Available online at www.elixirpublishers.com (Elixir International Journal)

Advanced Mathematics

Elixir Adv. Math. 92 (2016) 39036-39044

ISSN: 2229-712X

# On $T-$ curvature tensor in Trans-Sasakian manifolds

Abhishek Singh, Ramesh Kumar Pandey and Sachin Khare
Department of Mathematics, BBD University, Lucknow - 226005, UP India.

**ARTICLE INFO**

**Article history:**
Received: 8 January 2016;
Received in revised form:
9 March 2016;
Accepted: 14 March 2016;

**Keywords**

$T-$curvature tensor,
Trans-Sasakian manifold,
$\eta-$ Einstein manifold.

**ABSTRACT**

In this chapter, quasi $-T-$flat, the $\xi-$ $T-$flat, and $\varphi-T-$flat trans-Sasakian manifolds are studied. It is proved that quasi $-T-$flat, the $\xi-T-$flat, and the $\varphi-T-$flat trans-Sasakian manifolds are $\eta-$Einstein under the condition $\varphi(grad\alpha)=(2n-1)(grad\beta)$.



## 1. Introduction

Oubina [12] studied a new class of almost contact Riemannian manifold known as trans-Sasakian manifold which generalizes both $\alpha$ -Sasakian and $\beta$ -Kenmotsu structure. A trans-Sasakian structure of type $(0,0),(\alpha,0)$ and $(0,\beta)$ are cosympletic [1], $\alpha$ -Sasakian [2,19] and $\beta$ -Kenmotsu [2, 10] respectively. Sasakian, $\alpha$ -Sasakian, Kenmotsu, $\beta$ -Kenmotsu are particular cases of trans-Sasakian manifold of type $(\alpha,\beta)$. Szabo [20,21] has obtained some curvature identities and Nomizu [11] has studied some curvature properties. It is also known that a locally trans-Sasakian manifold of dimension $\geq 5$ is either cosympletic or $\alpha$ -Sasakian or $\beta$ -Kenmotsu manifold [9]. On other hand, $3-$dimensional proper trans-Sasakian manifold are constructed by Marrero [9]. De et al [5] etc. are also studied some important properties of trans-Sasakian manifold. In 2008, Tripathi and Dwivedi [22] defined and studied quasi projectively flat, $\xi-$ projectively flat and $\varphi-$ projectively flat almost contact metric manifold.

M.M. Tripathi and et al. [24] introduced the $T-$ curvature tensor which in particular cases reduces to known curvatures like conformal, concircular and projective curvature tensors and some recently introduced curvature tensors like $M-$ projective curvature tensor, $Wi-$curvature tensor $(i=0,...,9)$ and $Wj-$ curvature tensors $(j=0,1)$.

M.M. Tripathi and P. Gupta [24] also found some important results on $T-$ curvature tensor in $K-$ contact and Sasakian manifolds. However, the quasi$-T-$flat, the $\xi-T-$flat, and the $\varphi-T-$flat trans-Sasakian manifolds are almost not discussed so far. Let $M$ be a $(2n+1)-$ dimensional contact metric manifold. Since at each point $p\in M$, the tangent space $Tp(M)$ can be decomposed into the direct sum $Tp(M)=\varphi(Tp(M))\oplus\{\xi_p\}$, where $\{\xi_p\}$ is the $1$ -dimensional linear subspace of $Tp(M)$ generated by $\xi_p$, the $T-$ curvature tensor $T$ is a map

$T:Tp(M)\times Tp(M)\times Tp(M)\to\varphi(Tp(M))\oplus\{\xi_p\}$, such that

$$T(X,Y)Z$$
$$=a_0R(X,Y)Z+a_1S(Y,Z)X+a_2S(X,Z)Y$$
$$+a_3S(X,Y)Z+a_4g(Y,Z)QY+a_5g(X,Z)QY$$
$$+a_6g(X,Y)QZ+a_7r(g(Y,Z)X-g(X,Z)Y) \qquad (1)$$

where $a_0,a_1,a_2,a_3,a_4,a_5,a_6$ and $a_7$ are constants and $R,S,Q$ and $r$ are the Riemannian curvature-tensor, the Ricci-tensor, the Ricci-operator and the scalar curvature of the manifold respectively. It may be natural to consider the following particular cases:

1. If $T:Tp(M)\times Tp(M)\times Tp(M)\to\{\xi_p\}$ i.e., the projection of the image of $T$ in $\varphi(Tp(M))$ is zero, such that

$$g(T(X,Y)Z,\varphi W)=0; \qquad (2)$$

Tele:
E-mail address: sachinuptu1@gmail.com

2. If $T : Tp(M)\times Tp(M)\times Tp(M)\to \varphi(Tp(M))$ i.e., the projection of the image of $T$ in $\{\xi_p\}$ is zero, such that

$$g(T(X,Y)\xi, W) = 0; \tag{3}$$

3. If $T : \varphi(Tp(M))\times \varphi(Tp(M))\times \varphi(Tp(M))\to \{\xi_p\}$

i.e., when $C$ is restricted to $\varphi(Tp(M))\times \varphi(Tp(M))\times \varphi(Tp(M))$, the projection of the image of $T$ in $\varphi(Tp(M))$ is zero, such that

$$g(T(\varphi X,\varphi Y)\varphi Z, \varphi W) = 0; \tag{4}$$

A Riemannian manifold, satisfying the cases $1,2$ and $3$ is called quasi $-T-$ flat, the $\xi-\ T-$ flat, and the $\varphi - T-$ flat respectively. The present work is organized as under Section $-2$ contains necessary details about trans-Sasakian manifolds. Some basic results are also given in section $-2.$ In Section $-3,$ quasi $-T-$ symmetric trans-Sasakian manifold are discussed. Section $-4$ contains $\xi-\ T-$ flat trans-Sasakian manifold and section $-5$ contains $\varphi - T-$ flat trans-Sasakian manifolds.

## 1. Preliminaries

In this section, we recall some general definitions and basic formulas which will be used later on. For this, we recommend the reference [2]. A $(2n+1)$ -dimensional Riemannian manifold $(M,g)$ is said to be an almost contact metric manifold if there exist a $(1,1)$ tensor field $\varphi$, a unique global non-vanishing structural vector field $\xi$ (called the vector field) and a $1$-form $\eta$ such that

$$\varphi^2 X = -X + \eta(X)\xi, \tag{5}$$

$$(a)\ \ \eta(\xi) = 1,\ \ (b)\ \ g(X,\xi) = \eta(X), \qquad (c)\ \ \eta(\varphi X) = 0,\ \ (d)\ \ \varphi\xi = 0, \tag{6}$$

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{7}$$

$$d\eta(X,Y) = g(X,\varphi Y), \qquad \eta o \varphi = 0, \tag{8}$$

Such a manifold is called contact manifold if $\eta\wedge(d\eta)^n \neq 0,$ where $n$ is $n^{\text{th}}$ exterior power. For contact manifold we also have $d\eta = \Phi$, where $\Phi(X,Y) = g(\varphi X,Y)$ is called fundamental $2$ -form on $M$. An almost contact metric manifold is called trans-Sasakian manifold of type $(\alpha,\beta)$ if

$$(\nabla_X\varphi)(Y) = \alpha(g(X,Y)\xi - \eta(X)Y) + \beta(g(\varphi X,Y)\xi - \eta(X)\varphi Y), \tag{9}$$

for all vector field $X,Y$ on $M,$ where $\alpha$ and $\beta$ are some smooth real valued functions. Trans-Sasakian manifolds of type $(1,0)$ and $(0,1)$ are called Sasakian and Kenmotsu manifold respectively.

An almost contact metric manifold $M$ is said to be $\eta$ -Einstein if its Ricci-tensor $S$ is of the form

$$S(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y), \tag{10}$$

where $A$ and $B$ are smooth functions on $M.$ A $\eta$ -Einstein manifold becomes Einstein if $B=0.$

If $\{e_1, e_2, ......, e_{2n}, \xi\}$ is a local orthonormal basis of vector fields in a $(2n+1)-$ dimensional almost contact metric manifold $M,$ then $\{\varphi e_1, \varphi e_2, ......, \varphi e_{2n}, \xi\}$ is also a local orthonormal basis. It is easy to verify that

$$\sum_{i=1}^{2n} g(e_i,e_i) = \sum_{i=1}^{2n} g(\varphi e_i,\varphi e_i) = 2n \tag{11}$$

$$\begin{aligned}\sum_{i=1}^{2n} g(e_i,Y)S(X,e_i) &= \sum_{i=1}^{2n} g(\varphi e_i,Y)S(X,\varphi e_i)\\ &= S(X,Y) - S(X,\xi)\eta(Y)\end{aligned} \tag{12}$$

for all $X,Y\in T(M).$ In view of $(8)$ and (12), we have

$$\begin{aligned}\sum_{i=1}^{2n} g(e_i,\varphi Y)S(\varphi X,e_i) &= \sum_{i=1}^{2n} g(\varphi e_i,\varphi Y)S(\varphi X,\varphi e_i)\\ &= S(\varphi X,\varphi Y)\end{aligned} \tag{13}$$

