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π-Closure Local Functions in Ideal Topological Spaces

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ABSTRACT

In this paper we formulate a new local function called π-closure local function and we construct π-closure compatible spaces using π-open sets. Further we introduce an operator $\Psi_{\Upsilon}(A)$ for each A $\in$ P(X) by utilizing Υ(A). Moreover we characterize the properties of π-closure local function and investigate their relationship with other types of similar functions.

Introduction

Ideal topology is a topological space endowed with an additional structure namely the ideal. Kuratowski [11] introduced the concept of local functions in ideal topological spaces. The notion of Kuratowski operator plays a vital role in defining ideal topological space which has its application in localization theory in set topology by Vaidyanathaswamy [15]. Ideals have been frequently used in the fields closely related to topology such as real analysis measure theory and lattice theory. In 1990, Jankovic and Hamlett [7, 8] developed new topologies from old via ideals and introduced I-open sets with respect to an ideal I in 1992. Compatibility of the topology τ with an ideal I was first defined by Njastad [13] in 1996. In this paper we define π-closure local function and its properties in ideal topological spaces. Moreover the relationships with other local functions are investigated.

Preliminaries

Throughout this paper (X, τ) is a topological space on which no separation axioms are assumed unless explicitly stated. The notation $(X, \tau, \mathcal{J})$ will denote the topological space (X, τ) and an ideal $\mathcal{J}$ on X with no separation properties assumed. For A $\subseteq$ (X, τ), Cl(A) and Int(A) respectively denote the closure and interior of A with respect to τ. N(x) denotes the open neighbourhood system at a point $x \in$ X and P(X) denotes the power set of X.

Definition. 2.1[11]

An ideal $\mathcal{J}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies the following properties:

(1) $A \in \mathcal{J}$ and $B \subseteq A$ implies $B \in \mathcal{J}$.

(2) $A \in \mathcal{J}$ and $B \in \mathcal{J}$ implies $A \cup B \in \mathcal{J}$.

An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{J}$ on X and is denoted by $(X, \tau, \mathcal{J})$.

Definition. 2.2[11]

For a subset A of X, $A^*(\mathcal{J}) = \{x \in X: U \cap A \notin \mathcal{J}$ for each neighbourhood U of $x\}$ is called the local function of A with respect to $\mathcal{J}$ and τ. We simply write A^* instead of $A^*(\mathcal{J})$.

Definition. 2.2[11]

It is well known that $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(\mathcal{J})$ which finar than τ.

Definition. 2.3[11]

A basis $\beta(\mathcal{J}, \tau)$ for $\tau^*(\mathcal{J})$ can be described as follows: $\beta(\mathcal{J}, \tau) = \{U - E: U \in \tau$ and $E \in \mathcal{J}\}$.

Definition. 2.4

A subset A of an ideal topological space $(X, \tau, \mathcal{J})$ is

(1) $*$-perfect [6], if $A = A^*$

(2) $*$- closed [7], if $A^* \subseteq A$

(3) *-dense [9], if $Cl^*(A) = X$

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(4) τ^*-closed set [7], if $A = Cl^*(A)$

Definition. 2.5[17]

A subset A of a space (X, τ) is said to be regular open set, if $A = int(cl(A))$.

Definition. 2.6[15]

Finite union of regular open sets in (X, τ) is π-open in (X, τ).The complement of π -open in (X, τ) is π -cloπsed in (X, τ).

Definition. 2.7[10]

Let $(X, \tau, \mathcal{J})$ be an ideal topological space and A be a subset of X. Then $A^{*S}(\mathcal{J}, \tau) = \{x \in X \mid A \cap U \notin \mathcal{J}$ for every $U \in SO(X, x)\}$ is called the semi local function of A with respect to $\mathcal{J}$ and τ, where $SO(X, x) = \{U \in SO(X) \mid x \in U\}$.

Definition. 2.8[1]

Let $(X, \tau, \mathcal{J})$ be an ideal topological space. For a subset A of X, we define the following operator:

$\Gamma(A)(\mathcal{J}, \tau) = \{x \in X \mid A \cap cl(U) \notin \mathcal{J}$ for every $U \in \tau(x)\}$ is called the local closure function of A with respect to $\mathcal{J}$ and τ, where $\tau(x) = \{U \in \tau : x \in U\}$.

Definition. 2.9[3]

Given a space $(X, \tau, \mathcal{J})$, a set operator $()^{*p}: P(X) \to P(X)$ is called the pre-local function of $\mathcal{J}$ with respect to τ is defined as follows; for $A \subseteq X$, $(A)^{*p}(\mathcal{J}, \tau) = \{x \in X / U_x \cap A \notin \mathcal{J}$, for every $U_x \in PN(x)\}$, where $PN(x)\} = \{U \in PO(x) \mid x \in U\}$.

Definition. 2.10[2]

Given a space $(X, \tau, \mathcal{J})$, a set operator $()^{*\pi}: P(X) \to P(X)$ is called the π-local function of $\mathcal{J}$ with respect to τ is defined as follows; for $A \subseteq X$, $(A)^{*\pi}(\mathcal{J}, \tau) = \{x \in X / U_x \cap A \notin \mathcal{J}$, for every $U_x \in \pi N(x)\}$, where $\pi N(x)\} = \{U \in \pi O(x) \mid x \in U\}$.

