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Sum of Special Types of Sequences of Numbers

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ABSTRACT
In this article we introduce different proofs to the total sum of special sequences given by muslem mathematicians during the golden age of Islamic civilization.

Introduction

The existence of sequences in Babylonian Mathematics is evident(certain); the archaeologists Otonkipaor has investigated an artifact at the Louvre containing two mathematical issues, one of which identifying a Geometric Sequence equation:

$$1+2+2^2+\cdots 2^9 = 2^9+2^9-1 = 2^{10}-1$$

The second refers to the sum of squares of natural numbers to ten , as follows:

$$1\times 1+2\times 2+\cdots+10\times 10 = 55\left(1\times\frac{1}{3}+10\times\frac{2}{3}\right) = 385$$

On this basis, some scientists believed that the Babylonians knowing that

$$\sum_{i=1}^{n} i^2 = \left(1\times\frac{1}{3}+n\times\frac{2}{3}\right)(1+2+\cdots+n) = \frac{n(n+1)(2n+1)}{6}$$

It is said that the Babylonians were aware of the relationship :

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$$

We would also like to point out that the sum of the first ten natural numbers

$$1+2+3+\cdots+10 = \frac{10(10+1)}{2} = 55$$

is known to Greek mathematicians also . Some scientists says that the relationship

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

is known to Pythagoras and Archimedes .

Note that

$$\sum_{i=1}^{n} i = 1+2+3+........+n$$

and

$$\sum_{i=1}^{n} i = (n-1)+n+\cdots+3+2+1$$

Therefore

$$2\sum_{i=1}^{n} i = (n+1)+(n+1)+.......(n+1)$$

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Hence

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Arabic manuscripts and texts includes many types of sequences (numerical series) , and the rules which gives the general term and the total sum of the sequence. Abu Bakr Muhammad al-Karkhi[1] consider, [6,7] the numerical series $a, a+c, a+2c, \ldots$ He deduce that the general term $g_n = a+(n-1)c$, and the sum of the first, n terms,

$$S_n = \frac{n}{2}(a+g_n)$$

Main Results

Note that al-karkhi is the first Arab prove by induction that

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} i \times \frac{2n+1}{3} = \frac{n(n+1)}{2} \times \frac{2n+1}{3} = \frac{n(n+1)(2n+1)}{6} \qquad (1)$$

Ibn al-Haitham[2] proof of (1), see [3,5,19],as follows

(a) $P_n = (n+1)S_n = S_n^{(2)} + S_n + S_{n-1} + \cdots\cdots + S_1$

where

$$S_n = \sum_{i=1}^{n} i$$

(b) $$S_n^{(2)} = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) By induction: If n=1, then

$$P_1 = (1+1).S_1 = (1+1).1 = 1^2 + 1 = S_1^{(2)} + S_1$$

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1) Al Karkhi (Alkrgi): Abu Bakr Muhammad Ibn al-Hasan was born in Karkh from the outskirts of Baghdad, lived and put the most important production in Baghdad at the end of the tenth and the begining of Eleventh century.He has spent apart of his life in the mountainous areas, where he worked in engineering, this work appeare in his book "About drilling of wells,.He "died in Baghdad in (421 AH = 1020 AD), considered by some as one of the greatest Mathematician who have had a real impact in the progress of Mathematical Sciences, he has several books, including: a book in the Indian account, which speaks for the extraction approximate polynomial roots, and a book in the induction , and Alkafi book which contains rules of the product signs and unknowns, sums of the Algebraic terms and the laws of the last term and the total sum in numerical sequence , and the square root of Algebraic amounts. while in his book Alfkhry in algebra he study many problems, he is the first Arab proved that $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$ and $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.In his book ‘Ellal Algeber wa almukabla” ، he set out rules to solve the equations of the second degree as well as multiplication and division and addition and subtraction rules formulas for two rational numbers and proved those rules algebraically.