If $M$ is a trans-Sasakian manifold, then it is known that

$$R(X,\xi)\xi = (\alpha^2 - \beta^2 - \alpha\beta)[X - \eta(X)\xi], \tag{14}$$

for all $X\in T(M).$

$$S(X,\xi) = (2n(\alpha^2-\beta^2) - \xi\beta)\eta(X) - (2n-1)X\beta - (\varphi X)\alpha \tag{15}$$

and

$$S(\xi,\xi)=2n(\alpha^2-\beta^2-\xi\beta), \tag{16}$$

From $(16)$, we have

$$\sum_{i=1}^{2n}S(e_i,e_i)=\sum_{i=1}^{2n}S(\varphi e_i,\varphi e_i)=r-2n(\alpha^2-\beta^2-\xi\beta) \tag{17}$$

where $r=trace(Q)$ is the scalar curvature. In a trans-Sasakian manifold, we have

$$g(R(\xi,Y)Z,\xi)=(\alpha^2-\beta^2-\xi\beta)g(\varphi Y,\varphi Z), \tag{18}$$

for all $Y,Z\in T(M)$. Consequently,

$$\sum_{i=1}^{2n}g(R(e_i,Y)Z,e_i)=\sum_{i=1}^{2n}g(R(\varphi e_i,Y)Z,\varphi e_i) \\ =S(Y,Z)-(\alpha^2-\beta^2-\xi\beta)g(\varphi Y,\varphi Z), \tag{19}$$

$$\begin{cases}\sum_{i=1}^{2n}g(R(e_i,\varphi Y)\varphi Z,e_i)=\sum_{i=1}^{2n}g(R(\varphi e_i,\varphi Y)\varphi Z,\varphi e_i)\\ \quad =S(\varphi Y,\varphi Z)-(\alpha^2-\beta^2-\xi\beta)g(\varphi Y,\varphi Z),\end{cases} \tag{20}$$

**2. quasi $T-$ flat trans-Sasakian manifold manifolds**

We begin with the following lemma:

**Lemma 1** The quasi-$T-$ flat trans-Sasakian manifold takes the form of $(10)$ under the condition

$$\varphi(grad\alpha)=(2n-1)(grad\beta).$$

**Proof.** Let $M$ be a $(2n+1)$ -dimensional trans-Sasakian manifold.

We write the curvature tensor $T$ in its $(0,4)$ form as follows

$$\begin{cases} g(T(X,Y)Z,W)=a_0g(R(X,Y)Z,W)\\ +a_1S(Y,Z)g(X,W)+a_2S(X,Z)g(Y,W)\\ +a_3S(X,Y)g(Z,W)+a_4g(Y,Z)S(X,W)\\ +a_5g(X,Z)S(Y,W)+a_6g(X,Y)S(Z,W)\\ +a_7r(g(Y,Z)g(X,W)-g(X,Z)g(Y,W))\end{cases} \tag{21}$$

From (21), we have

$$\begin{cases} g(T(\varphi X,Y)Z,\varphi W)=a_0g(R(\varphi X,Y)Z,\varphi W)\\ +a_1S(Y,Z)g(\varphi X,\varphi W)+a_2S(\varphi X,Z)g(Y,\varphi W)\\ +a_3S(\varphi X,Y)g(Z,\varphi W)+a_4g(Y,Z)S(\varphi X,\varphi W)\\ +a_5g(\varphi X,Z)S(Y,\varphi W)+a_6g(\varphi X,Y)S(Z,\varphi W)\\ +a_7r(g(Y,Z)g(\varphi X,\varphi W)-g(\varphi X,Z)g(Y,\varphi W))\end{cases} \tag{22}$$

for all $X,Y,Z,W\in T(M)$. If $\{e_1,e_2,......,e_{2n},\xi\}$ is a local orthonormal basis of vector fields in $M$. Now on putting $X=W=e_i$ and taking summation over $1\le i\le 2n$ in $(22)$ we have

$$\sum_{i=1}^{2n}g(T(\varphi e_i,Y)Z,\varphi e_i)$$
$$=\sum_{i=1}^{2n}(a_0g(R(\varphi e_i,Y)Z,\varphi e_i)+a_1S(Y,Z)g(\varphi e_i,\varphi e_i)$$
$$+a_2S(\varphi e_i,Z)g(Y,\varphi e_i)+a_3S(\varphi e_i,Y)g(Z,\varphi e_i)$$
$$+a_4g(Y,Z)S(\varphi e_i,\varphi e_i)+a_5g(\varphi e_i,Z)S(Y,\varphi e_i)$$
$$+a_6g(\varphi e_i,Y)S(Z,\varphi e_i)$$
$$+a_7r(g(Y,Z)g(\varphi e_i,\varphi e_i)-g(\varphi e_i,Z)g(Y,\varphi e_i) \tag{23}$$

for all $Y,Z\in T(M)$. Using $(11)$, (12), (17) and $(19)$, we have

$$\sum_{i=1}^{2n} g(T(\varphi e_i, Y)Z, \varphi e_i)$$
$$= a_0(S(Y,Z) - (\alpha^2 - \beta^2 - \xi\beta)g(\varphi Y, \varphi Z))$$
$$+ 2na_1 S(Y,Z) + a_2(S(Y,Z) - S(Z,\xi)\eta(Y))$$
$$+ a_3(S(Y,Z) - S(Y,\xi)\eta(Z)) + a_4 g(Y,Z)(r - 2n(\alpha^2 - \beta^2 - \xi\beta))$$
$$+ a_5(S(Y,Z) - S(Y,\xi)\eta(Z))$$
$$+ a_6(S(Y,Z) - S(Z,\xi)\eta(Y))$$
$$+ a_7 r((2n-1)g(Y,Z) - \eta(Y)\eta(Z)) \quad (24)$$

