



On the Convergence and Accuracy of the Adomian Decomposition and Picard Iterative Methods Applying to Nonlinear Ordinary Differential Equations

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ABSTRACT

In this work, the Adomian decomposition (ADM) and Picard's Iterative Methods were used to solve nonlinear ordinary differential equations analytically and numerically using the Trapezoidal rule approach, and the results are compared for accuracy and rate of convergence. Though a little modification by the use of contraction principle was made to the Picard Iteration Method in order to accelerate the convergence of the method and found out that the ADM converges faster than the Picard's method.

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Introduction

In the 1980's George Adomian introduced a new powerful method for solving nonlinear functional equations. Since then, this method has been known as the Adomian decomposition method (ADM). The technique is based on a decomposition of a solution of a nonlinear operator equation in a series of functions. Each term of the series is obtained from a polynomial generated from an expansion of analytic functions into a power series. The Adomian technique is very simple in an abstract formulation but the difficulty arises in calculating the polynomials and in proving the convergence of the series of the functions [1].

Convergence of the Adomian method when applied to some classes of ordinary and partial differential equations was discussed by many authors. [2] Proved the convergence of the Adomian method for differential and operator equations. [3] investigated convergence of the Adomian decomposition method when applied to time-dependent heat, wave and beam equations for both forward and backward time evolution. He showed that the convergence was faster for forward problems than for backward problems. [4] implemented the Adomian method for variable-depth shallow water equations with a source term and illustrated the convergence numerically. A comparative study between the ADM and the Sine-Galerkin method for solving population growth model was performed by [5] while that between ADM and Runge-Kutta method for solving system of ordinary differential equations was performed by [6]. [7] discussed applications of ADM to a class in acoustics. [8] Compared ADM and Taylor series method by using some particular example and showed that the decomposition method produced reliable results with few iterations, whereas Taylor series method suffered from computational difficulties. [9] modified the ADM to accelerate the convergence of the series solution.

In this work, we intend to study the convergence and accuracy of Adomian decomposition method as compare to Picard's iteration method. Both analytical and numerical comparison will be made. The numerical comparison is classified into two different categories, namely the accuracy and the rate of convergence of the two methods. The first requires the use of trapezoidal rule and the latter the use of a relatively new method using Banach fixed point theorem.

The work is structured as follows: section 2 contains the methodology. It gives detailed description of how the ADM works with nonlinear ODEs. Both analytic and numerical comparison between ADM and Picard's method is made. Formalism for the convergence of both methods is discussed. A relatively new but simple method is introduced to determine the rate of convergence of ADM and Picard's method. Numerical algorithms for ADM and Picard's method using trapezoidal rule is presented. Section 3 presents the applications and results of ADM and Picard's method to nonlinear ODEs. Both analytic and numerical results are considered. The section demonstrates how to use the trapezoidal rule on ADM and Picard's method.

Methodology

Nonlinear ODEs by Adomian Decomposition Method

It is well known that nonlinear ordinary differential equations are in general difficult to handle [10]. The Adomian decomposition method which is one of the few recent methods is powerful in handling such difficulty. [1] reviewed the basic method, by considering an abstract system of nonlinear differential equation:

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$$\frac{dy}{dt} = f(t, y), y \in \mathbb{R}^d, f: \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d \quad (2.1)$$

With initial condition

$$y(0) = y_0 \in \mathbb{R}^d, \quad (2.2)$$

To apply the ADM on nonlinear ordinary differential equations, we consider the equation

$$Ly + R(y) + F(y) = g(t) \quad (2.3)$$

Where L the highest order derivative in the equation, R is the remainder of the differential operator, $F(y)$ expresses the nonlinear terms and $g(t)$ is an inhomogeneous term.

If L is a first order operator defined by

$$L = \frac{d}{dt} \quad (2.4)$$

then, it is assumed L is invertible and the inverse operator L^{-1} is given by

$$L^{-1}(\cdot) = \int_0^t (\cdot) dt \quad (2.5)$$

so that

$$L^{-1}Ly = y(t) - y(0) \quad (2.6)$$

However, if L is a second order differential operator given by

$$L = \frac{d^2}{dt^2} \quad (2.7)$$

The

$$L^{-1}(\cdot) = \int_0^t \int_0^t (\cdot) dt dt \quad (2.8)$$

which follows that

$$L^{-1}L = y(t) - y(0) - ty'(0) \quad (2.9)$$

In the same procedures, we can see that for third order differential operator, we have

$$L = \frac{d^3}{dt^3} \quad (2.10)$$

$$L^{-1}(\cdot) = \int_0^t \int_0^t \int_0^t (\cdot) dt dt dt$$

So that

$$L^{-1}L = y(t) - y(0) - ty'(0) - \frac{1}{2!}t^2y''(0) \quad (2.11)$$

Following the same procedures as outlined above, higher order operator and the related inverse operators can easily be defined.

Now applying L^{-1} and the initial condition to both sides of (2.3) we have

Where Φ_0 is as follows:

$$\left\{ \begin{array}{l} y(0) \text{ for } L = \frac{d}{dt} \\ y(0) + ty'(0) \text{ for } L = \frac{d^2}{dt^2} \\ y(0) + ty'(0) + \frac{1}{2!}t^2y''(0) \text{ for } L = \frac{d^3}{dt^3} \\ y(0) + ty'(0) + \frac{1}{2!}t^2y''(0) + \frac{1}{3!}t^3y'''(0) \text{ for } L = \frac{d^4}{dt^4} \\ y(0) + ty'(0) + \frac{1}{2!}t^2y''(0) + \frac{1}{3!}t^3y'''(0) + \frac{1}{4!}t^4y^{(4)}(0) \text{ for } L = \frac{d^5}{dt^5} \end{array} \right. \quad (2.13)$$

The Adomian decomposition method as applicable to nonlinear ODEs admits the decomposition of y into an infinite series of components

$$y(t) = \sum_{n=0}^{\infty} y_n \quad (2.14)$$

and the nonlinear term $F(y)$ as

$$F(y) = \sum_{n=0}^{\infty} A_n \quad (2.15)$$

Where A_n 's are the Adomian polynomials.