(2) Ibn al-Haytham, Abu Ali al-Hasan Ibn al-Hasan, born in Basra, 965 m, and died in Egypt about 1039 m . Alhasan made significant contributions to optics, number theory, geometry, mechanics ,astronomy and natural philosophy. Alhasan derive a formula for the sum of fourth powers, his method can be readily generalized to find the formula for the sum of any integral powers, He used his result on sums of integral powers to perform what would now be called an integration, where the formulas for the sums of integral squares and fourth powers allowed him to calculate the volume of a paraboloid. Alhazen eventually solved the problem using conic sections and a geometric proof. Alhazen explored what is now known as the Euclidean parallel postulate, the fifth postulate in Euclid's *Elements*, using a proof by contradiction, and in effect introducing the concept of motion into geometry. He formulated the Lambert quadrilateral, which Boris Abramovich Rozenfeld names the "Ibn al-Haytham–Lambert quadrilateral". His theorems on quadrilaterals, including the Lambert quadrilateral, were the first theorems on elliptical geometry and hyperbolic geometry. These theorems, along with his alternative postulates, such as Playfair's axiom, can be seen as marking the beginning of non-Euclidean geometry. His work had a considerable influence on its development among the later Persian geometers Omar Khayyám and Nasīr al-Dīn al-Tūsī, and the European geometers Witelo, Gersonides, and Alfonso. His contributions to number theory include his work on perfect numbers. In his *Analysis and Synthesis*, Alhazen may have been the first to state that every even perfect number is of the form $2^{n-1}(2^n - 1)$ where $2^n - 1$ is prime, but he was not able to prove this result successfully (Euler later proved it in the 18th century).

Alhazen solved problems involving congruences using what is now called Wilson's theorem. In his *Opuscula*, Alhazen considers the solution of a system of congruences, and gives two general methods of solution. His first method, the canonical method, involved Wilson's theorem, while his second method involved a version of the Chinese remainder theorem.

and the result is true when n=1.If n=2,we have

$$P_2=(1+2)S_2=(1+2)(1+2)=2^2+1^2+(2+1)+1=S_2^{(2)}+S_2+S_1$$

and the result is true when n=2. Now suppose that

$$P_m=(m+1)S_m=S_m^{(2)}+S_m+S_{m-1}+\cdots\cdots+S_1$$

Then

$$P_{m+1}=[(m+1)+1]S_{m+1}=(m+1)S_{m+1}+S_{m+1}=[S_m+(m+1)](m+1)+S_{m+1}$$

Therefore

$$P_{m+1}=(m+1)S_m+(m+1)^2+S_{m+1}=S_m^{(2)}+S_m+S_{m-1}+\cdots\cdots+S_1+(m+1)^2+S_{m+1}/$$

$$=S_m^{(2)}+(m+1)^2+S_{m+1}+S_m+S_{m-1}+\cdots\cdots+S_1=S_{m+1}^{(2)}+S_{m+1}+S_m+\cdots\cdots+S_1$$

and the result is true when n=m+1.Hence

$$P_n=(n+1)S_n=S_n^{(2)}+S_n+S_{n-1}+\cdots+S_1$$

is true for all $n\in\aleph$

(b) Since

$$(n+1)S_n=S_n^{(2)}+S_n+S_{n-1}+\cdots\cdots+S_1$$

$$=S_n^{(2)}+\frac{1}{2}\left[n(n+1)+n(n-1)+\cdots\cdots+2(2+1)+1(1+1)\right]$$

$$=S_n^{(2)}+\frac{1}{2}\left[n(n+1)+[(n-1)+1](n-1)+\cdots\cdots+2(2+1)+1(1+1)\right]$$

$$=S_n^{(2)}+\frac{1}{2}\left[n^2+(n-1)^2+\cdots\cdots+2^2+1^2+n+(n-1)+\cdots\cdots+2+1\right]$$

$$=S_n^{(2)}+\frac{1}{2}\left[S_n^{(2)}+S_n\right]=\frac{3}{2}S_n^{(2)}+\frac{1}{2}S_n$$

it follows,

$$\left(n+\frac{1}{2}\right)S_n=\frac{3}{2}S_n^{(2)}$$

Hence,

$$S_n^{(2)}=\frac{2}{3}\left(n+\frac{1}{2}\right)S_n=\frac{2}{3}\left(n+\frac{1}{2}\right).\frac{n(n+1)}{2}=\frac{n(n+1)(2n+1)}{6}$$

The Moroccan al-sumaal[1] proofs of (1) (see, albaher fi algeber, [8]) , as follows:

(a)

$$(n+2)\sum_{i=1}^{n}i=(n+2)\times\frac{n(n+1)}{2}=n\left[\sum_{i=1}^{n}i+(n+1)\right]=n\sum_{i=1}^{n}i$$

(b) $(n+1)(n+2)=(n+1)^2+(n+1)$

Hence

$$(n+2)(n+1)+n(n+1)=(n+1)^2+(n+1)+n(n+1)$$

$$=(n+1)^2+(n+1)(n+1)=2(n+1)^2,\ \forall n\in\aleph$$

(c)Since,

$$n\sum_{i=1}^{n+1}i=n\left(\sum_{i=1}^{n-2}i+3n\right)$$

Hence

$$n\sum_{i=1}^{n+1}i=n\sum_{i=1}^{n-2}i+3n^2$$

To prove that (1), note that

$$\left(2n+1\right)\sum_{i=1}^{n} i = n\sum_{i=1}^{n+1} i + \left(n+1\right)\sum_{i=1}^{n-1} i$$

but

$$(n+1)\sum_{i=1}^{n-1} i = (n-1)\sum_{i=1}^{n} i = (n-1)\sum_{i=1}^{n-3} i + 3(n-1)^2$$

by(c). However

$$n\sum_{i=1}^{n+1} i = n\sum_{i=1}^{n-2} i + 3n^2$$

by(c). There

$$\left(2n+1\right)\sum_{i=1}^{n} i = 3n^2 + 3\left(n-1\right)^2 + n\sum_{i=1}^{n-2} i + \left(n-1\right)\sum_{i=1}^{n-3} i$$

$$\left(2n+1\right)\sum_{i=1}^{n} i = 3n^2 + 3\left(n-1\right)^2 + 3\left(n-2\right)^2 + \left(n-2\right)\sum_{i=1}^{n-4} i \left(n-3\right)\sum_{i=1}^{n-5} i$$

From (a),(b)and(c)we get

$$\left(2n+1\right)\sum_{i=1}^{n} i = 3n^2 + 3(n-1)^2 + \ldots + 3\times 2^2 + 3\times 1^2 = 3\sum_{i=1}^{n} i^2$$

Hence

$$\sum_{i=1}^{n} i^2 = \frac{2n+1}{3}\times\sum_{i=1}^{n} i = \frac{n\left(n+1\right)\left(2n+1\right)}{6}$$

Ibn al-Banna al-murrakishi[3] , proof of (1) see his book (Rafue alhigab en wiguh (faces) aemal alhisab "[9,10,11] , as follows:

Let

$$a_1 = \frac{1^2}{1},$$

$$a_2 = \frac{1^2+2^2}{1+2} = \frac{5}{3},$$

$$a_3 = \frac{1^2+2^2+3^2}{1+2+3} = \frac{7}{3}$$

$$a_n = \frac{\sum_{i=1}^{n} i^2}{\sum_{i=1}^{n} i}$$

Therefore

$a_1, a_2, a_3, \ldots\ldots a_n$ is an arithmetic series its first term is one ,its basis $\frac{2}{3}$ and the last term is (a_n)

Hence

[1)]Ibn al-Banna al-Murrakishi (1256-1321) Ahmed Ibn Mohammed Ibn Uthman al-Azdi, also known as Abu al-Abbas , was born in Marrakesh , his father was builder , excelled in mathematics, astronomy and Islamic scholar. Ibn al-Banna' wrote a large number of treatises, encompassing such varied topics as Algebra, Astronomy, Linguistics, Rhetoric, and Logic. One of his works, called *ʿamal al-ḥisāb*, includes topics such as fractions, sums of squares and cubes etc .Yet another work by Ibn al-Banna' was *Rafʿ al-Ḥijāb (Lifting the Veil)* which included topics such as computing square roots of a number and theory of continued fractions.

$$a_n = 1 + (n-1) \times \frac{2}{3} = \frac{2n+1}{3}$$

And

$$\frac{\sum_{i=1}^{n} i^2}{\sum_{i=1}^{n} i} = \frac{2n+1}{3}$$

Therefore

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} i \times \left(\frac{2n+1}{3} \right)$$

Hence

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

The relation $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2 = \left[\frac{n(n+1)}{2} \right]^2$ (2)

