A brief discussion on Graph Theory and some of its Applications in different field of mathematics

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ABSTRACT

Graph theory is a branch of Mathematics that deals with the study how networks can be encoded and their properties can be measured. Graph theory is becoming most important as it is most commonaly used in too many other areas of Mathematics, science and Engineering. It is very a actively used in many areas such as biochemistry, Communication networks and Coding theory Algorithms and Computation and operations research . Combinatorial methods found in graph theory have also been used to prove some fundamental results in pure mathematics. This paper, giving a general view of these facts, includes new graph theoretical proofs of Fermat's Little Theorem.

Introduction

The origins of graph theory can be traced to Leonhard Euler who devised in 1735 a problem that came to be known as the "Seven Bridges of Konigsberg". In this problem, someone had to cross once all the bridges only once and in a continuous sequence, a problem the Euler proved to have no solution by representing it as a set of nodes and links. This led the foundation of graph theory.
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Keeping in view of above applications, we will discuss new graph theoretical proofs of Fermat's Little Theorem. In this paper we will also studing the timetabling problem and the assignment of frequencies in GSM mobile phone networks. This work is new and very useful for the studing the another applications of graph theory. This type of work has not been previously discussed in the literature.

Fundamental Concept of Graph Theory

Graph:
A graph *G* is a set of vertex (nodes) *v* connected by edges (links) *e*. Hence we can write $G=(v , e)$.

Vertex A

Node which will be denoted by *v* is an meeting point point of a graph. It is the concept of a location such as a city, an administrative division, a road intersection or a transport terminal.
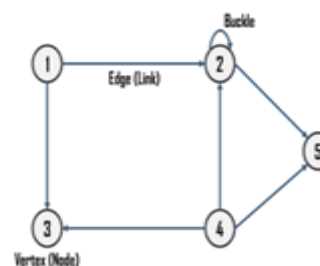
Edge

An edge which will be denoted by *e* is a link between two nodes. The link (*i* , *j*) is of initial extremity *i* and of terminal extremity *j*. A link is the direction that is represented as an arrow.

Some Basic Classifications of Graph Representation of a Network.

This simple graph has the following definition: G = (v,e)
where,
v = (1,2,3,4,5)
e = (1,2), (1,3), (2,2), (2,5), (4,2), (4,3), (4,5)

Planar Graph:

A graph in which all the intersections of two edges are a node. Since this type of graph is placed within a plane, its topology is two-dimensional. This is most difficult in the case of power grids, road and railway networks, whereas large care must be inferred to the definition of nodes.

Non-Planar Graph:

A graph in which there are no nodes at the intersection of at least two edges. This type of graph implies a third dimension in the topology of the graph since there is the possibility of having a movement "passing over" another movement such as for air and maritime transport.

Geometrical Representation of Planar Graph and Non-Planar Graph

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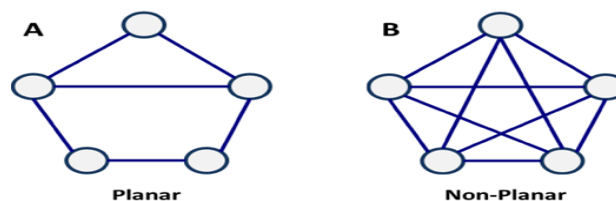
Graph **A** is planar because there is no link is overlapping with another. Graph **B** is non-planar because several links are overlapping.
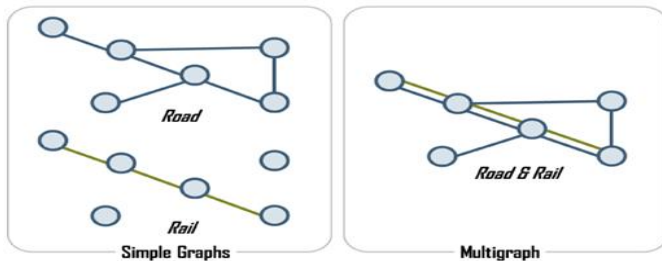
Simple Graph

A graph that includes only one type of link between its vertices. For example a rail network is a simple graph.