Now substituting (2.14) and (2.15) into (2.12) we have

$$\sum_{n=0}^{\infty} y_n = \Phi_0 - L^{-1}g(t) - L^{-1}R\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \quad (2.16)$$

Adomian consider the solution $y(t)$ in the series of functions:

$$y(t) = y_0 + \sum_{n=1}^{\infty} y_n, n \geq 1 \quad (2.17)$$

and write the nonlinear function $f(t, y)$ as the series of functions

$$f(t, y) = \sum_{n=0}^{\infty} A_n(t, y_0, y_1, \dots, y_n) \quad (2.18)$$

The dependence of A_n on t and y_0 may be non-polynomial. A_n is formally obtained by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} f\left(t, \sum_{i=0}^{\infty} \lambda^i y_i\right)_{\lambda=0}, n \quad (2.19)$$

where λ is a formal parameter. Functions A_n are polynomials in (y_0, y_1, \dots, y_n) , which are referred to as the Adomian polynomials.

In what follows, we shall consider a scalar differential equation and set $d=1$. The first four Adomian polynomials for $d = 1$ are listed as follows

$$\begin{cases} A_0 = f(t, y_0) \\ A_1 = y_1 f'(t, y_0) \\ A_2 = y_2 f'(t, y_0) + \frac{1}{2} y_1^2 f''(t, y_0) \\ A_3 = y_3 f'(t, y_0) + y_1 y_2 f''(t, y_0) + \frac{1}{6} y_1^3 f'''(t, y_0) \end{cases} \tag{2.20}$$

Where the primes denotes the derivatives with respect to y .

It was shown by [2] that the Adomian polynomials A_n are defined explicitly by the formulae:

$$A_n = \sum_{k=1}^n \frac{1}{k!} f^{(k)}(t, y_0) \left(\sum_{n_1 + n_2 + \dots + n_k = n} y_{n_1} \dots y_{n_k} \right), n \geq 1 \tag{2.21}$$

or equivalently by

$$A_n = \sum_{k=n}^{\infty} f^{(k)}(t, y_0) \frac{y_1^{k_1} \dots y_n^{k_n}}{k_1! \dots k_n!}, n \geq 1 \tag{2.22}$$

Where

$$k = k_1 + \dots + k_n \text{ and } nk = k_1 + 2k_2 + \dots + nk_n.$$

[11] proved a bound for Adomian polynomials by

$$A_n \leq \frac{(n+1)^n}{(n+1)!} M^{n+1} \tag{2.23}$$

where $\sup_{t \in J} f^{(k)}(t, y_0) \leq M$ for a given time interval $J \subset \mathbb{R}$. Equations (2.17) and (2.18) lead to a recursive equation in the form:

$$y_{n+1}(t) = \int_0^t A_n(s, y_0(s), y_1(s), \dots, y_n(s)) ds, n \geq 0 \tag{2.24}$$

Comparison between the ADM and the Picard Method

The Adomian decomposition method was first compared with Picard’s method by [12] on number of examples. [13] showed that the Adomian method for linear differential equations was equivalent to the classical method of successive approximations (Picard). However, this equivalence does not hold for nonlinear differential equations. Here, we shall compare the two methods and show differences of the decomposition method.

Picard’s method was introduced by Emile Picard in 1891 and it is used for the proof of existence and uniqueness of a solution of a system of differential equations.

In the analysis of the Picard’s method we assume that $f(t, y)$ satisfies a local Lipschitz condition in a ball around

$$\begin{aligned} t = 0 \text{ and } y_0: t \leq t_0, y - y_0, \\ \bar{y} - y_0 \leq \delta_0; f(t, y) - f(t, \bar{y}) \leq ky - \bar{y} \end{aligned} \tag{2.25}$$

where k is the Lipschitz constant and y is any norm in \mathbb{R}^d .

Let $y^{(0)} = y_0$ and define a recurrence relation

$$y^{(n+1)}(t) = y_0 + \int_0^t f(s, y^{(n)}(s)) ds, n \geq 0 \tag{2.26}$$

If t_0 is small enough, the new approximation $y^{(n+1)}(t)$ will belong to the same ball as $\bar{y} - y_0 \leq \delta_0$ for all $t \leq t_0$ and (2.24) is a contraction in the sense that

$$\int_0^t [f(s, y(s)) - f(s, \bar{y}(s))] ds \leq Q \sup_{0 \leq s \leq t} y(s) - \bar{y}(s) \tag{2.27}$$

where

$$Q = kt_0 < t, \text{ so that } t_0 < \frac{1}{k}$$

By the Banach fixed point theorem, there exists a unique solution $y(t)$ in $C([-t_0, t_0], B_{\delta_0}(y_0))$, where $B_{\delta_0}(y_0)$ is an open ball in \mathbb{R}^d centered at y_0 with radius δ_0 . Recall here that $C([-t_0, t_0], \mathbb{R}^d)$ with the norm

$$\|y\| = \sup_{t \in [-t_0, t_0]} y(t) \tag{2.28}$$

is a complete metric space. Since the integral of a continuous function is continuously differentiable function, $y(t)$ is actually in $C'([-t_0, t_0], B_{\delta_0}(y))$.

By contraction mapping principle, the error of the approximation solution $y^{(n)}(t)$ is estimated by:

$$E_n = \|y - y^{(n)}\| \leq \frac{Mk^n t_0^{n+1}}{(n+1)!} \tag{2.29}$$

$$M = \sup_{t \in [-t_0, t_0]} \sup_{y \in B_{\delta_0}(y_0)} f(t, y) \tag{2.30}$$

Convergence Analysis of ADM

It follows from (2.17) that A_n are polynomials in y_1, \dots, y_n and thus y_{n+1} is obtained from (2.22) explicitly, if we are able to calculate A_n . The first proof of the convergence of the ADM was given by [14] who used fixed point theorem for abstract functional equations.

We will consider a functional equation of the form:

$$y = y_0 + f(y), \quad y \in \mathbb{H}, \tag{2.31}$$

where, \mathbb{H} is a Hilbert space and $f: \mathbb{H} \rightarrow \mathbb{H}$.

Let
$$S_n = y_1 + y_2 + \dots + y_n \tag{2.32}$$

and
$$f_n(y_0 + S_n) = \sum_{i=1}^n A_i \tag{2.33}$$

The Adomian decomposition method is equivalent to determining the sequence $\{S_n\}_{n \in \mathbb{N}}$, defined by
$$S_{n+1} = f_n(y_0 + S_n), \quad S_0 = 0 \tag{2.34}$$

If there exist limit
$$S = \lim_{n \rightarrow \infty} S_n, \quad f = \lim_{n \rightarrow \infty} f_n$$

in a Hilbert space, \mathbb{H} then $S = f(y_0 + S)$ is also in \mathbb{H} .

