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Complement of the Boolean Function Graph $B(K_p, INC, \overline{K}_q)$ of a graph

S. Muthammai[1,*] and T.N. Janakiraman[2]

[1]Government Arts College for Women, Pudukkottai- 622 001, India.

[2]National Institute of Technology, Tiruchirappalli – 620 015, India.

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ABSTRACT

For any graph G, let V(G) and E(G) denote the vertex set and edge set of G respectively. The Boolean function graph $B(K_p, INC, \overline{K}_q)$ of G is a graph with vertex set $V(G)\cup E(G)$ and two vertices in $B(K_p, INC, \overline{K}_q)$ are adjacent if and only if they correspond to two adjacent vertices of G, two nonadjacent vertices of G or to a vertex and an edge incident to it in G, For brevity, this graph is denoted by $\overline{B}_4(G)$. In this paper, structural properties of the complement $\overline{B}_4(G)$ of $B_4(G)$ including eccentricity properties are studied. Also, domination number and neighborhood number are found.

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. A graph with p vertices and q edges is denoted by G(p, q). If v is a vertex of a connected graph G, its eccentricity e(v) is defined by $e(v) = \max \{d_G(v, u): u \in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. The minimum and maximum eccentricities are the radius and diameter of G, denoted r(G) and diam(G) respectively. When diam(G) = r(G), G is called a self-centered graph with radius r, equivalently G is r-self-centered. A connected graph G is said to be geodetic, if a unique shortest path joins any two of its vertices.

A vertex and an edge are said to cover each other, if they are incident. A set of vertices, which covers all the edges of a graph G is called a point cover for G. The smallest number of vertices in any point cover for G is called its point covering number and is denoted by $\alpha_0(G)$ or α_0. A set of vertices in G is independent, if no two of them are adjacent. The largest number of vertices in such a set is called the point independence number of G and is denoted by $\beta_0(G)$ or β_0.

Sampathkumar and Neeralagi [12] introduced the concept of neighborhood sets in graphs. A subset S of V(G) is a neighborhood set (n-set) of G, if $G = \cup_{v\in S}(<N[v]>)$, where <N[v]> is the subgraph of G induced by N[v]. The neighborhood number $n_0(G)$ of G is the minimum cardinality of an n-set of G.

The concept of domination in graphs was introduced by Ore [2]. A set $S \subseteq V$ is said to be a dominating set in G, if every vertex in V– S is adjacent to some vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. A dominating set with cardinality $\gamma(G)$ is referred as a γ-set. A dominating set S of a graph G is called an independent dominating set of G, if the induced subgraph <S> is independent.The minimum cardinality of an independent dominating set of G is called the independent domination number of G and is denoted by γ_i.

Whitney[14] introduced the concept of the line graph L(G) of a given graph G in 1932. The first characterization of line graphs is due to Krausz. The Middle graph M(G) of a graph G was introduced by Hamada and Yoshimura[6]. Chikkodimath and Sampathkumar [5] also studied it independently and they called it, the semi-total graph $T_1(G)$ of a graph G. Characterizations were presented for middle graphs of any graph, trees and complete graphs in [3]. The concept of total graphs was introduced by Behzad [4] in 1966. Sastry and Raju[13] introduced the concept of quasi-total graphs and they solved the graph equations for line graphs, middle graphs, total graphs and quasi-total graphs. Janakiraman et al., introduced the concepts of Boolean and Boolean function graphs [7-11].

The points and edges of a graph are called its elements. Two elements of a graph are neighbors, if they are either incident or adjacent. The Boolean Function graph $B(K_p, INC, \overline{K}_q)$ of G is a graph with vertex set $V(G)\cup E(G)$ and two vertices in $B(K_p, INC, \overline{K}_q)$ are adjacent if and only if they correspond to two adjacent vertices of G, two nonadjacent vertices of G or to a vertex and an edge incident to it in G. For brevity, this graph is denoted by $B_4(G)$. Two vertices in $\overline{B}_4(G)$ are adjacent if and only if they correspond to two adjacent edges of G, two nonadjacent edges of G or to a vertex and an edge not incident to it in G. In this paper, structural properties of the complement $\overline{B}_4(G)$ of $B_4(G)$ including eccentricity properties are studied. Domination and neighborhood numbers of $\overline{B}_4(G)$ are also found. For graph theoretic terminology, Harary [1] is referred.