Lemma. 2.10[7]

Let $(X, \tau, \mathcal{J})$ be an ideal space and A, B subsets of X.

(1) If $A \subset B$, then $A^* \subset B^*$.

(2) If $G \in \tau$, then $G \cap A^* \subset (G \cap A)^*$

(3) $A^* = Cl(A^*) \subset Cl(A)$

π-Closure Local Functions

Definition. 3.1

Let $(X, \tau, \mathcal{J})$ be an ideal topological space. For a subset A of X, we define the following operator: $\Upsilon(A)(\mathcal{J}, \tau) = \{x \in X : A \cap \pi cl(U) \notin \mathcal{J}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. In case there is no confusion $\Upsilon(A)(\mathcal{J}, \tau)$ is briefly denoted by $\Upsilon(A)$ and is called π-closure local function of A with respect to $\mathcal{J}$ and τ .

Preposition. 3.2

Let $(X, \tau, \mathcal{J})$ be an ideal topological space. Then

1. Every local function is π-closure local function.
2. Every π-local function is π-closure local function.
3. Every semi local function is π-closure local function.
4. Every pre local function is π-closure local function.

Proof

obvious

Remark. 3.3

The reverse implications need not be true as shown in the following examples.

Example. 3.4

$X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$ and $\mathcal{J} = \{\phi, \{a\}\}$. Take $A = \{c\}$. Then $A^* = \{c\}$ and $\Upsilon(A) = \{a, b, c, d\}$. Hence $\Upsilon(A) \not\subseteq A^*$.

Example. 3.5

$X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b, d\}\}$ and $\mathcal{J} = \{\phi, \{a\}\}$. Take $A = \{c\}$. Then $A^{*\pi} = \{c, d\}$ and $\Upsilon(A) = \{a, b, c, d\}$. Hence $\Upsilon(A) \not\subseteq A^{*\pi}$.

Example. 3.6

$X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{J} = \{\phi, \{a\}\}$. Take $A = \{b\}$. Then $A^{*s} = \{b\}$ and $\Upsilon(A) = \{b, c\}$. Hence $\Upsilon(A) \not\subseteq A^{*s}$.

Example. 3.7

$X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{d\}, \{a, c\}, \{a, c, d\}\}$ and $\mathcal{J} = \{\phi, \{c\}, \{d\}, \{c, d\}\}$. Take $A = \{a, b\}$. Then $A^{*p} = \{a, b\}$ and $\Upsilon(A) = \{a, b, c, d\}$. Hence $\Upsilon(A) \not\subseteq A^{*p}$.

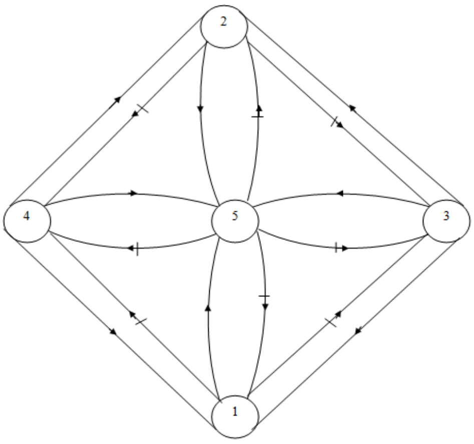
The following diagram represents the above results:

(1). Local function

(2). π-local function

(3). semi local function

(4). pre local function
(5). π-closure local function



Theorem. 3.8

Let (X, τ) be a topological space, $\mathcal{J}$ and $\mathscr{J}$ be two ideals on X, and let A and B be subsets of X. Then the following properties hold:

1. If $A \subseteq B$, then $\Upsilon(A) \subseteq \Upsilon(B)$.
2. If $\mathcal{J} \subseteq \mathscr{J}$, then $\Upsilon(A)(\mathcal{J}) \supseteq \Upsilon(B)(\mathscr{J})$.
3. $\Upsilon(A) = \pi cl(\Upsilon(A)) \subseteq \pi cl(A)$ and $\Upsilon(A)$ is π-closed.
4. If $A \in \mathcal{J}$, then $\Upsilon(A) = \emptyset$.

Proof

(1) Suppose that $x \notin \Upsilon(B)$. Then there exists $U \in \tau(x)$ such that $B \cap \pi cl(U) \in \mathcal{J}$. Since $A \cap \pi cl(U) \subseteq B \cap \pi cl(U)$, $A \cap \pi cl(U) \in \mathcal{J}$. Hence $x \notin \Upsilon(A)$. Thus $X - \Upsilon(B) \subseteq X - \Upsilon(A)$ or $\Upsilon(A) \subseteq \Upsilon(B)$.

(2) Suppose that $x \notin \Upsilon(A)(\mathcal{J})$. There exists $U \in \tau(x)$ such that $A \cap \pi cl(U) \in \mathcal{J}$. Since $\mathcal{J} \subseteq \mathscr{J}$, $A \cap \pi cl(U) \in \mathscr{J}$ and $x \notin \Upsilon(A)(\mathscr{J})$. Therefore, $\Upsilon(A)(\mathscr{J}) \subseteq \Upsilon(A)(\mathcal{J})$ or $\Upsilon(A)(\mathcal{J}) \supseteq \Upsilon(B)(\mathscr{J})$.