Multigraph

A graph that includes more than one types of links between its vertices. Some vertices may be connected to one link type while others can be connected to more than one that are running in parallel. For example a graph depicting a rail network with different links between vertices serviced by either or both modes is a multigraph.

Geometrical Representation of Simple Graph and Multigraph.

The multigraph is a combination of the two simple graphs as we cleared in the above definition of simple graph.

Links and their Structures

A transportation network enables flows of people, freight or information, which are occurring along its links. Graph theory must thus offer the possibility of representing movements as linkages

Connection

A set of two nodes as every node is linked to the other. Considers if a movement between two nodes is possible, whatever its direction. Knowing connections makes it possible to find if it is possible to reach a node from another node within a graph.

Path

A sequence of links that are traveled in the same direction. For a path to exist between two nodes, it must be possible to travel an uninterrupted sequence of links.
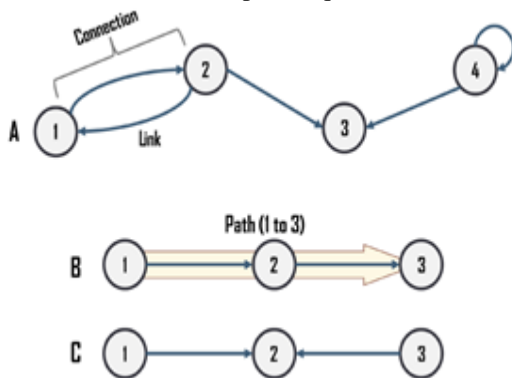
Here, On graph **A**, there are 5 links [(1,2), (2,1), (2,3), (4,3), (4,4)] and 3 connections [(1-2), (2-3), (3-4)]. On graph **B**, there is a path between 1 and 3, but on graph **C** there is no path between 1 and 3.

Chain. A sequence of links having a connection in common with the other. Direction does not matter.

Length of a Link, Connection or Path

Refers to the label associated with a link, a connection or a path. This label can be distance, the amount of traffic, the capacity or any attribute of that link. The length of a path is the number of links in this path.

Cycle

Refers to a chain where the initial and terminal node is the same and that does not use the same link more than once is a cycle.

Circuit

A path where the initial and terminal node corresponds. It is a cycle where all the links are traveled in the same direction. Circuits are very important in transportation because several distribution systems are using circuits to cover as much territory as possible in one direction (delivery route)
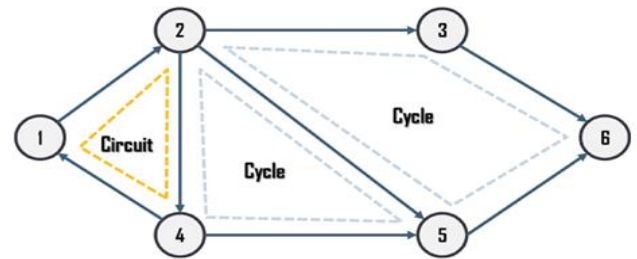
On this graph, 2-3-6-5-2 is a cycle but not a circuit. 1-2-4-1 is a cycle and a circuit.

Clique

A clique is a maximal complete subgraph where all vertices are connected.

Cluster. Also called community, it refers to a group of nodes having denser relations with each other than with the rest of the network. A wide range of methods are used to reveal clusters in a network, notably they are based on modularity measures (intra- versus inter-cluster variance).

Ego network

For a given node, the ego network corresponds to a sub-graph where only its adjacent neighbors and their mutual links are included.