Tele:
E-mail address: muthammai.sivakami@gmail.com

2. Previous Results

Observation 2.1. [11]

1. K_p is an induced subgraph of $B_4(G)$ and subgraph of $B_4(G)$ induced by q vertices is totally disconnected.
2. Number of vertices in $B_4(G)$ is p + q, since $B_4(G)$ contains vertices of both G and the line graph L(G) of G.
3. Number of edges in $B_4(G)$ is (p(p-1)/2) + 2q
4. For every vertex $v \in V(G)$, $d_{B4(G)}(v) = p - 1 + d_G(v)$

(a) If G is complete, then $d_{B4(G)}(v) = 2(p - 1)$.

(b) If G is totally disconnected, then $d_{B4(G)}(v) = p - 1$.

(c) If G has atleast one edge, then $2 \le d_{B4(G)}(v) \le 2(p - 1)$ and $d_{B4(G)}(v) = 1$ if and only if $G \cong 2K_1$.

5. For an edge $e \in E(G)$, $d_{B4(G)}(e) = 2$.
6. $B_4(G)$ is always connected.
7. Let G be a (p, q) graph with atleast one edge. If p is odd, then $B_4(G)$ is Eulerian if and only if G is Eulerian.
8. If G is r-regular ($r \ge 1$ and is odd), then $B_4(G)$ is Eulerian.
9. For any graph G, $B_4(G)$ is Hamiltonian if and only if
   (i) G is a path or a cycle on atleast three vertices
   (ii) Each component of $B_4(G)$ is K_1, K_2 or P_m, $m \ge 3$.
10. $B_4(G)$ is self-centered with radius 2 if and only if G is either C_3 or $C_3 \cup nK_1$, $n \ge 1$.
11. $B_4(G)$ is bi-eccentric with radius 1 and diameter 2, if and only if G is either $K_{1,n}$ or $K_{1,n} \cup mK_1$, $n \ge 2$, $m \ge 1$.
12. $B_4(G)$ is bi-eccentric with radius 2 and diameter 3, if and only if $\beta_0(G) \ge 2$.

3. Main Results

In this section, the properties of $\overline{B}_4(G)$ including traversability and eccentricity properties are studied.

Observation 3.1.

1. K_p is an induced subgraph of $\overline{B}_4(G)$ and the subgraph of $\overline{B}_4(G)$ induced by q vertices is totally disconnected.
2. Number of edges in $\overline{B}_4(G)$ is $(q(q-1)/2) + q(p-2)$
3. For every vertex $v \in V(G)$, degree v in $\overline{B}_4(G)$ is $q - d_G(v)$.
4. For an edge $e \in E(G)$, degree of e in $\overline{B}_4(G)$ is $p + q - 3$.
5. For any connected graph G with p vertices, $\overline{B}_4(G)$ is disconnected if and only if G is a star on p vertices**.**
6. If there exists a vertex $v \in V(G)$ such that v is not incident with exactly one edge in G, then v is a pendant vertex in $\overline{B}_4(G)$.
7. If v is a vertex in G such that v is incident with all the edges of G, then v is isolated in $\overline{B}_4(G)$.
8. If G is a graph with atleast three vertices, then each vertex of $\overline{B}_4(G)$ lies on a triangle and hence girth of $\overline{B}_4(G)$ is 2.
9. If G is a graph with atleast four vertices and atleast one edge, then $\overline{B}_4(G)$ is bi-regular if and only if G is regular and is regular if and only if G is totally disconnected.
10. If G is a graph with atleast three vertices, then $\overline{B}_4(G)$ has no cut vertices.
11. If G has atleast one edge, then vertex connectivity of $\overline{B}_4(G)$ is equal to the edge connectivity of $\overline{B}_4(G) = 2$.
12. Let G be a (p, q) graph with atleast one edge. If p is odd, then $\overline{B}_4(G)$ is Eulerian if and only if G is Eulerian.
13. If G is r-regular ($r \ge 1$ and is odd), then $\overline{B}_4(G)$ is Eulerian.
14. For any graph G, $\overline{B}_4(G)$ is geodetic if and only if G is either K_2 or nK_1, $n \ge 2$.
15. If G is a graph with atleast four vertices, then $\overline{B}_4(G)$ is P_4- free.