(3) We have $\Upsilon(A) \subseteq \pi cl(\Upsilon(A))$ in general. Let $x \in \pi cl(\Upsilon(A))$. Then $\Upsilon(A) \cap U \neq \emptyset$ for every $U \in \tau(x)$. Therefore there exists some $y \in \Upsilon(A) \cap U$ and $U \in \tau(y)$. Since $y \in \Upsilon(A)$, $A \cap \pi cl(U) \notin \mathcal{J}$ and hence $x \in \Upsilon(A)$. Hence we have $\pi cl(\Upsilon(A)) \subseteq \Upsilon(A)$. Therefore $\Upsilon(A) = \pi cl(\Upsilon(A))$. Again let $x \in \pi cl(\Upsilon(A)) = \Upsilon(A)$, then $A \cap \pi cl(U) \notin \mathcal{J}$ for every $U \in \tau(x)$. This implies $A \cap \pi cl(U) \neq \emptyset$ for every $U \in \tau(x)$. This shows that $\Upsilon(A) = \pi cl(\Upsilon(A)) \subseteq \pi cl(A)$ and $\Upsilon(A)$ is π-closed.

(4) Suppose that $x \in \Upsilon(A)$. Then for any $U \in \tau(x)$, $A \cap \pi cl(U) \notin \mathcal{J}$. But since $A \in \mathcal{J}$, $A \cap \pi cl(U) \in \mathcal{J}$ for every $U \in \tau(x)$. This is a contradiction. Hence $\Upsilon(A) = \emptyset$.

Lemma. 3.9

Let $(X, \tau, \mathcal{J})$ be an ideal topological space. If $U \in \tau(x)$, then $U \cap \Upsilon(A) = U \cap \Upsilon(U \cap A) \subseteq \Upsilon(U \cap A)$ for any subset A of X.

Proof

Suppose that $U \in \tau(x)$ and $x \in U \cap \Upsilon(A)$. Then $x \in U$ and $x \in \Upsilon(A)$. Let V be any open set containing x. Then $V \cap U \in \tau(x)$ and $\pi cl(V \cap U) \cap A \notin \mathcal{J}$ and hence $\pi cl(V) \cap (U \cap A) \notin \mathcal{J}$. This shows that $x \in \Upsilon(U \cap A)$. Hence we obtain $U \cap \Upsilon(A) \subseteq \Upsilon(U \cap A)$. Moreover $U \cap \Upsilon(A) \subseteq U \cap \Upsilon(U \cap A)$ and by Theorem 3.8 $\Upsilon(U \cap A) \subseteq \Upsilon(A)$ and $U \cap \Upsilon(U \cap A) \subseteq U \cap \Upsilon(A)$. Therefore $U \cap \Upsilon(A) = U \cap \Upsilon(U \cap A)$.

Theorem. 3.10

Let $(X, \tau, \mathcal{J})$ be an ideal topological space and A, B any subsets of X. Then the following properties hold:

1. $\Upsilon(\emptyset) = \emptyset$.
2. $\Upsilon(A) \cup \Upsilon(B) = \Upsilon(A \cup B)$.

Proof

(1) Obvious.

(2) It follows from Theorem 3.8 that $\Upsilon(A \cup B) \supseteq \Upsilon(A) \cup \Upsilon(B)$. To prove the reverse inclusion, let $x \notin \Upsilon(A) \cup \Upsilon(B)$. Then x belongs neither to $\Upsilon(A)$ nor to $\Upsilon(B)$. Therefore there exist $U_x, V_x \in \tau(x)$ such that $\pi cl(U_x) \cap A \in \mathcal{J}$ and $\pi cl(V_x) \cap B \in \mathcal{J}$. Since $\mathcal{J}$ is additive, $(\pi cl(U_x) \cap A) \cup (\pi cl(V_x) \cap B) \in \mathcal{J}$. Moreover, since $\mathcal{J}$ is hereditary and $(\pi cl(U_x) \cap A) \bigcup (\pi cl(V_x) \cap B) = [(\pi cl(U_x) \cap$

A) $\bigcup$ $\pi cl(V_x)] \cap [(\pi cl(U_x) \cap A) \bigcup B] = (\pi cl(U_x) \bigcup \pi cl(V_x)) \cap (A \bigcup \pi cl(V_x)) \cap (\pi cl(U_x) \bigcup B) \cap (A \cup B) \supseteq \pi cl(U_x \cap V_x) \cap (A \bigcup B)$. Then $\pi cl(U_x \cap V_x) \cap (A \bigcup B) \in \mathcal{J}$. Since $U_x \cap V_x \in \tau(x)$, $x \notin \Upsilon(A \bigcup B)$. Hence $(X - \Upsilon(A)) \cap (X - \Upsilon(B) \subseteq X - \Upsilon(A \bigcup B)$ or $\Upsilon(A \bigcup B) \subseteq \Upsilon(A) \bigcup \Upsilon(B)$. Hence we obtain $\Upsilon(A) \bigcup \Upsilon(B) = \Upsilon(A \bigcup B)$.

