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Properties of Two Dimensional Fractional Fourier-Mellin Transform

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ABSTRACT

Fourier-Mellin transform has many properties such as linearity, scaling, shifting, differentiation property etc. Mainly linearity and shifting property is used for image registration in medical field. Due to such properties transform has many applications like visual odometry, detection of human face, the comparison of plant leaves which is based on Fourier-Mellin transform. In present work we discussed about linearity, scaling, shifting, differentiation, first shifting property of two-dimensional fractional Fourier-Mellin Transform.

Introduction

Integral transform finds wide application in physics and particularly in electrical engineering, where the characteristic equations that describe the behavior of an electric circuit in the complex frequency domain correspond to linear combinations of exponentially damped, scaled, and time-shifted sinusoids in the time domain. Integral transforms find special applicability within other scientific and mathematical disciplines.

Linearity and shifting property is used for image registration in medical field. An image registration technique based on Fourier-Mellin transform which is used for recovering rotation, scale and translation parameters. Image registration technique operates in three phases scale, rotation and translation vector [1]. Tim Kazik presented in his work that a visual odometry method that estimates the location and orientation of a robotic recover platform using monocular ground facing camera. Visual odometry is based on Fourier-Mellin transform. This method gives more robust and accurate result than an optical flow based on visual odometry method [2].

Huy Tho Ho and Roland Goecke presented a novel method of estimating the optical flow of an image sequence using the Fourier Mellin Transform. They overcomes the limitations of phase correlation techniques by converting the Fourier transforms of image patches into log polar coordinates and, thus, being able to estimate not only the translation but also the scale and rotation motion of the patches [3].

Abbas Yaseri ,Seyed Mahmoud Anisheh introduced A new paper currency recognition method using the Fourier-Mellin transform . Due to Fourier-Mellin transform technique they observed that the output image is invariant to translation, rotation and scale [4]. A further development in the use of the Fourier-Mellin transform is its application into the radar classification of ships by Zwicke etal. Fourier-Mellin transform is used to identify plant leaves at various life stages based on the leaves shape or contour. Fourier-Mellin transform is also used in estimation of optical flow [5].

Motivated by the above work, we proved do research work on two dimensional fractional Fourier Transform, two dimensional fractional Mellin Transform and two dimensional fractional Fourier-Mellin Transform [5,7,8]. In this paper we have generalized two dimensional fractional Fourier-Mellin transform in the distributional sense. In this paper we proposed linearity, scaling, shifting, differentiation, first shifting property of two dimensional Fourier-Mellin transform.

Two-Dimensional Fractional Fourier-Mellin Transform

Definition of two-dimensional fractional Fourier-Mellin transform

The two-dimensional fractional Fourier-Mellin transform with parameters α and θ of $f(x,y,t,q)$ denoted by 2DFRFMT$\{f(x,y,t,q)\}$ performs a linear operation, given by the integral transform. 2DFRFMT

$$\{f(x,y,t,q)\} = F_{\alpha,\theta}(\xi,\eta,\lambda,\chi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} f(x,y,t,q)K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi)\,dxdydtdq \quad \text{----(1)}$$

Where

$$K_{\alpha,\theta}(x,y,t,q,\xi,\eta,\lambda,\chi) = \sqrt{\frac{1-icot\alpha}{2\pi}}$$

$$e^{\frac{1}{2sin\alpha}[(x^2+y^2+\xi^2+\eta^2)cos\alpha-2(x\xi+y\eta)}\, t^{\frac{2\pi i\lambda}{sin\theta}-1}$$

$$q^{\frac{2\pi i\chi}{sin\theta}-1}\, e^{\frac{\pi i}{tan\theta}[\lambda^2+\chi^2+log^2 t+log^2 q]}$$

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$= C_{1\alpha} e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)\cos\alpha - 2(x\xi+y\eta)} t^{C_{1\theta} i\lambda - 1}$
$q^{C_{1\theta} i\chi - 1} e^{C_{1\theta} i[\lambda^2+\chi^2+\log^2 t + \log^2 q]}$

where

$$C_{1\alpha} = \sqrt{\frac{1-i\cot\alpha}{2\pi}},\ C_{2\alpha} = \frac{1}{2\sin\alpha} C_{1\theta} = \frac{2\pi}{\sin\theta},\ C_{2\theta} = \frac{\pi}{\tan\theta} \quad 0 < \alpha < \frac{\pi}{2},\ 0 < \theta < \frac{\pi}{2}. \qquad ---(2)$$