In the following, a necessary and sufficient condition for $\overline{B}_4(G)$ to be Hamiltonian is proved.

Theorem 3.1.

Let G be any (p. q) graph which is not a star. Then $\overline{B}_4(G)$ is hamiltonian if and only if $q \ge p$ and $deg_G(v) \le 2$, for all v in G.

Proof.

Let G be any (p, q) graph such that $q \ge p$ and $deg_G(v) \le 2$, for all v in G.

Therefore, $deg_{\overline{B}4(G)}(v) = q - deg_G(v) \ge 2$. That is, for every vertex v in G, there exist atleast two edges in G not incident with v. By the construction of $\overline{B}_4(G)$, each vertex of L(G) in $\overline{B}_4(G)$ is adjacent to (p–2) vertices of G in $\overline{B}_4(G)$. A cycle of length 2p in $B_4(G)$ can be formed with p vertices of G and p vertices of L(G), each vertex of G followed by a vertex of L(G), that is, $v_1e_1v_2e_2 \ldots v_pe_pv_1$, where $e_i \in E(G)$ is not incident with v_i, v_{i+1}, i = 1, 2, …, (p–1) and e_p is not incident with v_p and v_1. Since each pair of vertices in L(G) is adjacent in $\overline{B}_4(G)$, the remaining (q–p) vertices of L(G) in $\overline{B}_4(G)$ can be placed suitably in the above cycle C_{2p}. Therefore, there exists a Hamiltonian cycle in $\overline{B}_4(G)$ and hence $\overline{B}_4(G)$ is Hamiltonian.

Conversely, let $\overline{B}_4(G)$ be Hamiltonian. Therefore degree of each vertex in $\overline{B}_4(G)$ must be atleast 2. That is, $deg_{\overline{B}4(G)}(v) \ge 2$, for all v in G, which implies $q - deg_G(v) \ge 2$.

That is, $deg_G(v) \le q - 2$.

Assume q < p. Since neither G nor $\overline{G}$ is a subgraph of $\overline{B}_4(G)$ and each vertex of G in $\overline{B}_4(G)$ is adjacent to (p–2) vertices of L(G) in $\overline{B}_4(G)$ in the hamiltonian cycle each vertex of G is followed by a vertex of L(G). This is not possible, if q < p. Therefore $q \ge p$.

Remark 3.1.

1. If G is a graph obtained by attaching pendant edges at a vertex of C_3, then $\overline{B}_4(G)$ contains a Hamiltonian path.
2. If G is a graph obtained by subdividing an edge of a star, then $\overline{B}_4(G)$ contains a hamiltonian path.

Theorem 3.2.

If G is a tree which is not a star, then $\overline{B}_4(G)$ contains a hamiltonian path.

Proof.

Let G be a tree which is not a star. Then $q = p–1$. Since G is not a star, $\overline{B}_4(G)$ contains no isolated vertices. As in Theorem 3.1., there exists a Hamiltonian path in $\overline{B}_4(G)$, $v_1e_1v_2e_2…v_{p-1}e_{p-1}v_p$, where $e_i \in E(G)$ is not incident with v_i, v_{i+1}, $i = 1, 2, …, (p–1)$.

Remark 3.2.

Let G be any graph such that $\overline{B}_4(G)$ is connected and $q = p – 1$. Then $\overline{B}_4(G)$ contains a Hamiltonian path.

In the following, point (line) covering number, (edge) independence number, chromatic number and neighborhood number are found for $\overline{B}_4(G)$.

Theorem 3.3.

Let G be any (p, q) graph with atleast three vertices and not totally disconnected.
Then $\alpha_0(\overline{B}_4(G)) = q$.

