t_0)\,d\mu(x) < \infty. \tag{4.2}$$

Proof:

For $0 < t < \infty$, if we assume that $u(x,t) \in H$ and $u(x,t) \geq 0$,then by Theorem 9.1 of [1], we have

$$u(x,t) = \int_0^\infty G(x,y;t)\,d\lambda(y), \ 0 < t < \infty, \tag{4.3}$$

for some $\lambda(y)$ increasing above. Further if (4.2) is also assumed to hold, then using (4.3), we find that

$$\begin{aligned}\int_0^\infty u(x,t_0)\,d\mu(x) &= \int_0^\infty d\mu(x)\int_0^\infty G(x,y;t_0)\,d\lambda(y)\\ &= \int_0^\infty d\lambda(y)\\ &< \infty.\end{aligned} \tag{4.4}$$

By virtue of (4.4), we may define

$$\psi(x) = \int_0^\infty h(xu)\,d\lambda(u), \quad 0 \leq x < \infty \tag{4.5}$$

Then we have

$$\begin{aligned}\int_0^\infty e^{-ty^2}h(xy)\,\psi(y)\,d\mu(y) &= \int_0^\infty e^{-ty^2}h(xy)\,d\mu(y)\int_0^\infty h(yu)\,d\lambda(u)\\ &= \int_0^\infty G(x,\mu;t)\,d\lambda(u)\\ &= u(x,t),\end{aligned} \tag{4.6}$$

where the change in order of integration is valid by (4.4). Hence the condition is sufficient.

Conversely, assume that (4.1) holds, with $\psi(x)$given by (4.5) for some bounded, non-decreasing function λ. Then as in (4.6), we find that

$$\begin{aligned}u(x,t) &= \int_0^\infty e^{-ty^2}\,h(xy)\,\psi(y),\ d\mu(y)\\ &= \int_0^\infty G\,(x,y;t)\,d\lambda(y)\ , \quad 0 < t < \infty\end{aligned}$$

so that an appeal to Theorem 9.1 of [1] confirms the fact that $u(x,t) \in H$ and $u(x,t) \geq 0$for $0 < t < \infty$. Further as

$$\int_0^\infty u(x,t_0)\,d\mu(x) = \int_0^\infty d\lambda(y) \tag{4.7}$$

And $\lambda(y)$ is non-decreasing bounded function, (4.2) holds for every$t_0 > 0$, and the proof is complete.

Note that the functions considered in this theorem form a proper subclass of the positive generalized temperature functions studied in [1]. An illustration is given by

$$u(x,t) = x^2 + 4(\alpha-\beta+1)t = \int_0^\infty G(x,y;t)\,d\lambda(y) \tag{4.8}$$

with

$$\lambda(y) = \frac{y^{2(\alpha-\beta+2)}}{2^{\alpha-\beta}2(\alpha-\beta+2)\Gamma(\alpha-\beta+1)} \tag{4.9}$$

so that clearly $\lambda(y)$increases above. It does not have a representation of the form (4.1).

On the other hand, for $a > 0$

$$u(x,t) = G(x;\ t+a) \tag{4.10}$$

satisfies (4.2) for every $t_0 > 0$.Indeed, we have

$$G(x;t+a) = \int_0^\infty h(xu)e^{-tu^2}\psi(u)\,d\mu(u), \tag{4.11}$$

with

$$\psi(x) = e^{-ax^2} = \int_0^\infty h(xy)\, d\lambda(y) \tag{4.12}$$

where

$d\lambda(y) = G(y; a)\, d\mu(y),$

so that λ is bounded and non-decreasing.

5. Temperatures in negative time

In this section, we investigate the question of integral representation for generalized temperature functions considered over negative time.In the event that the functions themselves are positive, we have the following result.