If $M$ satisfies $(2)$, then from $(24)$, we have

$$0 = a_0(S(Y,Z) - (\alpha^2 - \beta^2 - \xi\beta)g(\varphi Y, \varphi Z))$$
$$+ 2na_1 S(Y,Z) + a_2(S(Y,Z) - S(Z,\xi)\eta(Y))$$
$$+ a_3(S(Y,Z) - S(Y,\xi)\eta(Z)) + a_4 g(Y,Z)(r - 2n(\alpha^2 - \beta^2 - \xi\beta))$$
$$+ a_5(S(Y,Z) - S(Y,\xi)\eta(Z))$$
$$+ a_6(S(Y,Z) - S(Z,\xi)\eta(Y))$$
$$+ a_7 r((2n-1)g(Y,Z) - \eta(Y)\eta(Z)) \quad (25)$$

If $\varphi(grad\alpha) = (2n-1)(grad\beta)$. then $\xi\beta = 0$ [5], and by using (7) and (13) in (25) we get

$$(a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6)S(Y,Z)$$
$$= (a_2 + a_6)2n(\alpha^2 - \beta^2)\eta(Y)\eta(Z)$$
$$+ (a_3 + a_5)2n(\alpha^2 - \beta^2)\eta(Y)\eta(Z)$$
$$+ (a_0 + 2na_4)(\alpha^2 - \beta^2)g(Y,Z)$$
$$- (a_4 + (2n-1)a_7)rg(Y,Z)$$
$$+ a_7 r((2n-1)g(Y,Z) - \eta(Y)\eta(Z)) \quad (26)$$

If $a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6 \neq 0,$ from $(26)$ we have

$$\begin{cases} S(Y,Z) = \left\{ \dfrac{(a_0 + 2na_4)(\alpha^2 - \beta^2) - (a_4 + (2n-1)a_7)r}{(a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6)} \right\} g(Y,Z) \\ + \left\{ \dfrac{2n(\alpha^2 - \beta^2)(a_2 + a_3 + a_5 + a_6) - a_0(\alpha^2 - \beta^2) + a_7 r}{(a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6)} \right\} \eta(Y)\eta(Z) \end{cases} \quad (27)$$

we can rewritten $(26),$ as

$$S = A_0 g \times A_1 \eta \otimes \eta \quad (28)$$

where

$$A_0 = \frac{(a_0 + 2na_4)(\alpha^2 - \beta^2) - (a_4 + (2n-1)a_7)r}{(a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6)}$$

&

$$A_1 = \frac{2n(\alpha^2 - \beta^2)(a_2 + a_3 + a_5 + a_6) - a_0(\alpha^2 - \beta^2) + a_7 r}{(a_0 + 2na_1 + a_2 + a_3 + a_5 + a_6)}$$

Therefore $M$ is an $\eta-$Einstein manifold. In particular, $M$ becomes an Einstein manifold provided. in (28), we get the required lemma. In view of **lemma 1,** we can state that:

**Theorem 1** *A quasi* $T-$*flat trans-Sasakian manifold is* $\eta-$*Einstein if* $\varphi(grad\alpha) = (2n-1)(grad\beta).$

### *3.* $\xi - T -$*flat trans-Sasakian manifolds*

We begin with the following lemma:

**Lemma 2** *The* $\xi - T -$ *flat trans-Sasakian manifold takes the form of* $(10)$ *under the condition*

$$\varphi(grad\alpha) = (2n-1)grad\beta.$$

**Proof.** Let M be $(2n+1)-$ dimensional trans-Sasakian manifold. Putting $Z = \xi$ in $(1)$, we have

$$g(T(X,Y)\xi,W)$$
$$= a_0 g(R(X,Y)\xi,W) + a_1 S(Y,\xi)g(X,W)$$
$$+ a_2 S(X,\xi)g(Y,W) + a_3 S(X,Y)g(\xi,W)$$
$$+ a_4 g(Y,\xi)S(X,W) + a_5 g(X,\xi)S(Y,W)$$
$$+ a_6 g(X,Y)S(\xi,W) + a_7 r(g(Y,\xi)g(X,W) - g(X,\xi)g(Y,W) \quad (29)$$

for all $X,Y,W \in T(M)$. If $\{e_1, e_2, ......, e_{2n}, \xi\}$ is a local orthonormal basis of vector fields in $M$. Now on putting $Z = \xi$ and taking summation over $1 \le i \le 2n$ in $(29)$, we have