The Adomian decomposition method converges if
$$\|f\| \leq 1, \|f_n - f\| = \epsilon_n \rightarrow 0 \text{ as } n \rightarrow \infty \tag{2.35}$$

These two conditions are rather restrictive. The first implies a constraint on the nonlinear function while the second implies the convergence of the series, $\sum_{n=0}^{\infty} A_n$. Although to satisfy the two conditions for the convergence is difficult. Hence we shall introduce a new formalism for determining the convergence of the Adomian decomposition method.

Formalism of the Convergence of ADM and Picard’s Method

We consider an operator, f_n from a Hilbert space, \mathbb{H} into \mathbb{H} and y an exact solution of (2.1). The Adomian series

$$\sum_{n=0}^{\infty} y_n = \phi_0 - L^{-1}g(t) - L^{-1}R\left(\sum_{i=0}^{\infty} y_i\right) - L^{-1}\left(\sum_{i=0}^{\infty} A_i\right) \tag{2.36}$$

converges to y if given

$$y_{n+1}(t) = \int_0^t A_n(s, y_0(s), y_1(s), \dots, y_n(s)) ds \tag{2.37}$$

$$y_n(t) = \int_0^t A_n(s, y_0(s), y_1(s), \dots, y_{n-1}(s)) ds \tag{2.38}$$

we can find $0 \leq \alpha_n < 1$ such that
$$\|y_{n+1}\| \leq \alpha_n \|y_n\| \quad n \in \mathbb{N} \tag{2.39}$$

In sufficiently closer condition, the Picard method, being associated to contraction operator, converges in Hilbert space \mathbb{H} if given

$$y^{(n+1)}(t) = y_0 + \int_{t_0}^t f(s, y^{(n)}(s)) ds, n \geq 0, \tag{2.40}$$

$$y^{(n)}(t) = y_0 + \int_{t_0}^t f(s, y^{(n-1)}(s)) ds \tag{2.41}$$

we can find $0 \leq \bar{\alpha}_n < 1$ such that
$$\|y^{(n)}\| \leq \bar{\alpha}_n \|y^{(n+1)}\| \tag{2.42}$$

Rate of Convergence of ADM and Picard’s Method

Here we shall present a simple method to determine the rate of convergence of ADM and Picard’s method alike. We begin by assuming (2.39) and (2.42) are satisfied by the ADM and Picard’s method respectively. The rate of convergence, $\sum_{n=0}^{\infty} y_n$ and $\sum_{n=0}^{\infty} y^{(n)}$ for the ADM and the Picard’s method respectively depend on the values of α_n and $\bar{\alpha}_n$. The methodology implies that the smaller the value, the faster the rate of convergence to the exact solution. And that if by implication $\alpha_n < \bar{\alpha}_n$ n ,

then the rate of convergence of ADM is higher than that of Picard’s method [1].

Numerical Algorithms for ADM and Picard’s Method

Consider an abstract initial-value problem for system of nonlinear differential equation

$$\frac{dy}{dt} = f(t, y), \quad y(0) = y_0 \tag{2.44}$$

where $y \in \mathbb{R}^d$, and $f: \mathbb{R} \times \mathbb{R}^d$.

Adomian Decomposition Method by Trapezoidal Rule

To solve (2.44) using the Adomian decomposition method numerically, we define elements of the Adomian series by the recursive equation (2.24) and apply the trapezoidal rule on interval $[0, T]$ with grid points at

$$t_m = mh, \quad m = 0, 1, 2, \dots, M$$

where $h = \frac{T}{M}$. Then

$$y_{n+1}(t_m) = \frac{h}{2} \left(A_n(0, y_0(0), \dots, y_n(0)) + A_n(t_m, y_0(t_m), \dots, y_n(t_m)) + 2 \sum_{j=1}^{m-1} A_n(t_j, y_0(t_j), \dots, y_n(t_j)) \right) \tag{2.45}$$

Where $y_0(t) = y_0$ and $y_n(0) = 0$ for $n \geq 1$.

After the Adomian polynomials are computed recursively in the explicit form for $n = 0, 1, 2, \dots, N$, we can now use the trapezoidal rule (2.45) on the grid $\{t_m\}_{m=0}^M$ by incrementing n from $n=0$ to $n=N$. Thus we define the n^{th} -partial sum of the Adomian series on the grid $\{t_m\}_{m=0}^M$ by

$$Y_n(t_m) = y_0 + \sum_{i=1}^n y_i(t_m) \quad (2.46)$$

2.4.2 Picard's Method by Trapezoidal Rule

[15] used Simpson's rule to achieve their results. However, in our case we are going to Trapezoidal rule to implement our claim. Solving (2.44) using the Picard's method numerically we consider the recursive equation (2.26) and apply the trapezoidal rule on the same interval $[0, T]$ and grid point as in case 2.4.1. The trapezoidal rule on Picard method is given in the form:

$$Y^{(n+1)}(t_m) = y_0 + \frac{h}{2} (f(0, Y^{(n)}(0)) + f(t_m, Y^{(n)}(t_m))) + \sum_{j=1}^{m-1} f(t_j, Y^{(n)}(t_j)) \quad (2.47)$$

which presents the result.

Applications and Results

Analytic Comparison between ADM and Picard's Method

Example 1

Consider the nonlinear ODE:

$$\frac{dy}{dt} = 2y - y^2, \quad y(0) = 1 \quad (3.1)$$

With exact solution:

$$y(t)_{\text{exact}} = 1 + \tanh(t) \quad (3.2)$$

By the Adomian decomposition method, we write (3.2) in the integral form:

$$y(t) = 1 + \int_0^t (2y(s) - y^2(s)) ds \quad (3.3)$$

and compute the Adomian polynomials for $f(t, y) = 2y - y^2$ in the form:

$$\begin{cases} A_0 = 2y_0 - y_0^2 \\ A_1 = 2y_1 - 2y_0y_1 \\ A_2 = 2y_2 - 2y_0y_2 - y_1^2 \\ A_3 = 2y_3 - 2(y_0y_3 + y_1y_2) \\ A_4 = 2y_4 - 2(y_0y_4 + y_1y_3) - y_2^2 \end{cases} \quad (3.4)$$

Using (2.24) we determine few terms of the Adomian series:

$$\begin{cases} y_0 = 1 \\ y_1 = t \\ y_2 = 0 \\ y_3 = \frac{-t^3}{3} \\ y_4 = 0 \\ y_5 = \frac{2t^5}{15} \end{cases} \quad (3.5)$$