Theorem 5.1

A necessary and sufficient condition that

$$u(x,t) = \int_0^\infty e^{ty^2}\, g(xy)\, d\lambda(y), \ -\infty < t < 0, \tag{5.1}$$

Where $\lambda(y)$ increasing above, is that, for $-\infty < t < 0$, $u(x,t) \in H$ and $u(x,t) \geq 0$.

Proof

If (5.1) holds with $\lambda(y)$ increasing above, then clearly $u(x,t) \geq 0$and since the kernelof the integral (5.1)belongs to H for each y,so is the integral by the validity of differentiation under the integral sign. Hence the condition is necessary.

Conversely, assuming that $u(x,t) \in H$ and $u(x,t) \geq 0$, for $-\infty < t < 0$, we have that

$$u^A(x,t) = G(x;y)\, u(x/t, -1/t) \tag{5.2}$$

Is non-negative and in H for $0 < t < \infty$. We may thus apply Theorem 9.1 of [1] to get

$$u^A(x,t) = \int_0^\infty G(x,y;t)\, db(y), \ \ 0 < t < \infty,$$

with$b(y)$ increasing above. Thus we have

$$u(x,t) = \int_0^\infty e^{ty^2}\, g(xy)\, d\lambda(y), \qquad -\infty < t < 0,$$

were $\lambda(y) = b(2y)$and the theorem is proved.

Now by applying the Theorem 5.1 to the function $u(x, t+c)$, we readily derive the following extension as the corollary.

Corollary 5.2

A necessary and sufficient condition that

$$u(x,t) = \int_0^\infty e^{ty^2}\, g(xy)\, d\lambda(y), \ \ -\infty < t < c, \tag{5.3}$$

with$\lambda(y)$ increasing above, is that for $-\infty < t < c$, $u(x,t)$be a non-negative generalized temperature function.

Example

Theorem 5.1 is illustrated by the example

$$u(x,t) = G(ix; -t) = \int_0^\infty e^{ty^2}\, g(xy)\, d\mu(y), \ -\infty < t < c \tag{5.4}$$

The function (5.4) does not satisfy the condition

$$\int_0^\infty u(x,t_0) e^{x^2/4t_0}\, d\mu(x) < \infty, \tag{5.5}$$

for any $t_0 < 0$. By adding such a restriction to the functions considered in Theorem 5.1, we obtain a subclass of temperature function which in addition to (5.1) have an alternative integral representation as given in the following result.

Theorem 5.3

A necessary and sufficient condition that

$$u(x,t) = \int_0^\infty G(ix, y; -t)\, \psi(y)\, d\mu(y), \ -\infty < t < 0, \tag{5.6}$$

where$\psi(x) = \hat{\lambda}^s(x)$, for some non-decreasing, bounded function λ, is that, for $-\infty < t < 0$, $u(x,t) \in H$, $u(x,t) \geq 0$, and for some $t_0 < 0$,

$$\int_0^\infty u(x,t_0) e^{x^2/4t_0}\, d\mu(x) < \infty. \tag{5.7}$$

Proof

To establish the necessity of the condition, assume (5.6) with

$$\psi(x) = \int_0^\infty h(xy)\, d\lambda(y), \tag{5.8}$$

Where $\lambda(y)$ is a non-decreasing bounded function. Then, substituting (5.8) in (5.6), we find that

$$\begin{aligned} u(x,t) &= \int_0^\infty G(ix, y; -t)\, d\mu(y) \int_0^\infty h(xy)\, d\lambda(z) \\ &= \int_0^\infty e^{tz^2} g(xz)\, d\lambda(z) \ \ , \ -\infty < t < 0, \end{aligned} \tag{5.9}$$

where the inversion of order of integration is valid, for $t < 0$, since

$$\int_0^\infty d\lambda(z) \int_0^\infty e^{y^2/4t}\, d\mu(y) < \infty, \ \ t < 0. \tag{5.10}$$

since (5.9) holds, an appeal to Theorem 5.1 yields the fact that $u(x,t) \in H$ and $u(x,t) \geq 0$ for $-\infty < t < 0$. Further