Proof.

Since the subgraph of $\overline{B}_4(G)$ induced by vertices of L(G) in $\overline{B}_4(G)$ is complete, $\alpha_0(K_q) = q –1$. Let $e_1, e_2, …, e_q$ be the q vertices of L(G) in $\overline{B}_4(G)$. Then $D = \{e_1, e_2, …, e_{q-1}\}$ is a line cover of K_q in $\overline{B}_4(G)$. The vertex $e_q \in V(\overline{B}_4(G))$ covers the edges in $\overline{B}_4(G)$ of the from (v, e_q), where $e_q \in E(G)$ is not incident with $v \in V(G)$. Therefore $D \cup \{e_q\} \subseteq V(\overline{B}_4(G))$ is a minimum point cover of $\overline{B}_4(G)$.

Remark 3.3.

Since $\alpha_0(\overline{B}_4(G)) + \beta_0(\overline{B}_4(G)) = p + q$, $\beta_0(\overline{B}_4(G)) = p$ and hence $\alpha_0(\overline{B}_4(G)) = \beta_0(\overline{B}_4(G))$ if and only if $p = q$.

Theorem 3.4.

Let G be a (p, q) graph which is not totally disconnected and is not a star. Then line covering number $\alpha_1(\overline{B}_4(G))$ is given by

$$\alpha_1(\overline{B}_4(G)) = \begin{cases} \min\{ q, \lceil (p + 2q)/2 \rceil, & \text{if } q \geq p. \\ p, & \text{if } q < p. \end{cases}$$

Proof

Case 1: $q \geq p$

Subcase 1.a: q is even

Let $v_1, v_2, …, v_p$ be the vertices of G in $\overline{B}_4(G)$ and $e_{jk} = (v_j, v_k)$ be an edge in G and $e_{jk} \in V(\overline{B}_4(G))$. Since $q \geq p$, for each edge $e \in E(G)$, there exists a vertex $v_i \in V(G)$ not incident with e. Then $\{(v_i, e_{jk}) \in E(\overline{B}_4(G)) : e_{jk} \in E(G)$ is not incident with $vi \in V(G)$ and e_{jk} are distinct$\}$ is a line cover for $\overline{B}_4(G)$. Therefore, $\alpha_1(\overline{B}_4(G)) \leq q$. Since the subgraph of $\overline{B}_4(G)$ induced by q vertices of L(G) in $\overline{B}_4(G)$ is complete, $\alpha_1(K_q) = q/2$ and p vertices of $\overline{B}_4(G)$ are covered by the edges (v_i, e_{jk}), $i = 1, 2, …, p$ in $\overline{B}_4(G)$. Therefore, $\alpha_1(\overline{B}_4(G)) \leq p + q/2$ and hence $\alpha_1(\overline{B}_4(G)) \leq \min \{q, p+ q/2\} = \min \{q, (2p+q)/2\}$.

Subcase 1.b: q is odd

Let $e_1, e_2, …, e_q$ be the q vertices of L(G) in $\overline{B}_4(G)$. Then $<\{ e_1, e_2, …, e_{q-1\}>} \cong K_{q-1}$ in $\overline{B}_4(G)$ and $\alpha_1(K_{q-1}) = (q–1)/2$. Let $v_p \in V(G)$ be not incident with $e_q \in E(G)$. Then the edges (v_p, e_q), (v_i, e_{jk}), $i = 1, 2, …, p−1$ cover the p vertices and the vertex e_q in $\overline{B}_4(G)$. Therefore, $\alpha_1(\overline{B}_4(G)) = p + (q–1)/2 = 2p + q – 1)/2$ and hence $\alpha_1(\overline{B}_4(G)) = \min \{ q, (2p + q– 1)/2$. Combining Subcase 1.1. and 1.2.,

$$\alpha_1(\overline{B}_4(G)) = \min\{ q, \lceil (p + 2q)/2 \rceil, \quad \text{if } q \geq p.$$

Case 2: $q < p$

Then the edges $(v_i, e_{jk}) \in E(\overline{B}_4(G))$, $i = 1, 2, …, p$, where $e_{jk} \in E(G)$ is not incident with $v_i \in E(G)$, cover the vertices of $\overline{B}_4(G)$. Therefore, $\alpha_1(\overline{B}_4(G)) = p$.