Proved (by induction) by al_karkhi,while Ibn al-Haytham, as follows:

$$(n+1)S_n^{(2)} = nS_n^{(2)} + S_n^{(2)}$$

But

$$nS_n^{(2)} = n\left(\sum_{i=1}^{n} i^2 \right) = n\left(n^2 + \sum_{i=1}^{n-1} i^2 \right)$$

$$= n^3 + n\sum_{i=1}^{n-1} i^2 = n^3 + \left[(n-1)+1 \right] \sum_{i=1}^{n-1} i^2 = n^3 + (n-1)\sum_{i=1}^{n-1} i^2 + \sum_{i=1}^{n-1} i^2$$

$$= n^3 + (n-1)^3 + (n-1)\sum_{i=1}^{n-2} i^2 + S_{n-1}^{(2)} = n^3 + (n-1)^3 + S_{n-1}^{(2)} + \left[(n-2)+1 \right] S_{n-2}^{(2)}$$

$$= n^3 + (n-1)^3 + \cdots\cdots + 2^3 + 1^3 + \sum_{i=1}^{n-1} S_i^{(2)} = S_n^{(3)} + \sum_{i=1}^{n-1} S_i^{(2)}$$

Therefore

$$(n+1)S_n^{(2)} = S_n^{(3)} + \sum_{i=1}^{n} S_i^{(2)} = S_n^{(3)} + \frac{1}{3}S_n^{(3)} + \frac{1}{2}S_n^{(2)} + \frac{1}{6}S_n$$

and

$$S_n^{(3)} = \frac{3}{4}\left(n + \frac{1}{2} \right) S_n^{(2)} - \frac{1}{8}S_n = \frac{3}{4}\left(n + \frac{1}{2} \right) \frac{n(n+1)(2n+1)}{6} - \frac{1}{8} . \frac{n(n+1)}{2}$$

$$= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 = \left[\frac{n(n+1)}{2} \right]^2 = \left(\sum_{i=1}^{n} i \right)^2$$

Al-sumaal al-maghrebi give in his book(albahir fi algaber) the following

Proof

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} \Leftrightarrow 2\sum_{i=1}^{n-1} i = n\ (n-1) \Leftrightarrow 2n\sum_{i=1}^{n-1} i = n^3 - n^2 \Leftrightarrow n^2 + 2\sum_{i=1}^{n-1} i = n^3 \ (1)$$

But ,

$$\left(\sum_{i=1}^{n} i \right)^2 = \left(n + \sum_{i=1}^{n-1} i \right)^2$$

Hence

$$\left(\sum_{i=1}^{n} i\right)^2 = n^2 + 2n\sum_{i=1}^{n-1} i + \left(\sum_{i=1}^{n-1} i\right)^2$$

Therefore,

$$\left(\sum_{i=1}^{n} i\right)^2 = n^3 + (n-1)^3 + \left(\sum_{i=1}^{n-2} i\right)^2$$

$$\left(\sum_{i=1}^{n} i\right)^2 = n^3 + (n-1)^3 + (n-2)^3 + \left(\sum_{i=1}^{n-3} i\right)^2 \text{ (by (1)).}$$

Therefore

$$\left(\sum_{i=1}^{n} i\right)^3 = n^3 + (n-1)^3 + (n-2)^3 + \cdots + 2^3 + 1^3 = \sum_{i=1}^{n} i^3 ,$$

But

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} .$$

Hence

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2 = \left[\frac{n(n+1)}{2}\right]^2$$

The relation

$$S_n^{(4)} = \sum_{i=1}^{n} i^4 = \frac{1}{5}\left[6\sum_{i=1}^{n} i - 1\right] \times \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \qquad (3)$$

Proved by Abu Sager Alqubaisi[4] (Amathematician of eleventh century) and Abu mansur Abd alQahir Albaghdadi[5] in(Altakmila fil Hisab) [23], Ibn al-Haytham and Alkashi[6] in *Key to Arithmetic*