The Test Function

An infinitely differentiable complex valued smooth function $\emptyset(x, y, t, q)$ on R^n belongs to $E(R^n)$, if for each compact set $I \subset S_{a,b}$, $J \subset S_{c,d}$

where

$S_{a,b} = \{x, y: x, y \epsilon R^n, |x| \le a, |y| \le b, a > 0, b > 0\}$
$S_{c,d} = \{t, q: t, q \epsilon R^n, |t| \le c, |q| \le d, c > 0, d > 0\}$

$$\gamma_{E,m,n,k,l}[\emptyset(x,y,t,q)] = \sup_{\substack{x,y \epsilon I \\ t,q \epsilon J}} \left| D^{m,n,k,l}_{x,y,t,q} \emptyset(x,y,t,q) \right| < \infty \qquad ---(3)$$

Thus $E(R^n)$, will denote the space of all $\emptyset(x, y, t, q) \epsilon E(R^n)$ with compact support contained in $S_{a,b} \cap S_{c,d}$.
Note that the space E is complete and therefore a Frechet space. Moreover, we say that $f(x, y, t, q)$ is a fractional Fourier-Mellin transformable if it is a member of E.

Distributional Two Dimensional Fractional Fourier-Mellin Transform (2DFRFMT)

The two dimensional distributional Fractional Fourier -Mellin transform of $f(x, y, t, q) \epsilon E^*(R^n)$ can be defined by

$$2DFRFMT\{f(x, y, t, q)\} = F_{\alpha,\theta}(\xi, \eta, \lambda, \chi)$$
$$= \langle f(x, y, t, q), K_{\alpha,\theta}(x, y, t, q, \xi, \eta, \lambda, \chi)\rangle \qquad ---(4)$$

where,

$$K_{\alpha,\theta}(x, y, t, q, \xi, \eta, \lambda, \chi) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\, e^{\frac{1}{2\sin\alpha}[(x^2+y^2+\xi^2+\eta^2)\cos\alpha - 2(x\xi+y\eta)}$$
$$t^{\frac{2\pi i\lambda}{\sin\theta}-1} q^{\frac{2\pi i\chi}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}[\lambda^2+\chi^2+\log^2 t+\log^2 q]}$$
$$= C_{1\alpha} e^{iC_{2\alpha}[(x^2+y^2+\xi^2+\eta^2)\cos\alpha - 2(x\xi+y\eta)}$$
$$t^{C_{1\theta} i\lambda-1} q^{C_{1\theta} i\chi-1} e^{C_{1\theta} i[\lambda^2+\chi^2+\log^2 t+\log^2 q]}$$

where

$$C_{1\alpha} = \sqrt{\frac{1-i\cot\alpha}{2\pi}},\ C_{2\alpha} = \frac{1}{2\sin\alpha}, C_{1\theta} = \frac{2\pi}{\sin\theta},\ C_{2\theta} = \frac{\pi}{\tan\theta} \quad 0 < \alpha < \frac{\pi}{2},\ 0 < \theta < \frac{\pi}{2}. \qquad ---(5)$$

Right hand side of equation (4) has a meaning as the application of $f(x, y, t, q) \epsilon E^*(R^n)$ to $K_{\alpha,\theta}(x, y, t, q, \xi, \eta, \lambda, \chi) \epsilon E$.
It can be extended to the complex space as an entire function given by

$$2DFRFMT\{f(x, y, t, q)\} = F_{\alpha,\theta}(\xi', \eta', \lambda', \chi')$$
$$= \langle f(x, y, t, q), K_{\alpha,\theta}(x, y, t, q, \xi', \eta', \lambda', \chi')\rangle \qquad ---(6)$$

The right hand side is meaningful because for each $\xi', \eta', \lambda', \chi' \epsilon C^n$, $K_{\alpha,\theta}(x, y, t, q, \xi', \eta', \lambda', \chi') \epsilon E$ as a function of x,y,t,q.