Remark 3.4.

Since $\alpha_1(\overline{B}_4(G)) + \beta_1(\overline{B}_4(G)) = p + q$ for a totally disconnected graph G and is not a star, $\beta_1(\overline{B}_4(G))$ is given by

$$\beta_1(\overline{B}_4(G)) = \begin{cases} \min \{ p, \lceil (q + 1)/2 \rceil, & \text{if } q \geq p \\ q, & \text{if } q < p \end{cases}$$

Remark 3.5.

When $G \cong K_{1,n}$, $(n \geq 2)$, $\overline{B}_4(K_{1,n})$ contains an isolated vertex and $\beta_1(\overline{B}_4(K_{1,n})) = n$.

In the following, chromatic number of $\overline{B}_4(G)$ is found.

Theorem 3.5.

Let G be a (p, q) graph. Then

$$\chi(\overline{B}_4(G)) = \begin{cases} q, & \text{if } \delta(G) \geq 1 \\ q + 1, & \text{if } \delta(G) = 0 \end{cases}$$

Proof.

Case 1: $\delta(G) \geq 1$

The subgraph of $\overline{B}_4(G)$ induced by all the vertices of L(G) in $\overline{B}_4(G)$ is a complete graph on q vertices and $\chi(K_q) = q$. That is, vertices of L(G) in $\overline{B}_4(G)$ are coloured with q colours. Let $v_i \in V(G)$ and $e_i \in E(G)$ be an edge incident with v_i. Since $\delta(G) \geq 1$, such an edge exists in G. Then $v_i, e_i \in V(\overline{B}_4(G))$. Now colour the vertex v_i in $\overline{B}_4(G)$ by the colour given to e_i in $\overline{B}_4(G)$, since v_i and e_i are independent vertices in $\overline{B}_4(G)$.Since no two vertices of G are adjacent in $\overline{B}_4(G)$, $\chi(\overline{B}_4(G)) = q$.

Case 2: $\delta(G) = 0$.

Let v be an isolated vertex in G. Then $v \in V(\overline{B}_4(G))$ is adjacent to all the vertices of L(G) in $\overline{B}_4(G)$. Therefore v can be coloured with a new colour. Hence $V(\overline{B}_4(G))$ is q + 1 colourable and $V(\overline{B}_4(G))$ cannot be coloured with fewer than q + 1 colours. Therefore, $\chi(\overline{B}_4(G)) = q + 1$.

In the following, domination number of $\overline{B}_4(G)$ is found.

Remark 3.6.

Since there is no vertex of degree $p + q - 1$ in $\overline{B}_4(G)$, $\gamma(\overline{B}_4(G)) \geq 2$.

Theorem 3.6.

For any graph G with atleast one edge, $\gamma(\overline{B}_4(G)) = 2$ if and only if $\beta_1(G) \geq 2$.

Proof.

Let $\gamma(\overline{B}_4(G)) = 2$. Then there exists a minimum dominating set D of $\overline{B}_4(G)$ containing two vertices.

Case 1: $D = \{v_1, v_2\} \subseteq V(G)$

Since no two vertices of G are adjacent in $\overline{B}_4(G)$ and G contains atleast one edge, D cannot be dominating set of $\overline{B}_4(G)$.

Case 2: $D = \{v, e\} \subseteq V(\overline{B}_4(G))$, where $v \in V(G)$ and $e \in E(G)$.

Let $e = (u, w)$, where $u, w \in V(G)$. Then $u, w \in V(\overline{B}_4(G))$ and are not adjacent to any of the vertices in D.

Case 3: D $= \{e_1, e_2\} \subseteq E(G)$

Then $\{e_1, e_2\} \subseteq V(\overline{B}_4(G))$. Since D is a dominating set of $\overline{B}_4(G)$, e_1 and e_2 are independent edges in G. Therefore $\beta_1(G) \geq 2$.