Thus

$$y(t) = 1 + t - \frac{t^3}{3} + \frac{2t^5}{15} - \dots = 1 + \tanh(t) \quad (3.6)$$

By Picard's method, we write (3.1) in the integral form:

$$y^{(n)} = 1 + \int_0^t (2y^{(n-1)}(s) - (y^{(n-1)}(s))^2) ds \quad (3.7)$$

We now obtain the successive approximations in the form:

$$\begin{cases} y^{(0)} = 1 = y_0 \\ y^{(1)} = 1 + t = y_0 + y_1 \\ y^{(2)} = 1 + t - \frac{t^3}{3} = y_0 + y_1 + y_2 \\ y^{(3)} = 1 + t - \frac{t^3}{3} + \frac{2t^5}{15} - \frac{t^7}{63} = y_0 + y_1 + y_2 + y_3 - \frac{t^7}{63} \end{cases} \quad (3.8)$$

Example 2

Consider

$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0 \quad (3.9)$$

With exact solution:

$$y(t)_{\text{exact}} = \tan(t) \quad (3.10)$$

By Adomian decomposition method, we write (3.9) in the operator form:

$$L_y = 1 + y^2, \quad y(0) = 0 \quad (3.11)$$

Applying L^{-1} to both sides of (3.11) and using the initial condition gives:

$$y = t + L^{-1}(y^2) \quad (3.12)$$

But from (2.14) and (2.15) we noted that:

$$y(t) = \sum_{n=0}^{\infty} A_n \quad (3.13)$$

$$y^2 = \sum_{n=0}^{\infty} A_n \quad (3.14)$$

Inserting (3.13) and (3.14) into (3.12) yields :

$$\sum_{n=0}^{\infty} y_n(t) = t + L^{-1} \left(\sum_{n=0}^{\infty} A_n \right) \quad (3.15)$$

Computing the Adomian polynomials for $f(t, y) = 1 + y^2$ in the form:

$$\begin{cases} A_0 = y_0^2 \\ A_1 = 2y_0y_1 \\ A_2 = 2y_0y_2 + y_1^2 \\ A_3 = 2y_0y_3 + 2y_1y_2 \\ A_4 = 2y_0y_4 + 2y_1y_3 + y_2^2 \end{cases} \quad (3.16)$$

Using (2.24) we determine few terms of the Adomian series:

$$\begin{cases} y_0 = 0 \\ y_1(t) = t \\ y_2(t) = \frac{t^2}{3} \\ y_3(t) = \frac{2t^5}{15} \\ y_4(t) = \frac{17t^7}{315} \end{cases} \quad (3.17)$$

Thus

$$y(t) = t + \frac{t^2}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \dots = \tan(t) \quad (3.18)$$

By Picard's method we can write (3.9) in the form:

$$y^{(n)} = \int_0^t (1 + (y^{(n-1)}(s))^2) ds \quad (3.19)$$

We obtain the successive approximation in the form:

$$\begin{cases} y^{(0)} = 0 = y_0 \\ y^{(1)} = t = y_1 \\ y^{(2)} = t + \frac{t^2}{3} = y_1 + y_2 \\ y^{(3)} = t + \frac{t^2}{3} + \frac{2t^5}{15} + \frac{t^7}{63} = y_1 + y_2 + y_3 + \frac{t^7}{63} \\ y^{(4)} = t + \frac{t^2}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \frac{62t^9}{2835} + \frac{109t^{11}}{51975} + \dots \\ = y_1 + y_2 + y_3 + y_4 + \frac{62t^9}{2835} + \frac{109t^{11}}{51975} + \dots \end{cases} \quad (3.20)$$

Numerical Comparison between ADM and Picard's Method

Here we present two categories of numerical results which make numerical comparison between the Adomian decomposition method and the Picard's iteration method. The first case compares the accuracy of the two methods and the second compares the rate of convergence of the methods.

Example 3

We reconsider the problem (3.9):

$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 0$$

To solve (3.9) numerically by Adomian decomposition method (ADM), we take the interval $[0,1]$; $m = 0,1,2, \dots, 5$; $T = 1$ and $\frac{T}{M} = \frac{1}{5} = 0.2$. Where h is the step size; m is the vertical separator of the grids. But $t_m = mh$ so that:

$$t_0 = 0, t_1 = 0.2, t_2 = 0.4, t_3 = 0.6, t_4 = 0.8, t_5 = 1.0$$

It now follows from (3.16) that

$$\begin{cases} A_0 = y_0^2 = t_m^2 \\ A_1 = 2y_0y_1 = \frac{2t_m^4}{3} \\ A_2 = 2y_0y_2 + y_1^2 = \frac{51t_m^6}{135} \\ A_3 = 2y_0y_3 + 2y_1y_2 = \frac{62t_m^8}{315} \end{cases} \quad (3.21)$$

Now applying (2.45) we have the successive approximations the form:

$$y_1(t_0) = \frac{h}{2} (A_0(0, y_0(0)) + A_0(t_0, y_0(t_0))) = \frac{0.2}{2} (t_0^2) = 0$$

$$y_1(t_1) = \frac{h}{2} (A_0(0, y_0(0)) + A_0(t_1, y_0(t_1))) = \frac{0.2}{2} (t_1^2) = 0$$

$$y_1(t_2) = \frac{h}{2} (A_0(0, y_0(0)) + A_0(t_2, y_0(t_2)) + 2A_0(t_1, y_0(t_1))) = \frac{0.2}{2} (t_2^2 + 2t_1^2) = 2.4 \times 10^{-2}$$

$$y_1(t_3) = \frac{h}{2} (A_0(0, y_0(0)) + A_0(t_3, y_0(t_3)) + 2(A_0(t_2, y_0(t_2)) + A_0(t_1, y_0(t_1)))) = \frac{0.2}{2} (0 + t_3^2 + 2(t_2^2 + t_1^2)) = 7.6 \times 10^{-2}$$

$$y_1(t_4) = \frac{h}{2} (A_0(0, y_0(0)) + A_0(t_4, y_0(t_4)) + 2(A_0(t_3, y_0(t_3)) + A_0(t_2, y_0(t_2)) + A_0(t_1, y_0(t_1)))) = \frac{0.2}{2} (0 + t_4^2 + 2(t_3^2 + t_2^2 + t_1^2)) = 1.76 \times 10^{-1}$$