Nodal region

A nodal region refers to a subgroup (tree) of nodes polarized by an independent node (which largest flow link connects a smaller node) and a number of subordinate nodes (which largest flow link connects a larger node). Single or multiple linkage analysis methods are used to reveal such regions by removing secondary links between nodes while keeping only the heaviest links.
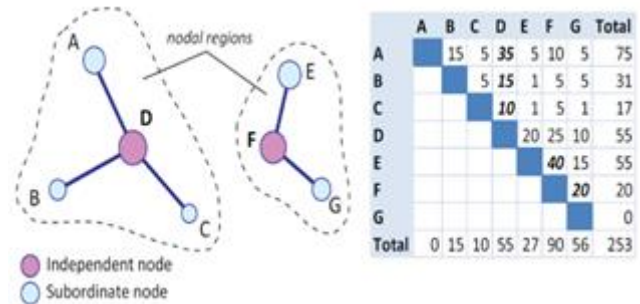
Nodal Region **A** refers to a subgroup (tree) of nodes polarized by an independent node (which largest flow link connects a smaller node) and a number of subordinate nodes (which largest flow link connects a larger node). Single or multiple linkage analysis methods are used to reveal such regions by removing secondary links between nodes while keeping only the heaviest links.

D and F are independent nodes because their largest flow is directed towards smaller nodes. A,B,C and E,G are subordinate nodes because their largest flow is directed

towards larger nodes (D and F). This algorithm can be also applied to directed graphs and may extend to secondary links (e.g. for each node, including up to 50% of its total traffic) so as to avoid losing too much information. Such methods proposed by Nystuen and Dacey (1961) are also labeled single linkage analysis (largest flows only) and multiple linkage analysis (largest flows over a certain threshold). They are often used to reveal functional regions based on flow patterns among localities.

Dual Graph

A method in space syntax that considers edges as nodes and nodes as edges. In urban street networks, large avenues made of several segments become single nodes while intersections with other avenues or streets become links (edges). This method is particularly useful to reveal hierarchical structures in a planar network.
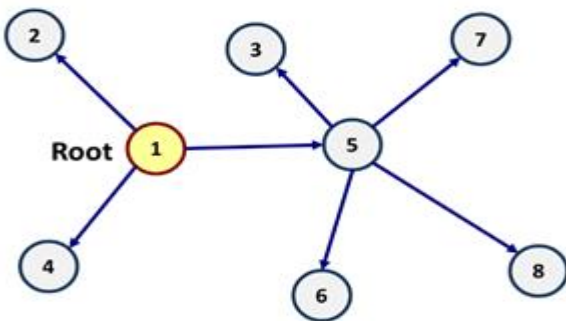
Root

A node r where every other node is the extremity of a path coming from r is a root. Direction has an importance. A root is generally the starting point of a distribution system, such as a factory or a warehouse.
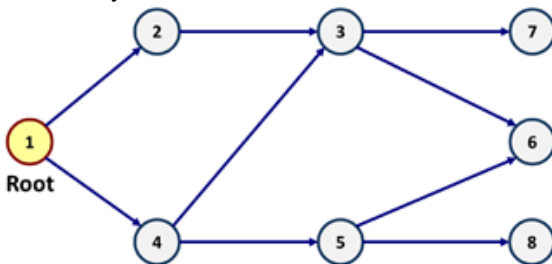
Node 1 is the only root of this graph because every other node is part of a path originating from node 1.

Trees

A connected graph without a cycle is a tree. A tree has the same number of links than nodes plus one, . ($e = v-1$). If a link is removed, the graph ceases to be connected. If a new link between two nodes is provided a cycle is created. A branch of root r is a tree where no links are connecting any node more than once. River basins are typical examples of tree-like networks based on multiple sources connecting towards a single estuary. This structure strongly influences river transport systems.

This graph is a tree having node 1 as a root.

Applications of Graph Theory

Graph theory is speedly moving into the main stream of mathematics because of its applications in different fields which include biochemistry, electrical engineering algorithms and computations and operations research etc.

There are so many vast applications of Grapy theory. A few of them had been discussed in this manuscript:

1. Fermat's (Little) Theorem

There are so many proofs of Fermat's Little Theorem. The first known proof was communicated by Euler in his letter of March 6, 1742 to Goldbach. The graph theoretic proof given below together with some number theoretic results, was used to prove Euler's generalization to non-prime modulus.