Conversely, assume $\beta_1(G) \geq 2$. Then there exist atleast two independent edges say e_1, e_2 in G. Let D′ be the set of vertices in $\overline{B}_4(G)$ corresponding to the edges e_1 and e_2 in G. Then D′ is a dominating set of $\overline{B}_4(G)$ and $\gamma(\overline{B}_4(G)) \leq 2$. But $\gamma(\overline{B}_4(G)) \geq 2$. Therefore $\gamma(\overline{B}_4(G)) = 2$.

Theorem 3.7.

If G is a graph with atleast one edge, then $\gamma(\overline{B}_4(G)) = \gamma_i(\overline{B}_4(G)) \leq 3$.

Proof.

Let $e = (u, v) \in E(G)$, where $u, v \in V(G)$.

Then $D = \{u, v, e\} \subseteq V(\overline{B}_4(G))$ is a dominating set of $\overline{B}_4(G)$ and is independent. Hence, $\gamma(\overline{B}_4(G)) = \gamma_i(\overline{B}_4(G)) \leq 3$.

Remark 3.7.

a. If G is a star or a cycle on three vertices, then $\gamma(\overline{B}_4(G)) = \gamma_i(\overline{B}_4(G)) = 3$.

b. For any graph G, let D be subset of E(G) with atleast two vertices and <D> is not a star. Then the set D′ vertices of $\overline{B}_4(G)$ corresponding to the edges of G is a dominating set of $\overline{B}_4(G)$ and D contains exactly two vertices, then D′ is a dominating set of $\overline{B}_4(G)$ if and only if edges of <D> are independent.

Theorem 3.8.

Let G be a graph which is not a star. Then $\gamma(\overline{B}_4(G)) = \gamma(G) = 2$ if and only if either there exists a dominating edge in G or G is a union of stars.

Proof.

Let there exist a dominating edge e in G. Then each vertex in G is adjacent to atleast one of the end vertices of e. Therefore, $\gamma(G) = 2$. Since G is a not a star, there exist atleast two independent edges in G. Let e_1, e_2 be the vertices in $\overline{B}_4(G)$ corresponding to the two independent edges in G. Then $D = \{e_1, e_2\}$ is a dominating set of $\overline{B}_4(G)$. Therefore, $\gamma(\overline{B}_4(G)) = 2$. Similarly, if G is a union of stars, then also $\gamma(\overline{B}_4(G)) = 2$.

Conversely, assume $\gamma(\overline{B}_4(G)) = \gamma(G) = 2$. Then there exists a dominating set D of G with two vertices. That is, either G contains a dominating edge or $G \cong K_{1,n} \cup K_{1,m}$, $n, m \geq 1$.

Theorem 3.9.

For any graph G, $\gamma(\overline{B}_4(G)) = \gamma(L(G)) = 2$ if and only if there exist two independent edges in G such that each edge in G is adjacent to atleast one of those independent edges, where L(G) is the line graph of G.

Proof.

Let there exist two independent edges in G such that each edge in G is adjacent to one of the independent edges. Let e_1 and e_2 be the corresponding vertices in L(G). Then $D = \{e_1, e_2\} \subseteq V(L(G))$ is a dominating set of both L(G) and $\overline{B}_4(G)$. Therefore $\gamma(\overline{B}_4(G)) = \gamma(G) = 2$.

Conversely, assume $\gamma(\overline{B}_4(G)) = \gamma(G) = 2$. Then there exists minimum dominating set D of both L(G) and $\overline{B}_4(G)$ containing two vertices. Let $D = \{e_1, e_2\} \subseteq V(L(G))$, where e_1 and e_2 are edges in G. Since D is a dominating set of L(G), each edge in G is adjacent to atleast one of e_1 and e_2. Similarly, since D is also a dominating set of $\overline{B}_4(G)$, e_1 and e_2 must be independent edges in G. Therefore, there exist two independent edges e_1 and e_2 in G such that each edge in G is adjacent to atleast one of e_1 and e_2.

Remark 3.8.

For any graph G, if $\gamma(L(G)) \geq 3$, then $\gamma(\overline{B}_4(G)) \neq \gamma(G)$.