Properties

Linearity Property

Prove that-

$$2DFRFMT\{A_1 f_1(x, y, u, v) + A_2 f_2(x, y, u, v)\}(p, q, r, s)$$
$$= A_1 2DFRFMT\{f_1(x, y, u, v)\}(p, q, r, s) + A_2 2DFRFMT\{f_2(x, y, u, v)\}(p, q, r, s)$$

Proof

$$2DFRFMT\{A_1 f_1(x, y, u, v) + A_2 f_2(x, y, u, v)\}(p, q, r, s)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \{A_1 f_1(x, y, u, v) + A_2 f_2(x, y, u, v)\} K_{\alpha,\theta}(x, y, u, v, p, q, r, s)\, dxdydudv$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \{A_1 f_1(x, y, u, v) K_{\alpha,\theta}(x, y, u, v, p, q, r, s)$$
$$A_2 f_2(x, y, u, v) K_{\alpha,\theta}(x, y, u, v, p, q, r, s)\}\, dxdydudv$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \{A_1 f_1(x, y, u, v) K_{\alpha,\theta}(x, y, u, v, p, q, r, s)\, dxdydudv$$
$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \{A_2 f_2(x, y, u, v) K_{\alpha,\theta}(x, y, u, v, p, q, r, s)\, dxdydudv$$
$$= A_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \{f_1(x, y, u, v) K_{\alpha,\theta}(x, y, u, v, p, q, r, s)\, dxdydudv$$

$$+A_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}\{f_2(x,y,u,v)K_{\alpha,\theta}(x,y,u,v,p,q,r,s)\,dxdydudv$$

$$=A_1 2DFRFMT\{f_1(x,y,u,v)\}(p,q,r,s)+A_2 2DFRFMT\{f_2(x,y,u,v)\}(p,q,r,s)$$

Scaling Property

$$2DFRFMT\{f(ax,by,cu,dv)\}(p,q,r,s)=\frac{1}{ab}c^{-\frac{2\pi ir}{\sin\theta}}d^{-\frac{2\pi is}{\sin\theta}}e^{\frac{\pi i}{\tan\theta}(\log^2 c+\log^2 d)}$$
$$2DFRFMT$$
$$\{e^{\frac{i}{2}\left[\left(\frac{1-a^2}{a^2}\right)m^2+\left(\frac{1-b^2}{b^2}\right)n^2\right]\cot\alpha-\frac{i}{2}\operatorname{cosec}\alpha\left[\left(\frac{1-a}{a}\right)mp+\left(\frac{1-b}{b}\right)nq\right]}$$
$$e^{-\frac{2\pi i}{\tan\theta}[\log k\log c+\log l\log d]}f(m,n,k,l)\}(p,q,r,s)$$

Proof

$$2DFRFMT\{f(ax,by,cu,dv)\}(p,q,r,s)$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}f(ax,by,cu,dv)$$
$$K_{\alpha,\theta}(x,y,u,v,p,q,r,s)dxdydudv$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}f(ax,by,cu,dv)\sqrt{\frac{1-i\cot\alpha}{2\pi}}\,e^{\frac{1}{2\sin\alpha}[(x^2+y^2+p^2+q^2)\cos\alpha-2(xp+yq)}u^{\frac{2\pi ir}{\sin\theta}-1}$$
$$v^{\frac{2\pi is}{\sin\theta}-1}e^{\frac{\pi i}{\tan\theta}[r^2+s^2+\log^2u+\log^2v]}\,dxdydudv \qquad ---(1)$$