Ibn al-Haytham proofs is as follows:

$$(n+1)S_n^{(3)} = S_n^{(3)} + nS_n^{(3)} = S_n^{(3)} + n\left(n^3 + S_{n-1}^{(3)}\right) = S_n^{(3)} + n^4 + nS_{n-1}^{(3)}$$

$$= S_n^{(3)} + n^4 + [(n-1)+1]S_{n-1}^{(3)} = S_n^{(3)} + n^4 + (n-1)S_{n-1}^{(3)} + S_{n-1}^{(3)}$$

$$= S_n^{(3)} + S_{n-1}^{(3)} + n^4 + (n-1)^4 S_{n-2}^{(3)}$$

$$= S_n^{(3)} + S_{n-1}^{(3)} + \cdots + S_2^{(3)} + S_1^{(3)} + n^4 + (n-1)^4 + \cdots + 2^4 + 1^4 = \sum_{i=1}^{n} S_i^{(3)} + S_n^{(4)}$$

But

$$\sum_{i=1}^{n} S_i^{(3)} = \frac{1}{4}S_n^{(4)} + \frac{1}{2}S_n^{(3)} + \frac{1}{4}S_n^{(2)}$$

Hence, Therefore

$$\left(n + \frac{1}{2}\right)S_n^{(3)} - \frac{1}{4}S_n^{(2)} = \frac{5}{4}S_n^{(4)}$$

and so

4) al-Qubaisi : Abdulaziz bin Usman al-Hashemi , astronomer and mathematician, writers and poets , attributed to Qabisa near Mosul or Samarra,. He wrote "The entrance to the industry the provisions of the Stars" ,

5) Abu Mansur 'Abd al-Qahir ibn Tahir al-Baghdadi: Born and raised in Baghdad. He wrote the treatise *al-Takmila fi'l-Hisab* which contains results in number theory, and comments on works by al-Khwarizmi

6) al-Kashi: Jamshid Ibn Mahmoud Ibn Masood, nicknamed, Ghiyāth al-Dīn was born in Kashan ment 654 AH, and lived in Samarkand, mathematician, astronomy, physics. He is the first to introduce decimal sign in Arithmetic operations.He wrote many letters and books in mathematics, astronomy including: “Alrisala Almuhitiya”in which he calculate the ratio between the circumference of a circle and its diameter $\pi = 3.14259265358979325$ and “mifthah Alhisab” containing some of his discoveries in Arithmetics, and a book”Nuzhat Alhadaek” in which he show that the orbits of the moon and Mercury are elliptical rather than circular.

$$S_n^{(4)} = \frac{4}{5}\left(n+\frac{1}{2}\right)\cdot\frac{n^2(n+1)^2}{4} - \frac{1}{5}\cdot\frac{n(n+1)(2n+1)}{6}$$

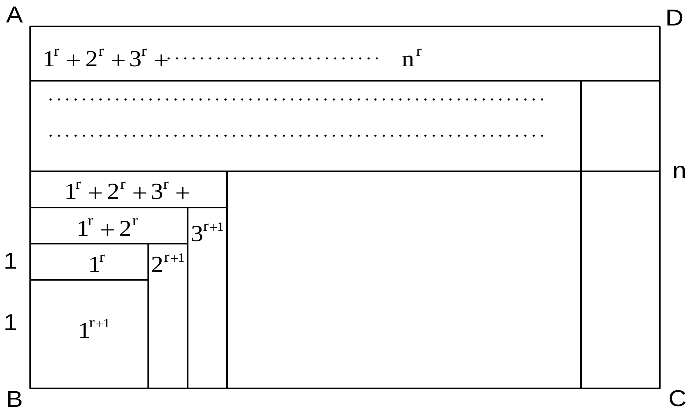
$$= \frac{1}{5}\cdot\frac{n(n+1)(2n+1)}{6}\left[3n(n+1)-1\right]$$

$$= \frac{1}{5}\sum_{i=1}^{n} i^2\left[6\sum_{i=1}^{n} i - 1\right] = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Note that Ibn al-Haytham prove the following

$$(n+1)\sum_{i=1}^{n} i^r = \sum_{i=1}^{n} i^{r+1}\left(\sum_{i=1}^{n} i^r\right) \forall r \in N$$

as follows



The area of the rectangle ABCD equal the areas of the interior rectangles. Therefore

$$(n+1)(1^r+2^r+\cdots+n^r) = 1\times 1^r + 2\times 2^r + \cdots + n\times n^r + 1^r + (1^r+2^r) + \cdots + (1^r+2^r+\cdots n^r)$$

$$(n+1)(1^r+2^r+\cdots+n^r) = 1^{r+1} + 2^{r+1} + \cdots + n^{r+1} + 1^r + (1^r+2^r) + \cdots + (1^r+2^r+\cdots n^r)$$