Theorem 3.10.

For any graph G, $4 \leq \gamma(B_4(G)) + \gamma(\overline{B}_4(G)) \leq \alpha_0(G) + 2$.

Proof.

Let $\beta_1(G) \geq 2$. Then $\gamma(\overline{B}_4(G)) = 2$ and $\gamma(B_4(G)) \leq \alpha_0(G)$.

Therefore $\gamma(B_4(G)) + \gamma(\overline{B}_4(G)) \leq \alpha_0(G) + 2$

Let $\beta_1(G) = 1$. If $G \cong K_{1,n}$, $n \geq 2$, then $\leq \gamma(B_4(G)) + \gamma(\overline{B}_4(G)) = 1 + 3 = 4$. If $G \cong C_3$, then $\gamma(B_4(G)) = \alpha_0(G) = 2$ and $\gamma(\overline{B}_4(G)) = 3$. Therefore $\gamma(B_4(G)) + \gamma(\overline{B}_4(G)) \geq 4$.

Hence $4 \leq \gamma(B_4(G)) + \gamma(\overline{B}_4(G)) \leq \alpha_0(G) + 2$.

The lower bound is attained, when $G \cong K_{1,n}$, $n \geq 2$ and the upper bound is attained for all graphs G with $\beta_1(G) \geq 2$ and $G \cong C_3$.

Theorem 3.11.

For any graph G with q edges, $\gamma(\overline{B}_4(G)) = \alpha_0(\overline{B}_4(G))$ if and only if G is one of the following graphs. $2K_2 \cup mK_1$, $C_3 \cup mK_1$, $K_{1,n} \cup mK_1$, $m \geq 0$, $n \geq 3$.

Proof.

$\gamma(\overline{B}_4(G)) = 2$ or 3. But $\alpha_0(\overline{B}_4(G)) = q$. Therefore, q = 2 or 3. If $\beta_1(G) \geq 2$, then $\gamma(\overline{B}_4(G)) = 2$ and $\alpha_0(\overline{B}_4(G)) = 2$. Therefore $G \cong 2K_2 \cup mK_1$, $m \geq 0$.

If $\beta_1(G) = 1$, then $\gamma(\overline{B}_4(G)) = \alpha_0(\overline{B}_4(G)) = 3$. Therefore $G \cong C_3 \cup mK_1$ or $K_{1,n} \cup mK_1$, $m \geq 0$, $n \geq 3$.

Theorem 3.12.

Let G be a graph with atleast three vertices and not totally disconnected.

Then $\gamma(\overline{B}_4(G)) = \beta_0(\overline{B}_4(G))$ if and only if G is one of the following graphs. C_3, P_3 and $K_2 \cup K_1$.

Proof.

$\gamma(\overline{B}_4(G)) = 2$ or 3. But $\beta_0(\overline{B}_4(G)) = p$. Therefore p = 2 or 3. Therefore $G \cong C_3$, P_3 or $K_2 \cup K_1$.

In the following neighborhood number $n_0(\overline{B}_4(G))$ of $\overline{B}_4(G)$ is found.

Theorem 3.13.

Let G be any graph containing atleast one edge. Then $n_0(\overline{B}_4(G)) = 2$ or 3.

Proof.

Case1: $\beta_1(G) \geq 2$**.**

Then there exist atleast two independent edges in G. Let e_1, e_2 be the vertices in $\overline{B}_4(G)$ corresponding to independent edges in G. Then $D = \{e_1, e_2\} \subseteq V(\overline{B}_4(G))$ is a dominating set of $\overline{B}_4(G)$.

Let $e_1 = (u_1, v_1)$, where $u_1, v_1 \in V(\overline{B}_4(G))$.

$N_{\overline{B}_4(G)}(e_1) = \{V(L(G) - \{e_1\}, V(G) - \{u_1, v_1\}\}$. Therefore, each edge in $\langle V(\overline{B}_4(G)) - D\rangle$ belongs to $\langle N(e_1)\rangle$ and hence D is an n-set for $\overline{B}_4(G)$ and $n_0(\overline{B}_4(G)) \leq 2$.