Lemma. 3.11

Let $(X, \tau, \mathcal{J})$ be an ideal topological space and A,B be subsets of X. Then $\Upsilon(A) - \Upsilon(B) = \Upsilon(A - B) - \Upsilon(B)$.

Proof

We have by Theorem 3.10 $\Upsilon(A) = \Upsilon[(A - B) \bigcup (A \cap B)] = \Upsilon(A - B) \bigcup \Upsilon(A \cap B) \subseteq \Upsilon(A - B) \bigcup \Upsilon(B)$. Thus $\Upsilon(A) - \Upsilon(B) \subseteq \Upsilon(A - B) - \Upsilon(B)$. By Theorem 3.8, $\Upsilon(A - B) \subseteq \Upsilon(A)$ and hence $\Upsilon(A - B) - \Upsilon(B) \subseteq \Upsilon(A) - \Upsilon(B)$. Hence $\Upsilon(A) - \Upsilon(B) = \Upsilon(A - B) - \Upsilon(B)$.

Corollary. 3.12

Let $(X, \tau, \mathcal{J})$ be an ideal topological space and A, B be subsets of X with $B \in \mathcal{J}$. Then $\Upsilon(A \bigcup B) = \Upsilon(A) = \Upsilon(A - B)$.

Proof

Since $B \in \mathcal{J}$, by Theorem 3.8, $\Upsilon(B) = \emptyset$. By Lemma 3.11,$\Upsilon(A) = \Upsilon(A - B)$ and by Theorem 3.10 $\Upsilon(A \cup B) = \Upsilon(A) \cup \Upsilon(B) = \Upsilon(A)$

π-Closure Compatibility of Topological Spaces

Definition. 4.1

Let $(X, \tau, \mathcal{J})$ be an ideal topological space. We say the τ is π-closure compatible with the ideal $\mathcal{J}$, denoted $\tau \sim^{\Upsilon} \mathcal{J}$ if the following holds: for every $A \subseteq X$ if for every $x \in A$ there exists $U \in \tau(x)$ such that $\pi cl(U) \cap A \in \mathcal{J}$, then $A \in \mathcal{J}$.

Remark. 4.2

1. If τ is compatible with $\mathcal{J}$, then τ is π-closure compatible with $\mathcal{J}$.
2. If τ is closure compatible with $\mathcal{J}$, then τ is π-closure compatible with $\mathcal{J}$.

Theorem. 4.3

Let $(X, \tau, \mathcal{J})$ be an ideal topological space, the following properties are equivalent:

1. $\tau \sim^{\Upsilon} \mathcal{J}$
2. If a subset A of X has a cover of open sets each of whose π-closure intersection with A is in $\mathcal{J}$, then $A \in \mathcal{J}$
3. For every $A \subseteq X$, $A \cap \Upsilon(A) = \emptyset$ implies that $A \in \mathcal{J}$
4. For every $A \subseteq X$, $A - \Upsilon(A) \in \mathcal{J}$
5. For every $A \subseteq X$, if A contains no nonempty subset B with $B \subseteq \Upsilon(B)$, then $A \in \mathcal{J}$.

Proof

(1) ⇒ (2): obvious.

(2) ⇒ (3)

Let $A \subseteq X$ and $x \in A$. Then $x \notin \Upsilon(A)$ and there exists $V_x \in \tau(x)$ such that $\pi cl(V_x) \cap A \in \mathcal{J}$. Therefore, we have $A \subseteq \bigcup\{V_x : x \in A\}$ and $V_x \in \tau(x)$ and by (2) $A \in \mathcal{J}$.

(3) ⇒ (4)

For any $A \subseteq X$, $A - \Upsilon(A) \subseteq A$ and $(A - \Upsilon(A)) \cap \Upsilon(A - \Upsilon(A)) \subseteq (A - \Upsilon(A)) \cap \Upsilon(A) = \emptyset$.

By (3) $A - \Upsilon(A) \in \mathcal{J}$.

(4) ⇒ (5)

By (4) for every $A \subseteq X$, $A - \Upsilon(A) \in \mathcal{J}$. Let $A = A - \Upsilon(A) \cup (A \cap \Upsilon(A))$ and by Theorem 3.10 (2) and Theorem 3.8(5), $\Upsilon(A) = \Upsilon(A - \Upsilon(A)) \bigcup \Upsilon(A \cap \Upsilon(A)) = \Upsilon(A \cap \Upsilon(A))$. Therefore, we have $A \cap \Upsilon(A) = A \cap \Upsilon(A \cap \Upsilon(A)) \subseteq \Upsilon(A \cap \Upsilon(A))$ and $A \cap \Upsilon(A) \subseteq A$. By the assumption $A \cap \Upsilon(A) = \emptyset$. Hence $A = A - \Upsilon(A) \in \mathcal{J}$.

(5) ⇒ (1)

Let $A \subseteq X$ and assume that for every $x \in A$, there exists $U \in \tau(x)$ such that $\pi cl(U) \cap A \in \mathcal{J}$. Then $A \cap \Upsilon(A) = \emptyset$. Suppose that A contains B such that $B \subseteq \Upsilon(B)$. Then $B = B \cap \Upsilon(B) \subseteq A \cap \Upsilon(A) = \emptyset$. Therefore, A contains no nonempty subset B with $B \subseteq \Upsilon(B)$. Hence $A \in \mathcal{J}$.