$$\int_0^\infty u(x,t)e^{x^2/4t}\, d\mu(x) = \int_0^\infty e^{tx^2}\, d\lambda(z) \int_0^\infty e^{x^2/4t}\, g(xz)\, d\mu(x)$$
$$= \int_0^\infty e^{tz^2} G(iz; -1/4t)\, d\lambda(z) \quad (5.11)$$
$$= (-2t)^{\alpha-\beta+1} \int_0^\infty d\lambda(z) < \infty,$$

so that (5.7) holds and the condition is necessary.

Conversely suppose that, for $-\infty < t < 0, u(x,t)$ is a non-negative generalized temperature function for which (5.7) holds for some $t_0 < 0$. By Theorem 5.1, we then have

$$u(x,t) = \int_0^\infty e^{ty^2} g(xy)\, d\lambda(y), \quad -\infty < t < 0 \quad (5.12)$$

For some $\lambda(y)$ increasing above. Since (5.7) holds for $t_0 < 0$, the left hand side of (5.11)is finite for t_0. Hencefrom the right hand side of (5.11), it follows that $\lambda(y)$ is bounded. Hence as $\lambda(y)$ is a bounded, monotonic increasing function, the integral $\int_0^\infty h(xy)\, d\lambda(y)$ exists and defines a function of x.

Let

$$\psi(x) = \int_0^\infty h(xy)\, d\lambda(y).$$

Then we derive the representation (5.6) by a computation as in the first part of the proof, and the theorem is established.

Now we give an application of Theorem 5.1 as the following theorem:

Theorem 5.4

If $u(x,t) \in H$ and $u(x,t) \geq 0$ for $-\infty < t \leq c$, and if

$$\underset{|x|\leq r}{Max}\, u(x,t) = M(r),$$

then

$$\liminf_{r\to\infty} \frac{\log M(r)}{r} \leq 0$$

implies that $u(x,t)$is constant for $-\infty < t \leq c$.

Proof

Without loss of generality, we may assume that $c = 0$.

Now, suppose that $y = y_0 > 0$ is a point of increase of $\lambda(y)$ given in Theorem 5.1. Then we have

$$u(x,0) = \int_0^\infty g(xy)\, d\lambda(y) \geq \int_{y_0-\delta}^{y_0+\delta} g(xy)\, d\lambda(y) \geq kg\big(x(y_0-\delta)\big),$$

where,

$$k = \lambda(y_0+\delta) - \lambda(y_0-\delta) > 0$$

and

$\delta$is such that $y_0 - \delta > 0$. Hence

$$M(r) \geq kg\left(r(y_0-\delta)\right)$$

so that

$$\liminf_{r\to\infty} M(r)/r = \infty$$

Contradicting the hypothesis. Hence $\lambda(y)$has at most one point of increase at $y = 0$ and $u(x,t)$ is constant. This completes the proof.

Remark

Note that any generalized temperature function which is uniformly bounded for $-\infty < t \leq c$ is necessarily constant. Somewhat different criteria for functions in H will also yield a representation of the form (5.1) as indicated in the following theorem.

Theorem 5.5

A necessary and sufficient condition that

$$u(x,t) = \int_0^\infty e^{ty^2}\, g(xy)\, d\lambda(y), \quad t < c < 0, \quad (5.13)$$

with

$$\int_0^\infty e^{cy^2} |d\lambda(y)| < \infty, \quad (5.14)$$

is that $u(x,t) \in H$ for $t < c < 0$, and that, for $t < c < 0$,

$$\int_0^\infty |u(x,t)|\, G(x; c-t)\, d\mu(x) < M. \quad (5.15)$$

Proof: If $u(x,t) \in H$ for $t < c < 0$, then $u^A(x,t) \in H$for $0 < t < -\frac{1}{c}$.Further, if (5.15) holds for $t < c < 0$, then

$$\int_0^\infty |u^A(x,t)|\, G\left(x; -\frac{1}{c} - t\right)\, d\mu(x)$$

$$= \left(-\frac{c}{2}\right)^{\alpha-\beta+1} \int_0^\infty |u(x,-1/t)|\, G(x;c+1/t)\, d\mu(x) \tag{5.16}$$
$$< \infty, \quad 0 < t < -1/c.$$