Putting

$ax=m,\quad by=n,\quad cu=k,\quad dv=l$

$adx=dm,\quad bdy=dn,\quad cdu=dk,\quad ddv=dl$

$x=\frac{m}{a},\quad y=\frac{n}{b},\quad u=\frac{k}{c},\quad v=\frac{l}{d}$

(1)$\Rightarrow$ $2DFRFMT\{f(ax,by,cu,dv)\}(p,q,r,s)$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty}f(m,n,k,l)$$
$$\sqrt{\frac{1-i\cot\alpha}{2\pi}}\,e^{\frac{1}{2\sin\alpha}[\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}+p^2+q^2\right)\cos\alpha-2(\frac{m}{a}p+\frac{n}{b}q)}$$
$$\left(\frac{k}{c}\right)^{\frac{2\pi ir}{\sin\theta}-1}\left(\frac{l}{d}\right)^{\frac{2\pi is}{\sin\theta}-1}e^{\frac{\pi i}{\tan\theta}[r^2+s^2+\log^2\left(\frac{k}{c}\right)+\log^2\left(\frac{l}{d}\right)]}$$
$$\frac{dm}{a}\frac{dn}{b}\frac{dk}{c}\frac{dl}{d}$$
$$=\frac{C_{1\alpha}}{abcd}\frac{1}{c^{\frac{2\pi ir}{\sin\theta}-1}d^{\frac{2\pi is}{\sin\theta}-1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}f(m,n,k,l)\;e^{\frac{i}{2}(m^2+n^2+p^2+q^2)\cot\alpha}e^{-\frac{i}{2}(mp+nq)\operatorname{cosec}\alpha}$$
$$e^{\frac{i}{2}[\left(\frac{(-a^2+1)m^2}{a^2}+\frac{(-b^2+1)n^2}{b^2}\right)\cot\alpha}$$
$$e^{-\frac{i}{2}(\frac{(-a+1)m}{a}p+\frac{(-b+1)n}{b}q)\operatorname{cosec}\alpha}(k)^{\frac{2\pi ir}{\sin\theta}-1}(l)^{\frac{2\pi is}{\sin\theta}-1}$$
$$e^{\frac{\pi i}{\tan\theta}[r^2+s^2+\log^2k+\log^2l]}e^{\frac{\pi i}{\tan\theta}[\log^2c+\log^2d]}$$
$$e^{\frac{-2\pi i}{\tan\theta}[\log k\log c+\log l\log d]}dmdndkdl$$
$$=\frac{C_{1\alpha}}{abcd}\frac{e^{\frac{\pi i}{\tan\theta}[\log^2c+\log^2d]}}{c^{\frac{2\pi ir}{\sin\theta}-1}d^{\frac{2\pi is}{\sin\theta}-1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty}f(m,n,k,l)\;e^{\frac{i}{2}(m^2+n^2+p^2+q^2)\cot\alpha-\frac{i}{2}(mp+nq)\operatorname{cosec}\alpha}$$
$$(k)^{\frac{2\pi ir}{\sin\theta}-1}(l)^{\frac{2\pi is}{\sin\theta}-1}e^{\frac{\pi i}{\tan\theta}[r^2+s^2+\log^2k+\log^2l]}$$
$$e^{\frac{i}{2}[\left(\frac{(-a^2+1)m^2}{a^2}+\frac{(-b^2+1)n^2}{b^2}\right)\cot\alpha}$$
$$e^{-\frac{i}{2}(\frac{(-a+1)m}{a}p+\frac{(-b+1)n}{b}q)\operatorname{cosec}\alpha\frac{-2\pi i}{\tan\theta}[\log k\log c+\log l\log d]}$$
$$dmdndkdl$$
$$=\frac{C_{1\alpha}}{ab}\frac{e^{\frac{\pi i}{\tan\theta}[\log^2c+\log^2d]}}{c^{\frac{2\pi ir}{\sin\theta}}d^{\frac{2\pi is}{\sin\theta}}}\;2DFRFMT\{\,e^{\frac{i}{2}[\left(\frac{(-a^2+1)m^2}{a^2}+\frac{(-b^2+1)n^2}{b^2}\right)\cot\alpha}$$
$$e^{-\frac{i}{2}\left(\frac{(-a+1)m}{a}p+\frac{(-b+1)n}{b}q\right)\operatorname{cosec}\alpha\frac{-2\pi i}{\tan\theta}[\log k\log c+\log l\log d]}$$
$$f(m,n,k,l)(p,q,r,s)$$

Differential Property

(i) Prove that

$2DFRFMT\{f'(x,y,u,v)\}(p,q,r,s)$
$= [-icot\alpha]2DFRFMT[xf(x,y,u,v)](p,q,r,s) + [ipcosec\alpha]\ 2DFRFMT[f(x,y,u,v)](p,q,r,s)$