Theorem 4.4

Let $(X, \tau, \mathcal{J})$ be an ideal topological space. If τ is π-closure compatible with $\mathcal{J}$, then the following equivalent properties hold:

1. For every $A \subseteq X$, $A \cap \Upsilon(A) = \emptyset$ implies that $\Upsilon(A) = \emptyset$
2. For every $A \subseteq X$, $\Upsilon(A - \Upsilon(A)) = \emptyset$
3. For every $A \subseteq X$, $\Upsilon(A \cap \Upsilon(A)) = \Upsilon(A)$

Proof

(1) ⇒ (2)

Assume that for every $A \subseteq X$, $A \cap \Upsilon(A) = \emptyset$ implies that $\Upsilon(A) = \emptyset$. Let $B = A - \Upsilon(A)$, then $B \cap \Upsilon(B) = (A - \Upsilon(A)) \cap \Upsilon(A - \Upsilon(A)) = (A \cap (X - \Upsilon(A))) \cap \Upsilon(A \cap (X - \Upsilon(A))) \subseteq [A \cap (X - \Upsilon(A))] \cap [\Upsilon(A) \cap (\Upsilon(X - \Upsilon(A)))] = \emptyset$. By (1) we have $\Upsilon(B) = \emptyset$. Hence $\Upsilon(A - \Upsilon(A)) = \emptyset$.

(2) ⇒ (3)

Assume for every $A \subseteq X$, $\Upsilon(A - \Upsilon(A)) = \emptyset$. $A = (A - \Upsilon(A)) \cup (A \cap \Upsilon(A))$. $\Upsilon(A) = \Upsilon[(A - \Upsilon(A)) \bigcup (A \cap \Upsilon(A))] = \Upsilon(A - \Upsilon(A)) \bigcup \Upsilon(A \cap \Upsilon(A)) = \Upsilon(A \cap \Upsilon(A))$.

(3) ⇒ (1)

Assume for every $A \subseteq X$, $A \cap \Upsilon(A) = \emptyset$ and $\Upsilon(A \cap \Upsilon(A)) = \Upsilon(A)$. This implies that $\emptyset = \Upsilon(\emptyset) = \Upsilon(A)$.

Theorem. 4.5

Let $(X, \tau, \mathcal{J})$ be an ideal topological space, then the following properties are equivalent:

1. $\pi cl(\tau) \cap \mathcal{J} = \emptyset$ where $\pi cl(\tau) = \{ \pi cl(V): V \in \tau\}$
2. If $I \in \mathcal{J}$, then $\pi int(I) = \emptyset$
3. For every $G \in \tau$, $G \subseteq \Upsilon(G)$
4. $X = \Upsilon(X)$

Proof

(1) ⇒ (2)

Let $\pi cl(\tau) \cap \mathcal{J} = \emptyset$ and $I \in \mathcal{J}$. Suppose that $x \in \pi\text{-}int(I)$. Then there exists $U \in \tau$ such that $x \in U \subseteq \pi cl(U) \subseteq I$. Since $I \in \mathcal{J}$ and hence $\emptyset \neq \{x\} \subseteq \pi cl(U) \in \pi cl(\tau) \cap I$. This is contrary to $\pi cl(\tau) \cap \mathcal{J} = \emptyset$. Therefore, $\pi int(I) = \emptyset$.

(2)⇒ (3)

Let $x \in G$. Assume $x \notin \Upsilon(G)$, then there exists $U_x \in \tau(x)$ such that $G \cap \pi cl(U_x) \in \mathcal{J}$ and hence $G \cap U_x \in \mathcal{J}$. By (2) $x \in G \cap U_x = \pi int(G \cap U_x) = \emptyset$. Hence $x \in \Upsilon(G)$ and $G \subseteq \Upsilon(G)$.

(3)⇒ (4)

Since X is clopen, then $X = \Upsilon(X)$.

(4)⇒ (1)

$X = \Upsilon(X) = \{x \in X: \pi cl(U) \cap X = \pi cl(U) \notin \mathcal{J}$ for each open set U containing x$\}$. Hence $\pi cl(\tau) \cap \mathcal{J} = \emptyset$.

Theorem. 4.6

Let $(X, \tau, \mathcal{J})$ be an ideal topological space, τ be π-closure compatible with $\mathcal{J}$. Then for every $G \in \tau$ and any subset A of X, $\pi cl(\Upsilon(G \cap A)) = \Upsilon(G \cap A) \subseteq \Upsilon(G \cap \Upsilon(A)) \subseteq \pi cl(G \cap \Upsilon(A))$.

Proof

By Theorem 4.4(3) and Theorem 3.8, we have $\Upsilon(G \cap A) = \Upsilon((G \cap A) \cap \Upsilon(G \cap A)) \subseteq \Upsilon(G \cap \Upsilon(A))$. Moreover by Theorem 3.8, $\pi cl(\Upsilon(G \cap A)) = \Upsilon(G \cap A) \subseteq \Upsilon(G \cap \Upsilon(A)) \subseteq \pi cl(G \cap \Upsilon(A))$.