$$\begin{aligned}
 y_1(t_2) &= \frac{h}{2} \left(A_0(0, y_0(0)) + A_0(t_2, y_0(t_2)) + 2 \left(A_0(t_1, y_0(t_1)) + A_0(t_2, y_0(t_2)) + A_0(t_2, y_0(t_2)) + A_0(t_4, y_0(t_4)) \right) \right) = 3.4 \times 10^{-1} \\
 y_2(t_0) &= \frac{h}{2} \left(A_1(0, y_1(0)) + A_1(t_0, y_1(t_0)) \right) = \frac{0.2}{2} \left(\frac{2t_0^4}{3} \right) = 0 \\
 y_2(t_2) &= \frac{h}{2} \left(A_1(0, y_1(0)) + A_1(t_2, y_1(t_2)) \right) = \frac{0.2}{2} \left(\frac{2t_2^4}{3} \right) = 0 \\
 y_2(t_2) &= \frac{h}{2} \left(A_1(0, y_1(0)) + A_1(t_2, y_1(t_2)) + 2A_1(t_2, y_1(t_2)) \right) = \frac{0.2}{2} \left(0 + \frac{2t_2^4}{3} + \frac{4t_2^4}{3} \right) = 1.92 \times 10^{-2} \\
 y_2(t_2) &= \frac{h}{2} \left(A_1(0, y_1(0)) + A_1(t_2, y_1(t_2)) + 2 \left(A_1(t_2, y_1(t_2)) + A_1(t_2, y_1(t_2)) \right) \right) = \frac{0.2}{2} \left(0 + \frac{2t_2^4}{3} + 2 \left(\frac{2t_2^4}{3} + \frac{2t_2^4}{3} \right) \right) = 1.2267 \times 10^{-2} \\
 y_2(t_4) &= \frac{h}{2} \left(A_1(0, y_1(0)) + A_1(t_4, y_1(t_4)) + 2 \left(A_1(t_2, y_1(t_2)) + A_1(t_2, y_1(t_2)) + A_1(t_2, y_1(t_2)) \right) \right) = \frac{0.2}{2} \left(0 + \frac{2t_4^4}{3} + 2 \left(\frac{2t_2^4}{3} + \frac{2t_2^4}{3} + \frac{2t_2^4}{3} \right) \right) = 4.8213332 \times 10^{-1} \\
 y_2(t_2) &= \frac{h}{2} \left(A_1(0, y_1(0)) + A_1(t_2, y_1(t_2)) + 2 \left(A_1(t_2, y_1(t_2)) + A_1(t_2, y_1(t_2)) + A_1(t_2, y_1(t_2)) + A_1(t_4, y_1(t_4)) \right) \right) \\
 &= \frac{0.2}{2} \left(0 + \frac{2t_2^4}{3} + 2 \left(\frac{2t_2^4}{3} + \frac{2t_2^4}{3} + \frac{2t_2^4}{3} + \frac{2t_2^4}{3} \right) \right) = 1.42186666 \times 10^{-1} \\
 y_2(t_0) &= \frac{h}{2} \left(A_2(0, y_2(0)) + A_2(t_0, y_2(t_0)) \right) = \frac{0.2}{2} \left(\frac{51t_0^6}{135} \right) = 0 \\
 y_2(t_2) &= \frac{h}{2} \left(A_2(0, y_2(0)) + A_2(t_2, y_2(t_2)) \right) = \frac{0.2}{2} \left(\frac{51t_2^6}{135} \right) = 2.41778 \times 10^{-8} \\
 y_2(t_2) &= \frac{h}{2} \left(A_2(0, y_2(0)) + A_2(t_2, y_2(t_2)) + 2A_2(t_2, y_2(t_2)) \right) = \frac{0.2}{2} \left(\frac{51t_2^6}{135} + \frac{102t_2^6}{135} \right) = 1.595733 \times 10^{-4} \\
 y_2(t_2) &= \frac{h}{2} \left(A_2(0, y_2(0)) + A_2(t_2, y_2(t_2)) + 2 \left(A_2(t_2, y_2(t_2)) + A_2(t_2, y_2(t_2)) \right) \right) = \frac{0.2}{2} \left(\frac{51t_2^6}{135} + 2 \left(\frac{51t_2^6}{135} + \frac{51t_2^6}{135} \right) \right) = 2.0768711 \times 10^{-2} \\
 y_2(t_4) &= \frac{h}{2} \left(A_2(0, y_2(0)) + A_2(t_4, y_2(t_4)) + 2 \left(A_2(t_2, y_2(t_2)) + A_2(t_2, y_2(t_2)) + A_2(t_2, y_2(t_2)) \right) \right) = \frac{0.2}{2} \left(\frac{51t_4^6}{135} + 2 \left(\frac{51t_2^6}{135} + \frac{51t_2^6}{135} + \frac{51t_2^6}{135} \right) \right) \\
 &= 1.3742648 \times 10^{-2} \\
 y_2(t_2) &= \frac{h}{2} \left(A_2(0, y_2(0)) + A_2(t_2, y_2(t_2)) + 2 \left(A_2(t_2, y_2(t_2)) + A_2(t_2, y_2(t_2)) + A_2(t_2, y_2(t_2)) + A_2(t_4, y_2(t_4)) \right) \right) \\
 &= \frac{0.2}{2} \left(\frac{51t_2^6}{135} + 2 \left(\frac{51t_2^6}{135} + \frac{51t_2^6}{135} + \frac{51t_2^6}{135} + \frac{51t_2^6}{135} \right) \right) = 6.1423644 \times 10^{-2} \\
 y_4(t_0) &= \frac{h}{2} \left(A_3(0, y_3(0)) + A_3(t_0, y_3(t_0)) \right) = 0 \\
 y_4(t_2) &= \frac{h}{2} \left(A_3(0, y_3(0)) + A_3(t_2, y_3(t_2)) \right) = 5.03873 \times 10^{-8} \\
 y_4(t_2) &= \frac{h}{2} \left(A_3(0, y_3(0)) + A_3(t_2, y_3(t_2)) + 2A_3(t_2, y_3(t_2)) \right) = 1.299992381 \times 10^{-5} \\
 y_4(t_2) &= \frac{h}{2} \left(A_3(0, y_3(0)) + A_3(t_2, y_3(t_2)) + 2 \left(A_3(t_2, y_3(t_2)) + A_3(t_2, y_3(t_2)) \right) \right) = 3.564901587 \times 10^{-4} \\
 y_4(t_4) &= \frac{h}{2} \left(A_3(0, y_3(0)) + A_3(t_4, y_3(t_4)) + 2 \left(A_3(t_2, y_3(t_2)) + A_3(t_2, y_3(t_2)) + A_3(t_2, y_3(t_2)) \right) \right) = 3.939902887 \times 10^{-2} \\
 y_4(t_2) &= \frac{h}{2} \left(A_3(0, y_3(0)) + A_3(t_2, y_3(t_2)) + 2 \left(A_3(t_2, y_3(t_2)) + A_3(t_2, y_3(t_2)) + A_3(t_2, y_3(t_2)) + A_3(t_4, y_3(t_4)) \right) \right) = 2.6924624 \times 10^{-2}
 \end{aligned}$$