Proof

$2DFRFMT\{f'(x,y,u,v)\}(p,q,r,s)$
$2DFRFMT\{f(ax,by,cu,dv)\}(p,q,r,s)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f'(x,y,u,v)K_{\alpha,\theta}(x,y,u,v,p,q,r,s)dxdydudv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f'(x,y,u,v)$$
$$\sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{i}{2sin\alpha}[(x^2+y^2+p^2+q^2)cos\alpha-2(xp+yq)]} u^{\frac{2\pi ir}{sin\theta}-1} v^{\frac{2\pi is}{sin\theta}-1}$$
$$e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2u+log^2v]}\,dxdydudv$$

$$= C_{1\alpha}e^{\frac{i}{2}(p^2+q^2)cot\alpha}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} e^{\frac{i}{2}[y^2cot\alpha-2yqcosec\alpha]}\,(u)^{\frac{2\pi ir}{sin\theta}-1}$$
$$(v)^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2u+log^2v]}$$
$$\left\{\int_{-\infty}^{\infty} e^{\frac{i}{2}x^2cot\alpha-ixpcosec\alpha} f'(x,y,u,v)\ dx\right\}dydudv$$

$$= C_{1\alpha}e^{\frac{i}{2}(p^2+q^2)cot\alpha}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} e^{\frac{i}{2}[y^2cot\alpha-2yqcosec\alpha]}\,(u)^{\frac{2\pi ir}{sin\theta}-1}$$
$$(v)^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2u+log^2v]}$$
$$\{-\int_{-\infty}^{\infty} e^{\frac{i}{2}x^2cot\alpha-ixpcosec\alpha}[ixcot\alpha]f(x,y,u,v)dx + \int_{-\infty}^{\infty} e^{\frac{i}{2}x^2cot\alpha-ixpcosec\alpha}[ipcosec\alpha]f(x,y,u,v)dx\}\ dydudv$$

$$= -C_{1\alpha}[icot\alpha]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} xf(x,y,u,v)\ e^{\frac{i}{2}(x^2+y^2+p^2+q^2)cot\alpha}$$
$$e^{-\frac{i}{2}(xp+yq)cosec\alpha}(u)^{\frac{2\pi ir}{sin\theta}-1}(v)^{\frac{2\pi is}{sin\theta}-1}$$
$$e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2u+log^2v]}dxdydudv$$
$$+[ipcosec\alpha]C_{1\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(x,y,u,v)$$
$$e^{\frac{i}{2}(x^2+y^2+p^2+q^2)cot\alpha}e^{-\frac{i}{2}(xp+yq)cosec\alpha}$$
$$(u)^{\frac{2\pi ir}{sin\theta}-1}(v)^{\frac{2\pi is}{sin\theta}-1}e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2u+log^2v]}dxdydudv$$
$$= [-icot\alpha]2DFRFMT[xf(x,y,u,v)](p,q,r,s) + [ipcosec\alpha]\ 2DFRFMT[f(x,y,u,v)](p,q,r,s)$$

(ii) Prove that

$2DFRFMT\{f'(x,y,u,v)\}(p,q,r,s)$
$= \frac{-2\pi i}{sin\theta}\{(cos\theta)2DFRFMT\left[\frac{logu}{u}f(x,y,u,v)\right](p,q,r,s)+\left(r-\frac{sin\theta}{2\pi i}\right)\ 2DFRFMT\left[\frac{1}{u}f(x,y,u,v)\right](p,q,r,s)\}$

Proof

$2DFRFMT\{f'(x,y,u,v)\}(p,q,r,s)$
$2DFRFMT\{f(ax,by,cu,dv)\}(p,q,r,s)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f'(x,y,u,v)$$
$$K_{\alpha,\theta}(x,y,u,v,p,q,r,s)dxdydudv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f'(x,y,u,v)$$
$$\sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{\frac{1}{2sin\alpha}[(x^2+y^2+p^2+q^2)cos\alpha-2(xp+yq)]}$$
$$u^{\frac{2\pi ir}{sin\theta}-1}v^{\frac{2\pi is}{sin\theta}-1}e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2u+log^2v]}\ dxdydudv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f'(x,y,u,v) C_{1\alpha}\, e^{\frac{1}{2sin\alpha}[(x^2+y^2+p^2+q^2)cos\alpha-2(xp+yq)]} (u)^{\frac{2\pi ir}{sin\theta}-1}(v)^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2u+log^2v]}\, dxdydudv$$

where,

$$C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} C_{1\alpha} e^{\frac{1}{2sin\alpha}[(x^2+y^2+p^2+q^2)cos\alpha-2(xp+yq)]}$$

$$(v)^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[s^2+log^2v]}$$

$$\{-\int_{0}^{\infty}(u)^{\frac{2\pi ir}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+log^2u]}[\left(\frac{2\pi i}{tan\theta}\frac{logu}{u}\right)+\frac{1}{u}(\frac{2\pi ir}{sin\theta}-1)]f(x,y,u,v)du\}\, dxdydv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} C_{1\alpha} e^{\frac{1}{2sin\alpha}[(x^2+y^2+p^2+q^2)cos\alpha-2(xp+yq)]}$$