Ψ_Υ-operator

Definition. 5.1

Let $(X, \tau, \mathcal{J})$ be an ideal topological space. An operator $\Psi_\Upsilon: P(X) \to \tau$ is defined as follows: for every $A \in X$, $\Psi_\Upsilon(A) = \{x \in X :$ there exists $U \in \tau(x)$ such that $\pi cl(U) - A \in \mathcal{J}\}$ and observe that $\Psi_\Upsilon(A) = X - \Upsilon(X - A)$.

Theorem. 5.2

Let $(X, \tau, \mathcal{J})$ be an ideal topological space. Then the following properties hold:

1. If $A \subseteq X$, then $\Psi_\Upsilon(A)$ is π-open.
2. If $A \subseteq B$, then $\Psi_\Upsilon(A) \subseteq \Psi_\Upsilon(B)$.
3. If $A, B \in P(X)$, then $\Psi_\Upsilon(A \cap B) = \Psi_\Upsilon(A) \cap \Psi_\Upsilon(B)$.
4. If $A \subseteq X$, then $\Psi_\Upsilon(A) = \Psi_\Upsilon(\Psi_\Upsilon(A))$ if and only if $\Upsilon(X - A) = \Upsilon(\Upsilon(X - A))$.
5. If $A \in \mathcal{J}$, then $\Psi_\Upsilon(A) = X - \Upsilon(X)$.
6. If $A \subseteq X$, $I \in \mathcal{J}$, then $\Psi_\Upsilon(A - I) = \Psi_\Upsilon(A)$.
7. If $A \subseteq X$, $I \in \mathcal{J}$, then $\Psi_\Upsilon(A \cup I) = \Psi_\Upsilon(A)$.
8. If $(A - B) \cup (B - A) \in \mathcal{J}$, then $\Psi_\Upsilon(A) = \Psi_\Upsilon(B)$.

Proof

(1) This follows from Theorem 3.8 (3).

(2) This follows from Theorem 3.8 (1).

(3) $\Psi_\Upsilon(A \cap B) = X - \Upsilon(X - (A \cap B)) = X - \Upsilon[(X - A) \cup (X - B)] = X - [\Upsilon(X - A) \cup \Upsilon(X - B)] = [X - \Upsilon(X - A)] \cap [X - \Upsilon(X - B)] = \Psi_\Upsilon(A) \cap \Psi_\Upsilon(B)$.

(4) This follows from the facts:

1. $\Psi_\Upsilon(A) = X - \Upsilon(X - A)$.
2. $\Psi_\Upsilon(\Psi_\Upsilon(A)) = X - \Upsilon[X - (X - \Upsilon(X - A))] = X - \Upsilon(\Upsilon(X - A))$.

(5) By Corollary 3.12 we obtain that $\Upsilon(X - A) = \Upsilon(X)$ if $A \in \mathcal{J}$. Then $X - \Upsilon(X - A) = X - \Upsilon(X)$. Hence $\Psi_\Upsilon(A) = X - \Upsilon(X)$.

(6) This follows from Corollary 3.12 and $\Psi_\Upsilon(A - I) = X - \Upsilon[X - (A - I)] = X - \Upsilon[(X - A) \cup I] = X - \Upsilon(X - A) = \Psi_\Upsilon(A)$.

(7) This follows from Corollary 3.12 and $\Psi_\Upsilon(A \cup I) = X - \Upsilon[X - (A \cup I)] = X - \Upsilon[(X - A) - I] = X - \Upsilon(X - A) = \Psi_\Upsilon(A)$.

(8) Assume $(A - B) \cup (B - A) \in \mathcal{J}$. Let $A - B = I$ and $B - A = J$. Observe that $I, J \in \mathcal{J}$ by heredity. Also observe that $B = (A - I) \cup J$. Thus $\Psi_\Upsilon(A) = \Psi_\Upsilon(A - I) = \Psi_\Upsilon[(A - I) \cup J] = \Psi_\Upsilon(B)$ by (6) and (7).

Corollary. 5.3

Let $(X, \tau, \mathcal{J})$ be an ideal topological space. Then $U \subseteq \Psi_\Upsilon(U)$ for every π-open set $U \subseteq X$.

Proof

We know that $\Psi_\Upsilon(U) = X - \Upsilon(X-U)$. Now $\Upsilon(X-U) \subseteq \pi cl(X-U) = X-U$, since $X-U$ is π-closed. Therefore $U = X - (X - U) \subseteq X - \Upsilon(X-U) = \Psi_\Upsilon(U)$.

Now we give an example of a set A which is not π-open but satisfies $A \subseteq \Psi_\Upsilon(A)$.

Example. 5.4

$X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, c\}, \{d\}, \{a, c, d\}\}$ and $\mathcal{J} = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Let $A = \{a\}$.

Then $\Psi_\Upsilon(\{a\}) = X - \Upsilon(X - \{a\}) = X - \Upsilon(\{b, c, d\}) = X - \{b, d\} = \{a, c\}$.Therefore $A \subseteq \Psi_\Upsilon(A)$, but A is not π-open.

Theorem. 5.5

Let $(X, \tau, \mathcal{J})$ be an ideal topological space and $A \subseteq X$. Then the following properties hold:

1. $\Psi_\Upsilon(A) = \cup\{U \in \tau : \pi cl(U) - A \in \mathcal{J}\}$.
2. $\Psi_\Upsilon(A) \supseteq \cup\{U \in \tau : (\pi cl(U) - A) \cup (A - \pi cl(U)) \in \mathcal{J}\}$.

