Available online at www.elixirpublishers.com (Elixir International Journal)

Statistics

Elixir Statistics 93C (2016) 39807-39813

ISSN: 2229-712X

Some Constructions of 3-Resolvable 2-Associates PBIB Designs

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ARTICLE INFO

Article history:
Received: 11 March 2016;
Received in revised form:
15 April 2016;
Accepted: 21 April 2016;

Keywords
Balanced Incomplete Block Design,
Partially Balanced Incomplete Block Design,
Group Divisible Design,
Resolvable Design,
α-Resolvable Design,
Symmetric Balanced Incomplete Block Design.

ABSTRACT
Some construction methods of 3-resolvable regular group divisible designs based on incidence matrix of known affine resolvable balanced incomplete block designs and 3-resolvable group divisible type partially balanced incomplete block designs with unequal block sizes within each resolution set based on incidence matrix of known symmetric balanced incomplete block designs are proposed with illustrations.

I Introduction

One of the earliest examples of a resolvable balanced incomplete block design is due to Kirkman [1] as school girl problem formulated in 1850 and pursued further in another paper (Kirkman, [2]). Kirkman himself gave partial solutions of the same. Many other mathematicians worked on this problem in the late 19^{th} and early 20^{th} century. Ray-Chaudhuri and Wilson [3] completely solved the problem. Yates [4], [5] has pointed out some statistical advantages of resolvable designs. The concept of resolvable balanced incomplete block designs was greatly enhanced by a combinatorial paper by Bose [6] and Shirkhande and Raghavarao [7].

A block design is said to be resolvable if the b blocks each of size k can be grouped into r resolution sets of b/r blocks each such that in each resolution set every treatment is replicated exactly once. Bose [6] proved that necessary condition for the resolvability of a balanced incomplete block design is $b \geq v+r-1$. A resolvable block design is said to be affine resolvable if and only if $b = v+r-1$ and any two blocks belonging to different resolution sets intersect in the same number, say, $q_2 = k^2/v$ of treatments.

The concept of resolvability and affine resolvability was generalized by Shirkhande and Raghavarao [7] to α - resolvability and affine α- resolvability. An incomplete block design with parameters v, $b = \beta t$, $r = \alpha t$, k is said to be α-resolvable if the b blocks can be divided into t sets of β blocks each, such that each treatment occurs α times in each resolution set. Further, α-resolvable incomplete block design is said to be affine α-resolvable if every two distinct blocks from the same α - resolution set intersect in the same number, say, q_1, of treatments, whereas every two blocks belonging to different α-resolution sets intersect in the same number, say, q_2, of treatments. The necessary and sufficient condition for the α - resolvable balanced incomplete block design to be affine α - resolvable with the block intersection numbers q_1 and q_2 is $q_1 = k(\alpha-1)/(\beta-1)$ and $q_2 = \alpha k/\beta = k^2/v$. There has been a very rapid development in this area of experimental designs. Some of the prominent work has been seen in Bailey et al. [8], Banerjee et al. [9], Caliński et al.[10], Kageyama [11]-[15], Kageyama et al.[16], [17], Rai et al. [18], Rudra et al. [19], Mukerjee et al. [20].

Let us consider v treatments arranged in b blocks, such that the j^{th} block contains k_j experimental units and i^{th} treatment appears r_i times in the entire design, i = 1, 2, ...,v; j = 1, 2, …, b. For any block design there exist a incidence matrix $N = [n_{ij}]$ of order v×b, where n_{ij} denotes the number of experimental units in the j^{th} block getting the i^{th} treatment. When $n_{ij} = 1$ or 0 ∀ i and j, the design is said to be binary. Otherwise it is said to be nonbinary. In this paper we consider binary block designs only. The following additional notations are used $\underline{k}=[k_1,k_2,....k_b]'$ is the column vector of block sizes, $\underline{r}=[r_1,r_2,.....r_v]'$ is the column vector of treatment replication, $K_{b\times b}=\text{diag }[k_1,k_2,....k_b]$, $R_{v\times v}=\text{diag }[r_1,r_2,.....r_v]$, $\sum r_i=\sum k_j=n$, the total number of experimental units, with this $N1_b=\underline{r}$ and $N'1_v=\underline{k}$, where 1_a is the $a\times 1$ vector of ones.