$$(v)^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[s^2+log^2v]}$$

$$\{-\int_{0}^{\infty}(u)^{\frac{2\pi ir}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+log^2u]}\frac{2\pi i}{u}$$

$$\left(logu\frac{cos\theta}{sin\,\theta}+\frac{r}{sin\theta}-\frac{1}{2\pi i}\right) \quad f(x,y,u,v)du\}\, dxdydv$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} C_{1\alpha} e^{\frac{1}{2sin\alpha}[(x^2+y^2+p^2+q^2)cos\alpha-2(xp+yq)]}$$

$$(u)^{\frac{2\pi ir}{sin\theta}-1}(v)^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+log^2u+s^2+log^2v]}$$

$$\frac{2\pi i}{usin\theta}[logucos\theta]f(x,y,u,v)\, dxdydudv$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} C_{1\alpha} e^{\frac{1}{2sin\alpha}[(x^2+y^2+p^2+q^2)cos\alpha-2(xp+yq)]}$$

$$(u)^{\frac{2\pi ir}{sin\theta}-1}(v)^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+log^2u+s^2+log^2v]}$$

$$\frac{2\pi i}{usin\theta}\left(r-\frac{sin\theta}{2\pi i}\right)f(x,y,u,v)\, dxdydudv$$

$$= \frac{-2\pi i}{sin\theta}\{(cos\theta)2DFRFMT\left[\frac{logu}{u}f(x,y,u,v)\right](p,q,r,s)+\left(r-\frac{sin\theta}{2\pi i}\right)\, 2DFRFMT\left[\frac{1}{u}f(x,y,u,v)\right](p,q,r,s)\}$$

Shifting Property

Prove that

$$2DFRFMT\{f(x-x_0,y-y_0,u,v)\}(p,q,r,s) = e^{-i[x_0p+y_0q]cosec\alpha}\, 2DFRFMT$$

$$\left\{e^{\frac{1}{2}[x_0(2g+x_0)+y_0(2h+y_0)q]cot\alpha} f(g,h,u,v)\right\}(p,q,r,s)$$

Proof

$$2DFRFMT\{f(x-x_0,y-y_0,u,v)\}(p,q,r,s)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(x-x_0,y-y_0,u,v) C_{1\alpha}\, e^{\frac{1}{2sin\alpha}[(x^2+y^2+p^2+q^2)cos\alpha-2(xp+yq)]}$$

$$(u)^{\frac{2\pi ir}{sin\theta}-1}(v)^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2u+log^2v]}\, dxdydudv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} f(x-x_0,y-y_0,u,v) C_{1\alpha} e^{\frac{i}{2}[(x^2+y^2+p^2+q^2)cot\alpha - e^{-i}(xp+yq)}$$

$$(u)^{\frac{2\pi ir}{sin\theta}-1}(v)^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2u+log^2v]}\, dxdydudv$$

Putting

$x - x_0 = \mathbf{g}$

$y - y_0 = h$

$\Rightarrow x = g + x_0\; y = h + y_0$

$dx = dg \quad dy = dh$

$$2DFRFMT\{f(x-x_0,y-y_0,u,v)\}(p,q,r,s)(u)^{\frac{2\pi ir}{sin\theta}-1}(v)^{\frac{2\pi is}{sin\theta}-1}$$

$e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2 u+log^2 v]} dgdhdudv$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} f(g,h,u,v)C_{1\alpha}\ e^{\frac{i}{2}[(g+x_0)+(h+y_0)+p^2+q^2]cot\alpha} e^{-i[(g+x_0)p+(h+y_0)q]cosec\alpha}$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} f(g,h,u,v)C_{1\alpha}\ e^{\frac{i}{2sin\alpha}[(g^2+h^2+p^2+q^2)cos\alpha-2(gp+hq)]}\ (u)^{\frac{2\pi ir}{sin\theta}-1}(v)^{\frac{2\pi is}{sin\theta}-1}$

$e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2 u+log^2 v]}\ e^{\frac{i}{2}[x_0(2g+x_0)+y_0(2h+y_0)q]cot\alpha} e^{-i[x_0p+y_0q]cosec\alpha} dgdhdudv$