Now, using (2.46), we obtain:

Table 1. Numerical result of (3.9) by ADM

$Y_n \setminus t$	0.0	0.2	0.4	0.6	0.8	1.0
1	0.0	4.0×10^{-4}	2.4×10^{-2}	7.6×10^{-2}	1.7600×10^{-1}	3.4×10^{-1}
2	0.0	4.1067×10^{-4}	2.5920×10^{-2}	8.8267×10^{-2}	2.2421×10^{-1}	4.8219×10^{-1}
3	0.0	4.1091×10^{-4}	2.6080×10^{-2}	9.0344×10^{-2}	2.3796×10^{-1}	5.4361×10^{-1}
4	0.0	4.1092×10^{-4}	2.6093×10^{-2}	9.0700×10^{-2}	2.4190×10^{-1}	5.7053×10^{-1}

To solve (3.9) numerically by Picard’s method, we consider the same interval [0,1] and the same grids as above. It follows from (3.20) that :

$$\begin{aligned}
 y^{(1)} &= t_m \\
 y^{(2)} &= t_m + \frac{t_m^3}{3} \\
 y^{(3)} &= t_m + \frac{t_m^3}{3} + \frac{2t_m^5}{15} + \frac{t_m^7}{63} \\
 y^{(4)} &= t_m + \frac{t_m^3}{3} + \frac{2t_m^5}{15} + \frac{17t_m^7}{315} + \frac{62t_m^9}{2835} + \frac{109t_m^{11}}{51975} + \frac{4t_m^{13}}{12285} + \frac{t_m^{15}}{59535}
 \end{aligned}$$

Applying (2.47) we obtain the successive approximations in the form:

$$\begin{aligned}
 Y^{(1)}(t_0) &= y_0 + \frac{h}{2} \left(f(0, y^{(0)}(0)) + f(t_0, y^{(0)}(t_0)) \right) = \frac{0.2}{2} (1 + 1) = 2.0 \times 10^{-1} \\
 Y^{(1)}(t_2) &= \frac{h}{2} \left(f(0, y^{(0)}(0)) + f(t_2, y^{(0)}(t_2)) \right) = 2.0 \times 10^{-1}
 \end{aligned}$$

$$\begin{aligned}
 Y^{(2)}(t_2) &= \frac{h}{2} \left(f(0, y^{(0)}(0)) + f(t_2, y^{(0)}(t_2)) + f(t_2, y^{(0)}(t_2)) \right) = 3.0 \times 10^{-1} \\
 Y^{(2)}(t_2) &= \frac{h}{2} \left(f(0, y^{(0)}(0)) + f(t_2, y^{(0)}(t_2)) + f(t_2, y^{(0)}(t_2)) + f(t_2, y^{(0)}(t_2)) \right) = 4.0 \times 10^{-1}
 \end{aligned}$$