Hence by Theorem 8.1 of [3], we have

$$u^A(x,t) = \int_0^\infty G(x,y;t)\, db(y)\,, \quad 0 < t < -\frac{1}{c}\,, \tag{5.17}$$

with

$$\int_0^\infty G(y;\,-1/c)|db(y)| < \infty\,. \tag{5.18}$$

But (5.17) gives

$$u(x,t) = \int_0^\infty e^{y^2 t}\, g(xy)\, db(2y), \quad t < c < 0,$$

or taking $\lambda(y) = b(2y)$, we can obtain

$$u(x,t) = \int_0^\infty e^{y^2 t}\, g(xy)\, d\lambda(y)\,, \quad t < c < 0 \tag{5.19}$$

with

$$\int_0^\infty G(2y;\,-1/c)|db(2y)| = (-c/2)^{\alpha-\beta+1} \int_0^\infty e^{cy^2} |d\lambda(y)| < \infty$$

so that sufficiency is established.

To prove the necessity of the condition, we note that if (5.13), (5.14) hold, then $u(x,t) \in H$for $u(x,t) \in H$for $t < c < 0$, since differentiation under the integral sign is valid. Further, for $t < c < 0$,

$$\begin{aligned}\int_0^\infty |u(x,t)|\, G(x;c-t)\, d\mu(x) &\le \int_0^\infty G(x;c-t)\, d\mu(x) \int_0^\infty e^{ty^2} g(xy)\, |d\lambda(y)| \\ &= \int_0^\infty e^{ty^2} |d\lambda(y)| \int_0^\infty g(xy)\, G(x;c-t)\, d\mu(x) \\ &= \int_0^\infty e^{cy^2} |d\lambda(y)| \\ &< \infty,\end{aligned}$$

and (5.15) holds and thus proof is completed.

Remarks

(i) Author claims that the results obtained in the present paper are more stronger than that of Haimo and Cholewinski[6].

(ii) In particular if we take $\alpha = v/2$ and $\beta = 1/2 - v/2$ throughout this paper then it reduces to the results studied inHaimo and Cholewinski[6].That is the results studied in [6] are particular case of the present paper for $\alpha = v/2,\ \beta = {}^1/_2 - v/2$.

References

1. F.M. Cholewinski and D.THaimo, The Weirstrass-Hankelconvolution transform, J. d'Analyse Math., Vol. 17, No. 1 (1966), 1-58.
2. D.T. Haimo, Integral equations associated with Hankel convolutions, Trans.Amer Math. Soc., 116(1965), 330-375.
3. D.T- Haimo, Generalized temperature functions,Duke Math. J., Vol. 116 (1965), 330-375.
4. D.T. Haimo, Functions with the Huygens property, Bull. Amer. Math. Soc., Vol. 71 (1965), 528-532.
5. D.T. Haimo, Expansions in terms of generalized heat polynomials and of their Appell transforms, J. Math. Mech., 115(1965) 735-758.
6. D.T. Haimo and F.M. Cholewinski, Integral representations of solutions of the generalized heat equation, Illinois-J. Math. Vol.10, Issue 4(1966), 623-638.
7. P.C. Rosenbloom and D.V. Widder, Expansions in terms of heat polynomials and associated functions, Trans. Amer. Math. Soc., Vol. 92(1959), 220-266.
8. D.V.Widder, The role of the Appell transformation in the theory of heat conduction, Trans. Amer. Math. Soc.,Vol. 95 (1962), 121-134.