Proof

(1) This follows immediately from the definition of Ψ_Υ-operator.

(2) Since $\mathcal{J}$ is heredity, it is obvious that $\cup\{U \in \tau: (\pi cl(U)-A)\cup(A-\pi cl(U)) \in \mathcal{J}\} \subseteq \cup\{U \in \tau : \pi cl(U) - A \in \mathcal{J}\} = \Psi_\Upsilon(A)$ for every $A \subseteq X$.

Theorem. 5.6

Let $(X, \tau, \mathcal{J})$ be an ideal topological space. If $\sigma = \{A \subseteq X : A \subseteq \Psi_\Upsilon(A)\}$. Then σ is a topology for X.

Proof

Let $\sigma = \{A \subseteq X : A \subseteq \Psi_\Upsilon(A)\}$. Since $\phi \in \mathcal{J}$, by Theorem 3.8(4) $\Upsilon(\phi) = \phi$ and $\Psi_\Upsilon(X) = X - \Upsilon(X - X) = X - \Upsilon(\phi) = X$. Moreover, $\Psi_\Upsilon(\phi) = X - \Upsilon(X - \phi) = X - X = \phi$. Therefore we obtain that $\phi \subseteq \Psi_\Upsilon(\phi)$ and $X \subseteq \Psi_\Upsilon(X) = X$, and thus ϕ and $X \in \sigma$. Now if $A, B \in \sigma$, then by Theorem 5.2 $A \cap B \subseteq \Psi_\Upsilon(A) \cap \Psi_\Upsilon(B) = \Psi_\Upsilon(A \cap B)$ which implies that $A \cap B \in \sigma$. If $\{A_\alpha : \alpha \in \Delta\} \subseteq \sigma$, then $A_\alpha \subseteq \Psi_\Upsilon(A_\alpha) \subseteq \Psi_\Upsilon(\bigcup A_\alpha)$ for every α and hence $\bigcup A_\alpha \subseteq \Psi_\Upsilon((\bigcup A_\alpha)$. This shows that σ is a topology.

Theorem. 5.7

Let $\sigma_0 = \{A \subseteq X : A \subseteq Int(Cl(\Psi_\Upsilon(A)))\}$ then σ_0 is a topology for X.

Proof

By Theorem 5.2, for any subset A of X, $\Psi_\Upsilon(A)$ is π-open and $\sigma \subset \sigma_0$. Therefore $\emptyset, X \in \sigma_0$. Let $A, B \in \sigma_0$. Then by Theorem 5.2, we have $A \cap B \subset Int(Cl(\Psi_\Upsilon(A))) \cap Int(Cl(\Psi_\Upsilon(B))) = Int(Cl(\Psi_\Upsilon(A) \cap \Psi_\Upsilon(B))) = Int(Cl(\Psi_\Upsilon(A \cap B)))$. Therefore, $A \cap B \in \sigma_0$. Let $A_\alpha \in \sigma_0$ for each $\alpha \in \Delta$. By Theorem 5.2, for each $\alpha \in \Delta$, $A_\alpha \subseteq Int(Cl(\Psi_\Upsilon(A\_))) \subseteq Int(Cl(\Psi_\Upsilon(\bigcup A_\alpha)))$. Hence $\bigcup A_\alpha \subset Int(Cl(\Psi_\Upsilon(\bigcup A_\alpha)))$. Hence $\bigcup A_\alpha \in \sigma_0$.This shows that σ_0 is a topology for X.

Theorem. 5.8

Let $(X, \tau, \mathcal{J})$ be an ideal topological space. Then $\tau \sim^{\Upsilon} \mathcal{J}$, if and only if $\Psi_\Upsilon(A) - A \in \mathcal{J}$ for every $A \subseteq X$.

Proof

Necessity

Assume $\tau \sim^{\Upsilon} \mathcal{J}$ and let $A \subseteq X$. Observe that $x \in \Psi_\Upsilon(A)-A$ if and only if $x \notin A$ and $x \notin \Upsilon(X - A)$ if and only if $x \notin A$ and there exists $U_x \in \tau(x)$ such that $\pi cl(U_x) - A \in \mathcal{J}$ if and only if there exists $U_x \in \tau(x)$ such that $x \in \pi cl(U_x) - A \in \mathcal{J}$. Now for each $x \in \Psi_\Upsilon(A) - A$ and $U_x \in \tau(x)$, $\pi cl(U_x) \cap (\Psi_\Upsilon(A)-A) \in \mathcal{J}$ by heredity. Hence $\Psi_\Upsilon(A)-A \in \mathcal{J}$ by assumption that $\tau \sim^{\Upsilon} \mathcal{J}$.

Sufficiency

Let A ⊆ X and assume that for each $x \in A$ there exists $U_x \in \tau(x)$ such that $\pi cl(U_x) \cap A \in \mathcal{J}$. Observe that $\Psi_\Upsilon(X - A) - (X - A) = A - \Upsilon(A) = \{x :$ there exists $U_x \in \tau(x)$ such that $x \in \pi cl(U_x) \cap A \in \mathcal{J}\}$.Thus we have $A \subseteq \Psi_\Upsilon(X - A) - (X - A) \in \mathcal{J}$. Hence $A \in \mathcal{J}$ by heredity of $\mathcal{J}$.