Since $\gamma(\overline{B}_4(G)) \geq 2$, $n_0(\overline{B}_4(G)) \geq 2$. Therefore, $n_0(\overline{B}_4(G)) = 2$.

Case 2: $\beta_1(G) = 1$

Then $G \cong C_3$ or $K_{1,n}$, $n \geq 1$

If $G \cong C_3$ or $K_{1,n}$, $n \geq 1$, then $n_0(\overline{B}_4(G)) = 3$.

In the following, distance between any two vertices of $\overline{B}_4(G)$ are found.

Lemma 3.1.

Let G be a graph which is not a star. If v_1 and v_2 are any two vertices in G, then distance $d(v_1, v_2)$ between v_1 and v_2 in $\overline{B}_4(G)$ is 2 or 3.

Proof.

Let $v_1, v_2 \in G$. Then $v_1, v_2 \in \overline{B}_4(G)$. Since no two vertices in G are adjacent in $\overline{B}_4(G)$, $d(v_1, v_2) \geq 2$ in $\overline{B}_4(G)$.

Let e be an edge in G not incident with both v_1 and v_2. Then v_1ev_2 is a geodesic path in $\overline{B}_4(G)$ and hence $d(v_1, v_2) = 2$ in $\overline{B}_4(G)$.

Assume each edge in G is incident with atleast one of v_1 and v_2. That is, $\{v_1, v_2\}$ is a point cover of G. Let e_1 be an edge in G incident with v_1 but not v_2 and e_2 be an edge in G incident with v_2 but not v_1. (This is possible, since G is not a star). Then $v_1e_2e_1v_2$ is a geodesic path in G.

Therefore, $d(v_1, v_2) = 3$ in $\overline{B}_4(G)$.

Lemma 3.2.

Let G be a graph which is not a star. If $v \in V(G)$ and $e \in E(G)$, then the distance between v, e in $\overline{B}_4(G)$ is atmost 2 and if $e_1, e_2 \in E(G)$, then distance between e_1 and e_2 in $\overline{B}_4(G)$ is 1.

Proof.

(i) Let $v \in V(G)$ and $e \in E(G)$. If e is not incident with v in G, then $d(v, e) = 1$ in $\overline{B}_4(G)$. Let e be incident with v in G. Since G is not a star, there exists an edge e_1 in G not incident with v in G.

Then $v, e, e_1 \in V(\overline{B}_4(G))$ and ve_1e is a geodesic path in $\overline{B}_4(G)$. Hence $d(v, e) = 2$ in $\overline{B}_4(G)$.

(ii) Let $e_1, e_2 \in E(G)$. Then $e_1, e_2 \in V(L(G))$. Since any vertices of L(G) in $\overline{B}_4(G)$ are adjacent,

$d(e_1, e_2) = 1$ in $\overline{B}_4(G)$.

Observation 3.2.

From Lemma 3.1. and Lemma 3.2., it is observed that, if G is not a star, then

a. Eccentricity of a vertex v in $V(\overline{B}_4(G)) \cap V(G)$ is 2 or 3 and eccentricity of a vertex e in $V(\overline{B}_4(G)) \cap V(L(G))$ is 2.

b. Radius of $\overline{B}_4(G)$ is 2 and diameter of $\overline{B}_4(G)$ is 2 or 3.

c. $\overline{B}_4(G)$ is self-centered with radius 2 if and only if for every pair of vertices u, v in G, there exists atleast one edge in G not incident with both u v in G. That is, there exists no point cover of G containing two vertices.

d. $\overline{B}_4(G)$ is bieccentric with radius 2 and diameter 3 if and only if there exists a point cover of G containing two vertices.

Example 3.1.

a. $\overline{B}_4(P_n)$ $(n \geq 6)$, $\overline{B}_4(C_n)$ $(n \geq 5)$, $\overline{B}_4(K_n)$ $(n \geq 4)$ are self-centered with radius 2.

b. $\overline{B}_4(C_n)$ (n = 3, 4) are bieccentric with radius 2 and diameter 3.

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