$$\begin{aligned}
 Y^{(1)}(t_4) &= \frac{h}{2} \left(f(0, y^{(0)}(0)) + f(t_4, y^{(0)}(t_4)) + f(t_1, y^{(0)}(t_1)) + f(t_2, y^{(0)}(t_2)) + f(t_3, y^{(0)}(t_3)) \right) = 5.0 \times 10^{-1} \\
 Y^{(2)}(t_2) &= \frac{h}{2} \left(f(0, y^{(0)}(0)) + f(t_2, y^{(0)}(t_2)) + f(t_1, y^{(0)}(t_1)) + f(t_2, y^{(0)}(t_2)) + f(t_3, y^{(0)}(t_3)) + f(t_4, y^{(0)}(t_4)) \right) = 6.0 \times 10^{-1} \\
 Y^{(2)}(t_0) &= y_0 + \frac{h}{2} \left(f(0, y^{(1)}(0)) + f(t_0, y^{(1)}(t_0)) \right) = \frac{0.2}{2} (2 + t_0^2) = 2.0 \times 10^{-1} \\
 Y^{(2)}(t_1) &= \frac{h}{2} \left(f(0, y^{(1)}(0)) + f(t_1, y^{(1)}(t_1)) \right) = \frac{0.2}{2} (2 + t_1^2) = 2.04 \times 10^{-1} \\
 Y^{(2)}(t_2) &= \frac{h}{2} \left(f(0, y^{(1)}(0)) + f(t_2, y^{(1)}(t_2)) + f(t_1, y^{(1)}(t_1)) \right) = \frac{0.2}{2} (3 + t_1^2 + t_2^2) = 3.2 \times 10^{-1} \\
 Y^{(2)}(t_3) &= \frac{h}{2} \left(f(0, y^{(1)}(0)) + f(t_3, y^{(1)}(t_3)) + f(t_1, y^{(1)}(t_1)) + f(t_2, y^{(1)}(t_2)) \right) = \frac{0.2}{2} (4 + t_2^2 + t_1^2 + t_3^2) = 4.56 \times 10^{-1} \\
 Y^{(2)}(t_4) &= \frac{h}{2} \left(f(0, y^{(1)}(0)) + f(t_4, y^{(1)}(t_4)) + f(t_1, y^{(1)}(t_1)) + f(t_2, y^{(1)}(t_2)) + f(t_3, y^{(1)}(t_3)) \right) = \frac{0.2}{2} (5 + t_4^2 + t_1^2 + t_2^2 + t_3^2) = 6.2 \times 10^{-1} \\
 Y^{(2)}(t_2) &= \frac{h}{2} \left(f(0, y^{(1)}(0)) + f(t_2, y^{(1)}(t_2)) + f(t_1, y^{(1)}(t_1)) + f(t_2, y^{(1)}(t_2)) + f(t_3, y^{(1)}(t_3)) + f(t_4, y^{(1)}(t_4)) \right) \\
 &= 8.2 \times 10^{-1} \\
 Y^{(2)}(t_0) &= \frac{h}{2} \left(f(0, y^{(2)}(0)) + f(t_0, y^{(2)}(t_0)) \right) = \frac{0.2}{2} \left(2 + (t_0 + \frac{t_0^2}{3})^2 \right) = 2.0 \times 10^{-1} \\
 Y^{(2)}(t_1) &= \frac{h}{2} \left(f(0, y^{(2)}(0)) + f(t_1, y^{(2)}(t_1)) \right) = \frac{0.2}{2} \left(2 + (t_1 + \frac{t_1^2}{3})^2 \right) = 2.04107377 \times 10^{-1} \\
 Y^{(2)}(t_2) &= \frac{h}{2} \left(f(0, y^{(2)}(0)) + f(t_2, y^{(2)}(t_2)) + f(t_1, y^{(2)}(t_1)) \right) = \frac{0.2}{2} \left(3 + (t_2 + \frac{t_2^2}{3})^2 + (t_1 + \frac{t_1^2}{3})^2 \right) = 3.21859555 \times 10^{-1} \\
 Y^{(2)}(t_3) &= \frac{h}{2} \left(f(0, y^{(2)}(0)) + f(t_3, y^{(2)}(t_3)) + f(t_1, y^{(2)}(t_1)) + f(t_2, y^{(2)}(t_2)) \right) = \frac{0.2}{2} \left(4 + (t_3 + \frac{t_3^2}{3})^2 + (t_1 + \frac{t_1^2}{3})^2 + (t_2 + \frac{t_2^2}{3})^2 \right) = 4.6701795 \times 10^{-1} \\
 Y^{(2)}(t_4) &= \frac{h}{2} \left(f(0, y^{(2)}(0)) + f(t_4, y^{(2)}(t_4)) + f(t_1, y^{(2)}(t_1)) + f(t_2, y^{(2)}(t_2)) + f(t_3, y^{(2)}(t_3)) \right) \\
 &= \frac{0.2}{2} \left(5 + (t_4 + \frac{t_4^2}{3})^2 + (t_1 + \frac{t_1^2}{3})^2 + (t_2 + \frac{t_2^2}{3})^2 + (t_3 + \frac{t_3^2}{3})^2 \right) = 6.61237333 \times 10^{-1} \\
 Y^{(2)}(t_2) &= \frac{h}{2} \left(f(0, y^{(2)}(0)) + f(t_2, y^{(2)}(t_2)) + f(t_1, y^{(2)}(t_1)) + f(t_2, y^{(2)}(t_2)) + f(t_3, y^{(2)}(t_3)) + f(t_4, y^{(2)}(t_4)) \right) \\
 &= \frac{0.2}{2} \left(6 + (t_2 + \frac{t_2^2}{3})^2 + (t_1 + \frac{t_1^2}{3})^2 + (t_2 + \frac{t_2^2}{3})^2 + (t_3 + \frac{t_3^2}{3})^2 + (t_4 + \frac{t_4^2}{3})^2 \right) = 9.3901511 \times 10^{-1} \\
 Y^{(4)}(t_0) &= \frac{h}{2} \left(f(0, y^{(2)}(0)) + f(t_0, y^{(2)}(t_0)) \right) = \frac{0.2}{2} \left(2 + (t_0 + \frac{t_0^2}{3} + \frac{2t_0^4}{15} + \frac{t_0^7}{63})^2 \right) = 2.0 \times 10^{-1} \\
 Y^{(4)}(t_1) &= \frac{h}{2} \left(f(0, y^{(2)}(0)) + f(t_1, y^{(2)}(t_1)) \right) = \frac{0.2}{2} \left(2 + (t_1 + \frac{t_1^2}{3} + \frac{2t_1^4}{15} + \frac{t_1^7}{63})^2 \right) = 2.20270953 \times 10^{-1} \\
 Y^{(4)}(t_2) &= \frac{h}{2} \left(f(0, y^{(2)}(0)) + f(t_2, y^{(2)}(t_2)) + f(t_1, y^{(2)}(t_1)) \right) = \frac{0.2}{2} \left(3 + (t_2 + \frac{t_2^2}{3} + \frac{2t_2^4}{15} + \frac{t_2^7}{63})^2 + (t_1 + \frac{t_1^2}{3} + \frac{2t_1^4}{15} + \frac{t_1^7}{63})^2 \right) = 3.6254342 \times 10^{-1} \\
 Y^{(4)}(t_3) &= \frac{h}{2} \left(f(0, y^{(2)}(0)) + f(t_3, y^{(2)}(t_3)) + f(t_1, y^{(2)}(t_1)) + f(t_2, y^{(2)}(t_2)) \right) \\
 &= \frac{0.2}{2} \left(4 + (t_3 + \frac{t_3^2}{3} + \frac{2t_3^4}{15} + \frac{t_3^7}{63})^2 + (t_1 + \frac{t_1^2}{3} + \frac{2t_1^4}{15} + \frac{t_1^7}{63})^2 + (t_2 + \frac{t_2^2}{3} + \frac{2t_2^4}{15} + \frac{t_2^7}{63})^2 \right) = 5.30824655 \times 10^{-1} \\
 Y^{(4)}(t_4) &= \frac{h}{2} \left(f(0, y^{(2)}(0)) + f(t_4, y^{(2)}(t_4)) + f(t_1, y^{(2)}(t_1)) + f(t_2, y^{(2)}(t_2)) + f(t_3, y^{(2)}(t_3)) \right) \\
 &= \frac{0.2}{2} \left(5 + (t_4 + \frac{t_4^2}{3} + \frac{2t_4^4}{15} + \frac{t_4^7}{63})^2 + (t_1 + \frac{t_1^2}{3} + \frac{2t_1^4}{15} + \frac{t_1^7}{63})^2 + (t_2 + \frac{t_2^2}{3} + \frac{2t_2^4}{15} + \frac{t_2^7}{63})^2 + (t_3 + \frac{t_3^2}{3} + \frac{2t_3^4}{15} + \frac{t_3^7}{63})^2 \right) = 7.32593269 \times 10^{-1} \\
 Y^{(4)}(t_2) &= \frac{h}{2} \left(f(0, y^{(2)}(0)) + f(t_2, y^{(2)}(t_2)) + f(t_1, y^{(2)}(t_1)) + f(t_2, y^{(2)}(t_2)) + f(t_3, y^{(2)}(t_3)) + f(t_4, y^{(2)}(t_4)) \right) \\
 &= \frac{0.2}{2} \left(6 + (t_2 + \frac{t_2^2}{3} + \frac{2t_2^4}{15} + \frac{t_2^7}{63})^2 + (t_1 + \frac{t_1^2}{3} + \frac{2t_1^4}{15} + \frac{t_1^7}{63})^2 + (t_2 + \frac{t_2^2}{3} + \frac{2t_2^4}{15} + \frac{t_2^7}{63})^2 + (t_3 + \frac{t_3^2}{3} + \frac{2t_3^4}{15} + \frac{t_3^7}{63})^2 + (t_4 + \frac{t_4^2}{3} + \frac{2t_4^4}{15} + \frac{t_4^7}{63})^2 \right) \\
 &= 9.80847238 \times 10^{-1}
 \end{aligned}$$