$$\sum_{i=1}^{2n} g(T(X,\xi)\xi,W)$$
$$= \sum_{i=1}^{2n} (a_0 g(R(X,\xi)\xi,W) + a_1 S(\xi,\xi)g(X,W)$$
$$+ a_2 S(X,\xi)g(\xi,W) + a_3 S(X,\xi)g(\xi,W)$$
$$+ a_4 g(\xi,\xi)S(X,W) + a_5 g(X,\xi)S(\xi,W)$$
$$+ a_6 S(\xi,W)g(X,\xi) + a_7 r(g(\xi,\xi)g(X,W) - g(X,\xi)g(\xi,W)) \quad (30)$$

for all $Y,Z \in T(M)$. Using $(8)$, $(15)$, $(16)$ and $(18)$, we have

$$\sum_{i=1}^{2n} T(X,\xi,\xi,W)$$
$$= \sum_{i=1}^{2n} [a_0((\alpha^2 - \beta^2 - \xi\beta)(g(X,W) - \eta(X)\eta(W))$$
$$+ a_1((2n(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (2n-1)X\beta - (\varphi X)\alpha)g(X,W)$$
$$+ a_2 2n(\alpha^2 - \beta^2 - \xi\beta)\eta(X)\eta(W) + a_3 2n(\alpha^2 - \beta^2 - \xi\beta)\eta(X)\eta(W)$$
$$+ a_4 S(X,W) + a_5 2n(\alpha^2 - \beta^2 - \xi\beta)\eta(X)\eta(W)$$
$$+ a_6 2n(\alpha^2 - \beta^2 - \xi\beta)\eta(X)\eta(W) + a_7 r(g(X,W) - \eta(X)\eta(W)) \quad (31)$$

If $M$ satisfies $(3)$, then from $(31)$, we have

$$a_0\{(\alpha^2 - \beta^2 - \xi\beta)(g(X,W) - \eta(X)\eta(W))$$
$$+ a_1((2n(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (2n-1)X\beta - (\varphi X)\alpha)g(X,W)$$
$$+ a_2 2n(\alpha^2 - \beta^2 - \xi\beta)\eta(X)\eta(W) + a_3 2n(\alpha^2 - \beta^2 - \xi\beta)\eta(X)\eta(W)$$
$$+ a_4 S(X,W) + a_5 2n(\alpha^2 - \beta^2 - \xi\beta)\eta(X)\eta(W)$$
$$+ a_6 2n(\alpha^2 - \beta^2 - \xi\beta)\eta(X)\eta(W)$$
$$+ a_7 r(g(X,W) - \eta(X)\eta(W)) = 0 \quad (32)$$

If $\varphi(grad\alpha) = (2n-1)(grad\beta)$, then $\xi\beta = 0$, which implies that

$$a_4 S(X,W)$$
$$= -(a_0(\alpha^2 - \beta^2) + a_7 r)g(X,W)$$
$$+ (a_0(\alpha^2 - \beta^2)\eta(X)\eta(W)$$
$$- 2na_1(\alpha^2 - \beta^2)g(X,W)$$
$$- 2na_2(\alpha^2 - \beta^2)g(X,W)$$
$$- 2na_3(\alpha^2 - \beta^2)\eta(X)\eta(W)$$

$$-2na_5(\alpha^2-\beta^2)\eta(X)\eta(W)$$
$$-2na_6(\alpha^2-\beta^2)\eta(X)\eta(W)$$
$$+a_7r\eta(X)\eta(W) \tag{33}$$

If $a_4 \neq 0$ then from $(33)$, we get

$$S(X,W)=-\left(\frac{(a_0+2na_1)(\alpha^2-\beta^2)+a_7r}{a_4}\right)g(X,W)$$
$$-\left(\frac{(-a_0+2n(a_2+a_3+a_5+a_6))(\alpha^2-\beta^2)-a_7r)}{a_4}\right)\eta(X)\eta(W) \tag{34}$$

we can rewritten $(34)$, as

$$S=A_2g+A_3\eta\otimes\eta \tag{35}$$

where

$$A_2=-\frac{(a_0+2na_1)(\alpha^2-\beta^2)+a_7r)}{a_4}$$

&

$$A_3=-\left(\frac{(-a_0+2n(a_2+a_3+a_5+a_6))(\alpha^2-\beta^2)-a_7r}{a_4}\right)$$

in $(35)$, we get the required lemma.