Theorem (Fermat)

Let a be a natural number and let p be a prime such that a is not divisible by p. Then, $a^p - a$ is divisible by p. *Proof.* Consider the graph $G = (V, E)$, where the vertex set V is the set of all sequences $(a_1, a_2, ..., a_p)$ of natural numbers between 1 and a (inclusive), with $a_i \neq a_j$ for some $i \neq j$. Clearly, V has $a^p - a$ elements. Let $u = (u_1, u_2, ..., u_p)$, $v = (u_p, u_1, ..., u_{p-1}) \in V$. Then, we say $uv \in E$. With this assumption, each vertex of G is of degree 2. So, each component of G is a cycle of length p. Therefore, the number of components is $(a^p - a) / p$. That is, $p \mid (a^p - a)$.
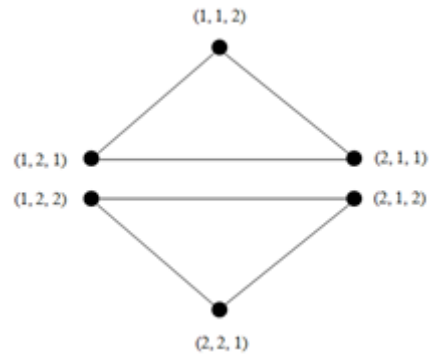

Figure . The graph G for a = 2 and p = 3.

2.The Timetabling Problem

If in a college there are n professors s1, s2, …, sn and n subjects q11, q2, …, qm to be taught. Given that professor mi is required (and able) to teach subject qj for pij periods (p = [pij] is called the teaching requirement matrix), the college administration wishes to make a timetable using the minimum possible number of periods. This is known as the timetabling problem and can be solved using the following strategy. Construct a bipartite multigraph G with vertices s1, s2, …, sn, q1, q2, …, qm such that vertices mi and qj are connected by pij edges. We presume that in any one period each professor can teach at most one subject and that each subject can be taught by at most one professor. Consider, first, a single period. The timetable for this single period corresponds to a matching in the graph and, conversely, each matching corresponds to a possible assignment of professors to subjects taught during this period. Thus, the solution to the timetabling problem consists of partitioning the edges of G into the minimum number of matchings.

3.Map Coloring and GSM Mobile Phone Networks

Given a map drawn on the plane or the surface of a sphere, the famous four color theorem asserts that it is always possible to properly color the regions of the map such that no two adjacent regions are assigned the same color, using at most four distinct colors . For any given map, we can construct its dual graph as follows. Put a vertex inside each region of the map and connect two distinct vertices by an edge if and only if their respective regions share a whole segment of their boundaries in common. Then, a proper vertex coloring of the dual graph yields a proper coloring of the regions of the original map.

The Groupe Spécial Mobile (GSM) was created in 1982 to provide a standard for a mobile telephone system. The first GSM network was launched in 1991 by Radiolinja in Finland with joint technical infrastructure maintenance from Ericsson. Today, GSM is the most popular mobile phones in the world, used by more than 2 billion people across more than 212 countries. GSM is a cellular network with its entire geographical range divided into hexagonal cells. Each cell has a communication tower which connects with mobile phones within the cell. All mobile phones connect to the GSM network by searching for cells in the immediate vicinity.

Conclusion

Graph theory deals with the study how networks can be encoded and their properties can be measured. Graph theory has broad area and there are so many applications of Grapy

theory. This paper, besides giving a general outlook of these facts, includes new graph theoretical proofs of Fermat's Little Theorem. In this paper we also discussed the timetabling problem and the assignment of frequencies in GSM mobile phone networks

References

1.Applications of Graph Theory,Shariefuddin Pirzada and Ashay Dharwadker, Journal of Korean so Fcietyor Industrial and Applied Mathematics,Vol II., no.4,2007
2.Graph Theory: Definition and Properties , Dr. Jean.