Table 2. Numerical result of (3.9) by Picard's method

$Y^{(n)} \setminus t$	0.0	0.2	0.4	0.6	0.8	1.0
1	2.0×10^{-1}	2.0×10^{-1}	3.0×10^{-1}	4.0×10^{-1}	5.0×10^{-1}	6.0×10^{-1}
2	2.0×10^{-1}	2.04×10^{-1}	3.2×10^{-1}	4.56×10^{-1}	6.2×10^{-1}	8.2×10^{-1}
3	2.0×10^{-1}	2.0411×10^{-1}	3.2186×10^{-1}	4.6702×10^{-1}	6.6124×10^{-1}	9.3902×10^{-1}
4	2.0×10^{-1}	2.2027×10^{-1}	3.6254×10^{-1}	5.3082×10^{-1}	7.3259×10^{-1}	9.8085×10^{-1}

Table 3. Numerical Comparison of Accuracy of ADM and PM

T	Exact	ADM	PM	E_n^{ADM}	E_n^{PM}
0.0	0.0	0.0	2.0×10^{-1}	0.0	2.0×10^{-1}
0.2	3.4907×10^{-1}	4.1092×10^{-1}	2.2027×10^{-1}	6.1850×10^{-4}	2.1678×10^{-1}
0.4	6.9814×10^{-1}	2.6093×10^{-1}	3.6254×10^{-1}	1.9111×10^{-4}	3.5556×10^{-1}
0.6	1.0472×10^{-1}	9.0700×10^{-2}	5.3082×10^{-1}	8.9653×10^{-4}	5.2035×10^{-1}
0.8	1.3964×10^{-1}	2.4190×10^{-1}	7.3259×10^{-1}	2.2793×10^{-1}	7.1863×10^{-1}
1.0	1.7455×10^{-1}	5.7053×10^{-1}	9.8085×10^{-1}	5.5308×10^{-1}	9.6339×10^{-1}

Rate of Convergence of ADM

Example 4

Reviewing (3.9), we noted that:

By the Adomian decomposition method, we have:

$$\begin{cases} y_1 = t \\ y_2 = \frac{t^2}{3} \\ y_3 = \frac{2t^5}{15} \\ y_4 = \frac{17t^7}{315} \end{cases} \tag{3.22}$$

By the Picard’s method, we have:

$$\begin{cases} y^{(1)} = t \\ y^{(2)} = t + \frac{t^2}{3} \\ y^{(3)} = t + \frac{t^2}{3} + \frac{2t^5}{15} + \frac{t^7}{63} \\ y^{(4)} = t + \frac{t^2}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \frac{62t^9}{2835} + \frac{109t^{11}}{51975} \end{cases} \tag{3.23}$$

Now setting, t= 0.1, it follows from (3.22) that:

$$\begin{aligned} \|y_1\| &= 1.0 \times 10^{-1} \\ \|y_2\| &= 3.333333333 \times 10^{-4} \\ \|y_3\| &= 1.333333333 \times 10^{-8} \\ \|y_4\| &= 5.396825397 \times 10^{-9} \end{aligned}$$

Similarly from (3.23), we have:

$$\begin{aligned} \|y^{(1)}\| &= 1.0 \times 10^{-1} \\ \|y^{(2)}\| &= 1.003333333 \times 10^{-1} \\ \|y^{(3)}\| &= 1.00334668 \times 10^{-1} \\ \|y^{(4)}\| &= 1.00334672 \times 10^{-1} \end{aligned}$$

Now from (2.39) we have:

$$\begin{aligned} \alpha_1 &= \frac{\|y_2\|}{\|y_1\|} = 3.333333333 \times 10^{-3} \\ \alpha_2 &= \frac{\|y_3\|}{\|y_2\|} = 3.999999999 \times 10^{-3} \\ \alpha_3 &= \frac{\|y_4\|}{\|y_3\|} = 4.047619049 \times 10^{-3} \end{aligned}$$

Similarly from (2.42), we have:

$$\begin{aligned} \bar{\alpha}_1 &= \frac{\|y^{(2)}\|}{\|y^{(1)}\|} = 9.96677744 \times 10^{-1} \\ \bar{\alpha}_2 &= \frac{\|y^{(3)}\|}{\|y^{(2)}\|} = 9.99986694 \times 10^{-1} \\ \bar{\alpha}_3 &= \frac{\|y^{(4)}\|}{\|y^{(3)}\|} = 9.9999996 \times 10^{-1} \end{aligned}$$

Table 4. Numerical comparison of rate of convergence of ADM and PM

y_n	ADM	Picard method	α_n	$\bar{\alpha}_n$
1	1.0×10^{-1}	1.0×10^{-1}		
			3.3333×10^{-3}	9.9668×10^{-1}
2	3.3333×10^{-4}	1.0033×10^{-1}		
			3.9999×10^{-3}	9.9998×10^{-1}
3	1.3333×10^{-8}	1.0033×10^{-1}		
			4.0476×10^{-3}	9.9999×10^{-1}
4	5.3968×10^{-9}	1.0033×10^{-1}		

Conclusion

The ADM has been successfully applied to finding the solutions of nonlinear ODE. The obtained results are compared with those of Picard iterations method. It is noted from the analytical results of the methods that the Picard’s method mixes up powers of the partial sum for the exact solutions, while the Adomian series is, in the other hand equivalent to the power series in time and the Adomian method requires analyticity of the function, $f(t, y)$, which is more restrictive than the Lipschitz condition required for the Picard method. It is also noted from the numerical results that the ADM presents more accurate results than the Picard’s method.

In a closely related outcome, the ADM has faster rate of convergence than the Picard’s method. Conclusively, the ADM is a powerful mathematical tool for solving nonlinear ordinary differential equations, and therefore can be widely applied in the field of science and engineering.

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