In view of **lemma** $2$, we can state that:

**Theorem 2** *A Ricci tensor quasi* $\xi-$ $T-$ *flat trans-Sasakian manifold is* $\eta-$*Einstein if*

$$\varphi(grad\alpha)=(2n-1)(grad\beta).$$

**4.** $\varphi-T-$ **flat trans-Sasakian manifold**

**Lemma 3** *The* $\varphi-T-$ *flat trans-Sasakian manifold takes the form of* $(10)$ *under the condition*

$$\varphi(grad\alpha)=(2n-1)grad\beta\cdot$$

**Proof.** Let M be $(2n+1)-$ dimensional trans-Sasakian manifold. From $(1)$, we have

$$T(\varphi X,\varphi Y,\varphi Z,\varphi W)$$
$$=a_0R(\varphi X,\varphi Y,\varphi Z,\varphi W)+a_1S(\varphi Y,\varphi Z)g(\varphi X,\varphi W)$$
$$+a_2S(\varphi X,\varphi Z)g(\varphi Y,\varphi W)+a_3S(\varphi X,\varphi Y)g(\varphi Z,\varphi W)$$
$$+a_4g(\varphi Y,\varphi Z)S(\varphi X,\varphi W)+a_5g(\varphi X,\varphi Z)S(\varphi Y,\varphi W)$$
$$+a_6g(\varphi X,\varphi Y)S(\varphi Z,\varphi W)$$
$$+a_7r(g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)-g(\varphi X,\varphi Z)g(\varphi Y,\varphi W) \tag{36}$$

for all $X,Y,Z,W\in T(M)$. If $\{e_1,e_2,......,e_{2n},\xi\}$ is a local orthonormal basis of vector fields in a $(2n+1)-$ dimensional almost contact manifold $M$, then $\{\varphi e_1,\varphi e_2,......,\varphi e_{2n},\xi\}$ is local orthonormal basis. From $(36)$, we have

$$\sum_{i=1}^{2n}T(\varphi e_i,\varphi Y,\varphi Z,\varphi e_i)$$
$$=\sum_{i=1}^{2n}[a_0R(\varphi e_i,\varphi Y,\varphi Z,\varphi e_i)+a_1S(\varphi Y,\varphi Z)g(\varphi e_i,\varphi e_i)$$
$$+a_2S(\varphi e_i,\varphi Z)g(\varphi Y,\varphi e_i)+a_3S(\varphi e_i,\varphi Y)g(\varphi Z,\varphi e_i)$$
$$+a_4g(\varphi Y,\varphi Z)S(\varphi e_i,\varphi e_i)+a_5g(\varphi e_i,\varphi Z)S(\varphi Y,\varphi e_i)$$
$$+a_6g(\varphi e_i,\varphi Y)S(\varphi Z,\varphi e_i)$$
$$+a_7r(g(\varphi Y,\varphi Z)g(\varphi e_i,\varphi e_i)-g(\varphi e_i,\varphi Z)g(\varphi Y,\varphi e_i) \tag{37}$$

Using $(11)$, $(13)$, $(19)$ and $(37)$, we have

$$\sum_{i=1}^{2n} T(\varphi e_i, \varphi Y, \varphi Z, \varphi e_i)$$
$$= \sum_{i=1}^{2n}[a_0(S(\varphi Y,\varphi Z)-(\alpha^2-\beta^2-\xi\beta)(g(\varphi Y,\varphi Z))$$
$$+2na_1 S(\varphi Y,\varphi Z)+a_2 S(\varphi Y,\varphi Z)+a_3 S(\varphi Y,\varphi Z)$$
$$+a_5 S(\varphi Y,\varphi Z)+a_6 S(\varphi Y,\varphi Z)+a_7 r(2ng(\varphi Y,\varphi Z)-g(\varphi Y,\varphi Z))] \tag{38}$$

If $M$ is $\varphi - T -$ flat, then from $(38),$ we have

$$0=(a_0+2na_1+a_2+a_3+a_5+a_6)S(\varphi Y,\varphi Z)$$
$$+(-a_0(\alpha^2-\beta^2)+(r-2n)(\alpha^2-\beta^2)a_4+(2n-1)a_7 r)g(\varphi Y,\varphi Z)) \tag{39}$$

If $a_0+2na_1+a_2+a_3+a_5+a_6 \neq 0,$ then from $(39),$ we get

$$S(\varphi Y,\varphi Z)=\left\{\frac{a_0(\alpha^2-\beta^2)-(r-2n)(\alpha^2-\beta^2)a_4+(2n-1)a_7 r)}{(a_0+2na_1+a_2+a_3+a_5+a_6)}\right\}g(\varphi Y,\varphi Z) \tag{40}$$

we can rewritten $(40),$ as

$$S(\varphi Y,\varphi Z)=A_4 g(\varphi Y,\varphi Z) \tag{41}$$

where

$$A_4=\left\{\frac{a_0(\alpha^2-\beta^2)-(r-2n)(\alpha^2-\beta^2)a_4+(2n-1)a_7 r)}{(a_0+2na_1+a_2+a_3+a_5+a_6)}\right\} \tag{42}$$

and Using $(4)$ and $(41)$ in $(38),$ we get

$$a_0 R(\varphi X,\varphi Y,\varphi Z,\varphi W)$$
$$=-(a_1 A_4+a_4 A_4+a_7 r)g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)$$
$$-(a_2 A_4+a_5 A_4-a_7 r)g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)$$
$$-(a_3 A_4+a_6 A_4)g(\varphi X,\varphi Y)g(\varphi Z,\varphi W) \tag{43}$$

in $(43),$ we get the required lemma.