Hence

$$(n+1)\sum_{i=1}^{n} i^r = \sum_{i=1}^{n} i^{r+1} + \sum_{m=1}^{n}\left(\sum_{i=1}^{m} i^r\right)$$

Note that:

If n=1,then

$$(n+1)\sum_{i=1}^{n} i = \sum_{i=1}^{n} i^2 + \sum_{m=1}^{n}\left(\sum_{i=1}^{m} i\right)$$

Hence

$$\frac{(n+1)^2}{2} = \sum_{i=1}^{n} i^2 + 1 + (1+2) + \cdots + (1+2+\cdots+n)$$

Therefore

$$\frac{(n+1)^2}{2} = \sum_{i=1}^{n} i^2 + 1 + 3 + 6 + \cdots + \frac{n(n+1)}{2}$$

And

$$\frac{(n+1)^2}{2} = \sum_{i=1}^{n} i^2 + \frac{n(n+1)(n+2)}{6}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)^2}{2} - \frac{n(n+1)(n+2)}{6} = \frac{n(n+1)(2n+1)}{6}$$

If n=2,then

$$(n+1)\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} i^3 + \sum_{m=1}^{n}\left(\sum_{i=1}^{m} i^2\right)$$

But

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Hence

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Now if we take from the following series

$1, 1+c, 1+2c, \ldots, 1+(n-1)c$

the following series :1,1+(1+c),1+(1+c)+(2+c),…

That is

$1, 2+c, 3+3c, 4+4c, \ldots$

which is a new series , its general term is

$$S = \sum_{i=1}^{n}\left[i + \frac{i(i-1)c}{2}\right] = \frac{n(n+1)}{2} + \frac{c}{2} \times \frac{n(n-1)(n+1)}{3} = \frac{n(n+1)}{2}\left[1 + \frac{(n-1)c}{3}\right]$$

This what we find in " *al-Takmila fi'l-Hisab"* to Ibn al-Qahir al-Baghdadi ,and Ibn Yaish Alumawi *"Marasim al-intisab fi'ilm al-hisab* ("On arithmetical rules and procedures"), It is usual to regard al-Umawi as a 14th century mathematician , and if c=1 , we get the sequence of the trigonometric numbers 1, 3, 6, 10, 15, 21, …

Which is affiliated to Phythagorian, and its general term is

$$g_n = \frac{n(n+1)}{2}$$

and the sum of the first n terms is

$$S_n = \frac{n(n+1)(n+2)}{6} \cdot$$

If c=2, we get what we called, the square numbers 1, 4, 9, 16, …. ,its general term is n^2 and the sum of the first n terms is

$$S_n = \frac{n(n+1)(2n+1)}{6}$$

If c=3, we get the so-called numbers Almkhmsat 1, 5, 12, 22…

its general term is

$$g_n = \frac{n(3n-1)}{2}$$

and the sum of its first n terms is

$$S_n = \frac{n^2(n+1)}{6}$$

Now if we take from the series $1, 2+c, 3+3c, 4+6c, \cdots$

the first term,first + second term,first + second + third term,and so on. This is:

$1, 3+c, 6+4c, 10+10c, \cdots$

Which the so-called hierarchical sequence numbers , which studied by al-Baghdadi, Ibn Yaish al-umawi and Ibn al banah al-marrakichi. Ibn Yaish al-umawi,[22], shows that the general term of this series is

$$g_n = \frac{n(n+1)[3+(n-1)c]}{6}$$

and the sum of the first n terms is

$$S_n = \frac{n(n+1)(n+2)[4+(n-1)c]}{24}$$

Finally we would like to point out that both al-umawi and al- Kashi emerging sequences such as i(i+1). That is like

$1 \times 2, 2 \times 3, \ldots$

and they give the following rules:

$$1 \times 2 + 2 \times 3 + \cdots + n(n-1) = \frac{n(n+1)(n+2)}{3}$$

$$1\times 3+3\times 5+\cdots+(2n-1)(2n+1)=\frac{(2n-1)(2n+1)(2n+3)}{6}+\frac{1}{2}$$

$$2\times 4+4\times 6+\cdots 2n(2n+2)=\frac{4n(n+1)(n+2)}{3}$$

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