$= 2DFRFMT\{e^{\frac{i}{2}[x_0(2g+x_0)+y_0(2h+y_0)q]cot\alpha} e^{-i[x_0p+y_0q]cosec\alpha} f(g,h,u,v)\}(p,q,r,s)$

$= e^{-i[x_0p+y_0q]cosec\alpha} 2DFRFMT$

$\left\{e^{\frac{i}{2}[x_0(2g+x_0)+y_0(2h+y_0)q]cot\alpha} f(g,h,u,v)\right\}(p,q,r,s)$

First Shifting Property

Prove that

$2DFRFMT\{e^{i(ax+by)}f(x,y,u,v)\}(p,q,r,s)$

$= e^{\frac{i}{2}[a^2+b^2]sin2\alpha} e^{i(ap+bq)cos\alpha} 2DFRFMT[f(x,y,u,v)]\{(p-asin\alpha),(q-bsin\alpha),r,s\}$

Proof

$2DFRFMT\{e^{i(ax+by)}f(x,y,u,v)\}(p,q,r,s)$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} e^{i(ax+by)}f(x,y,u,v)$

$K_{\alpha,\theta}(x,y,u,v,p,q,r,s)dxdydudv$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} e^{i(ax+by)}f(x,y,u,v)\sqrt{\frac{1-icot\alpha}{2\pi}}\ e^{\frac{1}{2sin\alpha}[(x^2+y^2+p^2+q^2)cos\alpha-2(xp+yq)]}$

$u^{\frac{2\pi ir}{sin\theta}-1} v^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2 u+log^2 v]} dxdydudv$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} e^{i(ax+by)}f(x,y,u,v)C_{1\alpha}$

$e^{\frac{1}{2sin\alpha}[(x^2+y^2+p^2+q^2)cos\alpha-2(xp+yq)]}(u)^{\frac{2\pi ir}{sin\theta}-1}(v)^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2 u+log^2 v]} dxdydudv$

where,

$C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$

$= C_{1\alpha}e^{\frac{i}{2}[p^2+q^2]cot\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} f(x,y,u,v)\ e^{\frac{i}{2}(x^2cot\alpha)-ixpcosec\alpha+iax}\ e^{\frac{i}{2}(y^2cot\alpha)-iyqcosec\alpha+iby}$

$(u)^{\frac{2\pi ir}{sin\theta}-1}(v)^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2 u+log^2 v]} dxdydudv = C_{1\alpha}e^{\frac{i}{2}[p^2+q^2]cot\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} f(x,y,u,v)$

$e^{\frac{i}{2}(x^2cot\alpha)-ixpcosec\alpha+iax}\ e^{\frac{i}{2}(y^2cot\alpha)-iyqcosec\alpha+iby}$

$(u)^{\frac{2\pi ir}{sin\theta}-1}(v)^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2 u+log^2 v]} dxdydudv = C_{1\alpha}e^{\frac{i}{2}[p^2+q^2]cot\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} f(x,y,u,v)$

$e^{\frac{i}{2}(x^2cot\alpha)-ixcosec\alpha(p-asin\alpha)}\ e^{\frac{i}{2}(y^2cot\alpha)-iycosec\alpha(q-bsin\alpha)}(u)^{\frac{2\pi ir}{sin\theta}-1}(v)^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2 u+log^2 v]} dxdydudv$

$= e^{\frac{i}{2}[a^2+b^2]sin\alpha cos\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} C_{1\alpha}f(x,y,u,v)$

$e^{\frac{i}{2}[x^2+y^2+(p-asin\alpha)^2+(q-bsin\alpha)^2]cot\alpha} e^{-i[x(p-asin\alpha)+y(q-bsin\alpha)]cosec\alpha}\ (u)^{\frac{2\pi ir}{sin\theta}-1}$

$(v)^{\frac{2\pi is}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}[r^2+s^2+log^2 u+log^2 v]} e^{i(ap+bq)cos\alpha} dxdydudv$

$= e^{\frac{i}{2}[a^2+b^2]sin2\alpha} e^{i(ap+bq)cos\alpha} 2DFRFMT[f(x,y,u,v)]\{(p-asin\alpha),(q-bsin\alpha),r,s\}$

Conclusion

In this paper we discussed on linearity, scaling, shifting, differentiation, first shifting property of two dimensional Fourier-Mellin transform.

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