A balanced incomplete block design is an arrangement of v symbols (treatment) into b sets (blocks) such that (i) each block contains k (< v) distinct treatments; (ii) each treatment appears in r (> λ) different blocks and (iii) every pair of distinct

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treatments appears together in exactly λ blocks. Here, the parameters of balanced incomplete block design (v, b, r, k, λ) are related by the following relations

$$vr = bk,\ r(k-1) = \lambda(v-1) \text{ and } b \geq v \text{ (Fisher's inequality).}$$

A balanced incomplete block design is said to be symmetric if b = v and $r = k$. In this case incidence matrix N is a square matrix i.e. $N'=N$. In case of symmetric balanced incomplete block design any two blocks have λ treatments in common.

A partially balanced incomplete block design based on an m-association scheme, with parameters v, b, r, k, λ_i (i=1,2,…,m), is a block design with v treatments and b blocks of size k each such that every treatment occurs in r blocks and two distinct treatments being i^{th} associate occur together in exactly λ_i (i=1,2,…,m) blocks.

A group divisible (GD) design is a 2-associates partially balanced incomplete block design based on group divisible association scheme, i.e. an arrangement of $v = mn$ treatments in b blocks such that each block contains $k(< v)$ treatments, each replicated r times, and the mn treatments can be divided into m groups of n treatments each, such that any two treatments occur together in λ_1 blocks if they belong to the same group and in λ_2 blocks if they belong to different groups. Furthermore, a group divisible (GD) design is said to be Singular (S) if $r - \lambda_1 = 0$; Semi regular (SR) if $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$; Regular (R) if $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$. For the definitions of partially balanced incomplete block design and group divisible design along with their combinatorial properties, refer, Raghavarao [21]. Group divisible designs and partially balanced incomplete block designs have been studied by Banerjee et al.[22]-[24], Bhagwandas, Banerjee and Kageyama [25], Kageyama et al. [26], Ghosh et al.[27], Mohan et al.[28], Vartak [29].

In this paper we have proposed construction methods of 3-resolvable regular group divisible type partially balanced incomplete block designs by using incidence matrix of known affine resolvable balanced incomplete block design. We have also proposed construction methods of 3-resolvble group divisible type partially balanced incomplete block designs with unequal block sizes within each resolution set by using incidence matrix of known symmetric balanced incomplete block design.

II Construction of design matrix: Method I

Let N be the $v\times b$ incidence matrix of an affine resolvable balanced incomplete block design D with parameters $v = 2k$, b = $2r$, $r = 2k-1$, k, $\lambda = k-1$. Then the following incidence matrix N^* yields 3- resolvable regular group divisible type partially balanced incomplete block design.

$$N^* = N \otimes \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \left(J_{v\times b} - N\right) \otimes \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \tag{1}$$

In incidence matrix N^*, $J_{v\times b}$ denotes a $v\times b$ matrix whose all elements are one and $A \otimes B$ denotes the Kronecker product.

The group divisible association scheme of the resultant design can be given by $v = mn$ treatments, divided into m groups of n treatments each (in the present construction method $m = 2$ and $n = 2k$). In the resultant design any of the treatment are replicating 3 times in each resolution set and b ≥ $v+r-1$. So the resultant design is also 3-resolvable.

Theorem 2.1

The existence of an affine resolvable balanced incomplete block design with parameters $v = 2k$, b = $2r$, $r = 2k-1$, k, $\lambda = k-1$ implies the existence of 3-resolvable regular group divisible design with parameters $v^* = 4k,\ b^* = 4r,\ r^* = 3r,\ k^* = 3k,$ $\lambda_1^* = 5k-3,\ \lambda_2^* = 4k-2,\ m = 2$ and $n = 2k$.