In view of **lemma 3,** we can state that:

**Theorem 3** A quasi $\varphi -$ $T -$ flat trans-Sasakian manifold is $\eta -$Einstein if $\varphi(grad\alpha)=(2n-1)(grad\beta).$

## 5. A Trans-Sasakian manifold with respect to the $\varphi -$ $T -$ flat

**Lemma 4** *The Ricci tensor of trans-Sasakian manifold takes the form of* $(10)$ *under the condition* $\varphi - T -$ *flat and* $\varphi(grad\alpha)=(2n-1)grad\beta\cdot$

**Proof.** Let $M$ be a $(2n+1)-$dimensional trans-Sasakian manifold. Now, assume that $M$ is $\varphi - T -$ flat. In a trans-Sasakian manifold, in view of $(5),$ $(6),$ $(8)$ and $(14)$ we can easily verify that

$$R(\varphi^2 X,\varphi^2 Y,\varphi^2 Z,\varphi^2 W)=R(X,Y,Z,W)$$
$$-g(Y,Z)\eta(X)\eta(W)+g(X,Z)\eta(Y)\eta(W)$$
$$+g(Y,W)\eta(X)\eta(Z)-g(X,W)\eta(Y)\eta(Z) \tag{44}$$

Replacing $X,Y,Z,W$ by $\varphi X,\varphi Y,\varphi Z,\varphi W$ respectively in $(4),$ we get

$$-a_0 R(\varphi^2 X,\varphi^2 Y,\varphi^2 Z,\varphi^2 W)$$
$$=(a_1 A_4+a_4 A_4+a_7 r)g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)$$
$$+(a_2 A_4+a_5 A_4-a_7 r)g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)$$
$$+(a_3 A_4+a_6 A_4)g(\varphi X,\varphi Y)g(\varphi Z,\varphi W) \tag{45}$$

From $(44)$ and $(45)$ , we get

$$\begin{cases} -a_0R(X,Y,Z,W) \\ =(A_4a_1+A_4a_4+a_7r)g(Y,Z)g(X,W) \\ +(A_4a_2+A_4a_5-a_7r)g(X,Z)g(Y,W) \\ +(A_4a_3+A_4a_6)g(X,Y)g(Z,W) \\ -(A_4a_3+A_4a_6)g(X,Y)\eta(Z)\eta(W) \\ -(A_4a_3+A_4a_6)g(Z,W)\eta(X)\eta(Y) \\ -(A_4a_1+A_4a_4+a_7r+a_0)\eta(Y)\eta(Z)g(X,W) \\ -(A_4a_1+A_4a_4+a_7r+a_0)\eta(X)\eta(W)g(Y,Z) \\ -(A_4a_2+A_4a_5+a_7r-a_0)\eta(Y)\eta(W)g(X,Z) \\ -(A_4a_2+A_4a_5+a_7r-a_0)\eta(X)\eta(Z)g(Y,W) \\ +(\sum_{i=1}^{6}a_1)A_4\eta(X)\eta(Y)\eta(Z)\eta(W) \end{cases} \tag{46}$$

for all $X,Y,Z,W \in T(M)$. If $\{e_1,e_2,......,e_{2n},\xi\}$ is a local orthonormal basis of vector fields in $M$. Now on putting $X=W=e_i$ and taking summation over $1 \le i \le 2n$ in $(46)$ we have

$$S(Y,Z)=A_5g(\varphi Y,\varphi Z)+(2n-1)\eta(Y)\eta(Z)$$

where

$$A_5=-\frac{1}{a_0}[(2n-1)(a_1A_4+a_4A_4+a_7r)+A_4(a_2+a_3+a_5+a_6-a_7r)-a_0-a_0(\alpha^2-\beta^2)] \tag{47}$$

Using $(47)$ and $(46)$ in $(28)$, we get

$$\begin{aligned} &T(X,Y,Z,\varphi W) \\ &=(a_1+a_4)(A_5-A_4)g(Y,Z)g(X,\varphi W) \\ &+(a_2+a_5)(A_5-A_4)g(X,Z)g(Y,\varphi W) \\ &+(a_3+a_6)(A_5-A_4)g(X,Y)g(Z,\varphi W) \\ &+((a_3+a_6)A_4+((2n-1)-A_5)a_3)g(Z,\varphi W)\eta(X)\eta(Y) \\ &+((a_1+a_4)A_4+a_7r-a_0+((2n-1)-A_5)a_1)g(X,\varphi W)\eta(Y)\eta(Z) \\ &+((a_2+a_5)A_4-a_7r-a_0+((2n-1)-A_5)a_2)g(Y,\varphi W)\eta(X)\eta(Z) \end{aligned} \tag{48}$$

in (48), we get the required lemma. In view of **lemma 4** we can state that:

**Theorem 4** A trans-Sasakian manifold is $\eta-$ Einstein if is $\varphi-T-$ flat and $\varphi(grad\alpha)=(2n-1)(grad\beta)$.