Proposition. 5.9

Let $(X, \tau, \mathcal{J})$ be an ideal topological space with $\tau \sim^\Upsilon \mathcal{J}$ $A \subseteq X$. If N is a nonempty open subset of $\Upsilon(A) \cap \Psi_\Upsilon(A)$, then $N - A \in \mathcal{J}$ and $\pi cl(N) \cap A \notin \mathcal{J}$.

Proof

If $N \subseteq \Upsilon(A) \cap \Psi_\Upsilon(A)$, then $N - A \subseteq \Psi_\Upsilon(A) - A \in \mathcal{J}$ by Theorem 5.8 and hence $N - A \in \mathcal{J}$ by heredity. Since $N \in \tau - \{\phi\}$ and $N \subseteq \Upsilon(A)$, we have $\pi cl(N) \cap A \notin \mathcal{J}$ by the definition of $\Upsilon(A)$.

In [11], Newcomb defines $A = B$ [mod $\mathcal{J}$] if $(A - B) \cup (B - A) \in \mathcal{J}$ and observes that = [mod $\mathcal{J}$] is an equivalence relation. By Theorem 5.2 (8), we have that if $A = B$ [mod$\mathcal{J}$] then $\Psi(A) = \Psi_\Upsilon(B)$.

Definition. 5.10

Let $(X, \tau, \mathcal{J})$ be an ideal topological space. A subset A of X is called a Baire set with respect to τ and I, denoted $A \in Br(X, \tau, \mathcal{J})$, if there exists a open set U such that $A = U$ [mod $\mathcal{J}$].

Lemma. 5.11

Let $(X, \tau, \mathcal{J})$ be an ideal topological space with $\tau \sim^\Upsilon \mathcal{J}$. If $U, V \in \tau$ and $\Psi_\Upsilon(U) = \Psi_\Upsilon(V)$, then $U = V$ [mod $\mathcal{J}$].

Proof

Since $U \in \tau$, by Corollary 5.3 we have $U \subseteq \Psi_\Upsilon(U)$ and hence $U - V \subseteq \Psi_\Upsilon(U) - V = \Psi_\Upsilon(V) - V \in \mathcal{J}$ by Theorem 5.8 Therefore, $U - V \in \mathcal{J}$. Similarly, $V - U \in \mathcal{J}$. Now $(U - V) \cup (V - U) \in \mathcal{J}$ by additivity. Hence $U = V$[mod $\mathcal{J}$].

Theorem. 5.12

Let $(X, \tau, \mathcal{J})$ be an ideal topological space with $\tau \sim^\Upsilon \mathcal{J}$. If $A, B \in Br(X, \tau, \mathcal{J})$, and $\Psi_\Upsilon(A) = \Psi_\Upsilon(B)$ then $A = B$ [mod $\mathcal{J}$].

Proof

Let $U, V \in \tau$ be such that $A = U$ [mod $\mathcal{J}$] and $B = V$ [mod $\mathcal{J}$]. Now $\Psi_\Upsilon(A) = \Psi_\Upsilon(U)$ and $\Psi_\Upsilon(B) = \Psi_\Upsilon(V)$ by Theorem 5.2(8). Since $\Psi_\Upsilon(A) = \Psi_\Upsilon(B)$ implies that $\Psi_\Upsilon(U) = \Psi_\Upsilon(V)$ and hence $U = V$ [mod $\mathcal{J}$] by Lemma 5.11. Hence $A = B$ [mod $\mathcal{J}$] by transitivity.

Theorem. 5.13

Let $(X, \tau, \mathcal{J})$ be an ideal topological space with $\tau \sim^\Upsilon \mathcal{J}$, where $\pi cl(\tau) \cap \mathcal{J} = \phi$. Then for $A \subseteq X$, $\Psi_\Upsilon(A) \subseteq \Upsilon(A)$.

Proof

Suppose $x \in \Psi_\Upsilon(A)$ and $x \notin \Upsilon(A)$. Then there exists a nonempty neighborhood $U_x \in \tau(x)$ such that $\pi cl(U_x) \cap A \in \mathcal{J}$. Since $x \in \Psi_\Upsilon(A)$, by Theorem 5.5 $x \in \cup\{U \in \tau : \pi cl(U) - A \in \mathcal{J}\}$ and there exists $V \in \tau(x)$ and $\pi cl(V) - A \in \mathcal{J}$. Now we have $U_x \cap V \in \tau(x)$, $\pi cl(U_x \cap V) \cap A \in \mathcal{J}$ and $\pi cl(U_x \cap V) - A \in \mathcal{J}$ by heredity. Hence by finite additivity we have $\pi cl(U_x \cap V) \cap A) \cup (\pi cl(U_x \cap V) - A) = \pi cl(U_x \cap V) \in \mathcal{J}$. Since $(U_x \cap V) \in \tau(x)$, which is contrary to $\pi cl(\tau) \cap \mathcal{J} = \phi$. Therefore, $x \in \Upsilon(A)$. This implies that $\Psi_\Upsilon(A) \subseteq \Upsilon(A)$.

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