Proof

Let N be the $v\times b$ incidence matrix of an affine resolvable balanced incomplete block design D with parameters $v = 2k$, b = $2r$, $r = 2k-1$, k, $\lambda = k-1$. Under the present method of construction, the design D^* yields the parameters $v^* = 4k$, $b^* = 4r$, $r^* = 3r$, $k^* = 3k$, are obvious by construction. Here $v^* = 4k$ treatments are divided into $m = 2$ groups of $n = 2k$ treatments each, such that any two treatments in the same group are first associates and any two treatments from different groups are second associates. Further, the parameters λ_1^* and λ_2^* can be determined as: In the present construction method any (θ, Ø) pair occurs as given below

$$\begin{array}{cccc} \overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^{(k-1)} & \overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^{k} & \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}^{k} & \overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}^{(k-1)} \\ \hline \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array} \tag{2}$$

From the structure given in (2), corresponding to given in (1), we can see that the inner product of first row with $(3,5,\ldots,4k-1)^{th}$ row will give us value of λ_1^*. So the value of λ_1^* can be calculated as $\lambda_1^* = 2(k-1)+k+k+(k-1) = 5k-3$.The inner product of first row with $(2,4,\ldots 4k)^{th}$ row will give us value of λ_2^*. So the value of λ_2^* can be calculated as $\lambda_2^* = (k-1)+k+k+(k-1) = 4k-2.$

Here $4k$ treatments in N^* can be partitioned into two groups of size $2k$ each and these two groups containing even and odd numbered treatments, respectively, which form a GD association scheme of the resultant design and $r^*k^* - v^*\lambda_2^* \geq 0$. Hence the resultant design is regular group divisible design. In the present construction method the resolution sets can be formed by pairing

$$\text{(i)}\ \frac{\overbrace{\begin{pmatrix}1&1\\1&0\end{pmatrix}}^{(k-1)}}{\begin{pmatrix}1&1\\1&0\end{pmatrix}}\ \text{with}\ \frac{\overbrace{\begin{pmatrix}0&1\\1&1\end{pmatrix}}^{(k-1)}}{\begin{pmatrix}0&1\\1&1\end{pmatrix}}\ \text{and (ii)}\ \frac{\overbrace{\begin{pmatrix}0&1\\1&1\end{pmatrix}}^{k}}{\begin{pmatrix}1&1\\1&0\end{pmatrix}}\ \text{with}\ \frac{\overbrace{\begin{pmatrix}1&1\\1&0\end{pmatrix}}^{k}}{\begin{pmatrix}0&1\\1&1\end{pmatrix}} \tag{3}$$

From structure given in (3), we have (k-1) resolution sets from (i) and k resolution sets from (ii). Thus 4r blocks are partitioned into (k–1)+k = 2k–1 = r resolution sets of 4 blocks each such that in each resolution set any of the v^* treatments is replicating 3 times and $b^* \geq v^* + r - 1$. Hence the resultant structure also preserves the properties of 3-resolvability. This completes the proof.

Corollary 2.2

The complementary design of D* is the disconnected design of group divisible type with parameters $v_* = 4k$, $b_* = 4r$, $r_* = r$, $k_* = k$, $\lambda_{*1} = 5k - 2r - 3$, $\lambda_{*2} = 0$.

Example 2.3

Let us consider an affine resolvable balanced incomplete block design with parameters $v = 4, b = 6, r = 3, k = 2, \lambda = 1$ with incidence matrix N given through the blocks [(1,2), (3,4)], [(1,3), (2,4)], [(1,4), (2,3)]. Then theorem 2.1 yields 3-resolvable regular group divisible (RGD) design with parameters $v^* = 8,\ b^* = 12,\ r^* = 9,\ k^* = 6,\ \lambda_1^* = 7$ and $\lambda_2^* = 6$. The incidence matrix of the resultant design is given as follows:

$$N^* = \left[\begin{array}{cccc|cccc|cccc}
1&1&0&1&1&1&0&1&1&1&0&1\\
1&0&1&1&1&0&1&1&1&0&1&1\\
1&1&0&1&0&1&1&1&0&1&1&1\\
1&0&1&1&1&1&1&0&1&1&1&0\\
0&1&1&1&1&1&0&1&0&1&1&1\\
1&1&1&0&1&0&1&1&1&1&1&0\\
0&1&1&1&0&1&1&1&1&1&0&1\\
1&1&1&0&1&1&1&0&1&0&1&1
\end{array}\right]$$

The GD association scheme of the above design can be written as

$$\begin{array}{llllll} G_1: & 1 & 3 & 5 & 7 \\ G_2: & 2 & 4 & 6 & 8 \end{array}$$

In the above design $b^* \geq v^* + r - 1$ and every treatment is replicated 3 times in each resolution set. Hence the design constructed above is 3-resolvable regular group divisible design. The complementary design of the design constructed above is the disconnected design of group divisible type with parameters $v_* = 8$, $b_* = 12$, $r_* = 3$, $k_* = 2$, $\lambda_{*1} = 1$ and $\lambda_{*2} = 0$.