**Acknowledgment**

The authors are thankful to Professor S. Ahmad Ali Dean School of Applied Sciences, B.B.D. University, Lucknow, providing suggestions for the improvement of this paper.

**References**

[1]D.E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Math. 509, springer Verlag, 1976.
[2]D.E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics, 203 (Boston, MA: Birkhauser Boston, Inc.) (2002).
[3]W. M. Boothby, and H. C. Wang, On contact manifolds, Ann. Math., 68 (1958), 721–734.
[4]J. L. Cabrerizo, L. M. Fernandez, M. Fernandez and G. Zhen, The structure of a class of $K-$contact manifolds, Acta Math. Hungar., 82 (4) (1999), 331–340.
[5]U.C. De, and M.M. Tripathi, Ricci tensor in 3-dimensional trans-Sasakian Manifolds, Kyungpook Math. J., 2, (2005), 247-255.
[6]M. K. Dwivedi, and J.-S. Kim, On conharmonic curvature tensor in $K-$contact and Sasakian manifolds, Bull. Malays. Math. Sci. Soc. 34(1) (2011), 171-180.
[7]L. P. Eisenhart, Riemannian Geometry, Princeton University Press, 1949.
[8]Y. Ishii, On conharmonic transformations, Tensor (N.S.), 7 (1957), 73-80.
[9]J.C. Marrero, The local structure of trans-Sasakian manifold, Ann. Mat. Pura Appl., 162(4)(1992),77-86.
[10]K. Kenmotsu, A class of almost contact Riemannian manifolds, Tohoku Math. J., 24 (1972), 93-103.[52]
[11]K. Nomizu, On hypersurfaces satisfying certain condition on the curvature tensor, Tohoku Math. J., 15, (1963), 148- 161.

[12]K. Ogiue, On fibrings of almost contact manifolds, Kodai Math. Sem. Rep., 17 (1965) 53–62.
[13]J.A. Oubi˜na, New Classes of almost contact metric structures, Pub. Math. Debrecen 32(1985), 187-193.
[14]G. P. Pokhariyal and R. S. Mishra, Curvature tensors and their relativistic significance, Yokohama Math. J., 18 (1970), 105–108.
[15]G. P. Pokhariyal, and R. S. Mishra, Curvature tensors and their relativistic significance II, Yokohama Math. J., 19 (2) (1971), 97–103.
[16]G. P. Pokhariyal, Relativistic significance of curvature tensors, Internat. J. Math. Math. Sci., 5 (1) (1982), 133–139.
[17]B. Prasad, A pseudo projective curvature tensor on a Riemannian manifold, Bull. Calcutta Math. Soc., 94 (3) (2002), 163–166.
[18]R. Prasad, Pankaj, M.M. Tripathi, and S.S. Shukla, On some special type of trans-Sasakian Manifolds, Rev. Mat. Univ. Parma, 8(2), (2009),1-17.
[19]S. Sasaki, On differentiable manifolds with certain structure which are closed related to an almost contact structure, Tohoku Math. Journal. 12 (1960), 459-476.
[20]Z.I. Szabo, Structure theorems on Riemannian space satisfying $R(X,Y).R=0,$ The Local version, J. Diff. Geom., 17, (1982),531-582.
[21]Z.I. Szabo, Classification and contraction of complete hypersurfaces satisfying $R(X,Y).R=0,$ Acta Sci Math., 7, (1984), 321-348.
[22]M. M. Tripathi, and M. K. Dwivedi, The structure of some classes of $K-$contact manifolds,Proc. Indian Acad. Sci. (Math. Sci.),118 (3) (2008), 371-379.
[23]M. M. Tripathi, and P. Gupta, $T-$curvature tensor on a semi-Riemannian manifold, J. Adv.Math. Stud. 4 (1) (2011), 117-129.
[24]M. M. Tripathi, and P. Gupta, $T-$curvature tensor in $K-$contact and sasakian manifolds, International Electronic Journal of Geometry 4 (2011) 32.
[25]K. Yano, Concircular Geometry I. Concircular transformations, Math. Institute, Tokyo Im-perial Univ. Proc., 16 (1940), 195-200.
[26]K. Yano, and S. Bochner, Curvature and Betti numbers, Annals of Mathematics Studies 32,Princeton University Press, 1953.
[27]K. Yano and S. Sawaki, Riemannian manifolds admitting a conformal transformation group,J. Di . Geom., 2 (1968), 161-184.
[28]G. Zhen, J. L. Cabrerizo, L. M. Fernandez, and M. Fernandez, On $\xi-$ conformally flat contactmetric manifolds, Indian J. Pure Appl. Math., 28 (1997), 725-734.
[29]G. Zhen, On conformal symmetric $K-$contact manifolds, Chinese Quart. J. Math., 7 (1992), 5-10.