Example 2.4

Let us consider an affine resolvable balanced incomplete block design with parameters $v = 8, b = 14, r = 7, k = 4, \lambda = 3$ with incidence matrix N given through the blocks [(1,2,4,7), (0,3,5,6)], [(2,3,5,7), (1,4,6,0)], [(3,4,6,7), (2,5,0,1)], [(4,5,0,7), (3,6,1,2)], [(5,6,1,7), (4,0,2,3)], [(6,0,2,7), (5,1,3,4)], [(0,1,3,7), (6,2,4,5)]. Then theorem 2.1 yields 3-resolvable regular group divisible (RGD) design with parameters $v^* = 16,\ b^* = 28,\ r^* = 21,\ k^* = 12,\ \lambda_1^* = 17$ and $\lambda_2^* = 14$. The incidence matrix of the resultant design is given as follows:

$$N^* = \left[\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc|cccc}
0&1&1&1&0&1&1&1&0&1&1&1&1&1&0&1&0&1&1&1&1&1&0&1&1&1&0&1\\
1&1&1&0&1&1&1&0&1&1&1&0&1&0&1&1&1&1&1&0&1&0&1&1&1&0&1&1\\
1&1&0&1&0&1&1&1&0&1&1&1&0&1&1&1&1&1&0&1&0&1&1&1&1&1&0&1\\
1&0&1&1&1&1&1&0&1&1&1&0&1&1&1&0&1&0&1&1&1&1&1&0&1&0&1&1\\
1&1&0&1&1&1&0&1&0&1&1&1&0&1&1&1&0&1&1&1&1&1&0&1&0&1&1&1\\
1&0&1&1&1&0&1&1&1&1&1&0&1&1&1&0&1&1&1&0&1&0&1&1&1&1&1&0\\
0&1&1&1&1&1&0&1&1&1&0&1&0&1&1&1&0&1&1&1&0&1&1&1&1&1&0&1\\
1&1&1&0&1&0&1&1&1&0&1&1&1&1&1&0&1&1&1&0&1&1&1&0&1&0&1&1\\
1&1&0&1&0&1&1&1&1&1&0&1&1&1&0&1&0&1&1&1&0&1&1&1&0&1&1&1\\
1&0&1&1&1&1&1&0&1&0&1&1&1&0&1&1&1&1&1&0&1&1&1&0&1&1&1&0\\
0&1&1&1&1&1&0&1&0&1&1&1&1&1&0&1&1&1&0&1&0&1&1&1&0&1&1&1\\
1&1&1&0&1&0&1&1&1&1&1&0&1&0&1&1&1&0&1&1&1&1&1&0&1&1&1&0\\
0&1&1&1&0&1&1&1&1&1&0&1&0&1&1&1&1&1&0&1&1&1&0&1&0&1&1&1\\
1&1&1&0&1&1&1&0&1&0&1&1&1&1&1&0&1&0&1&1&1&0&1&1&1&1&1&0\\
1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1\\
1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1
\end{array}\right]$$

The GD association scheme of the above design can be written as

$$\begin{array}{lllllllll} G_1: & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ G_2: & 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 \end{array}$$

In the above design $b^{*} \geq v^{*} + r - 1$ and every treatment is replicated 3 times in each resolution set. Hence the design constructed above is 3-resolvable regular group divisible design. The complementary design of the design constructed above is the disconnected design of group divisible type with parameters $v_{*} = 16$, $b_{*} = 28$, $r_{*} = 7$, $k_{*} = 4$, $\lambda_{*1} = 3$ and $\lambda_{*2} = 0$.

III. Construction of design matrix: Method II

If N be the $v \times b$ incidence matrix of a symmetric balanced incomplete block design D with parameters v = b, r = k, λ. Then the following incidence matrix N' yields a 3-resolvable group divisible type partially balanced incomplete block design with unequal block sizes within each resolution set.

$$N^{'} = \underbrace{N \otimes \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \left(J_{v\times b} - N\right)\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{I-part} \,\vdots\, \underbrace{N \otimes \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \left(J_{v\times b} - N\right)\otimes\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{II-part} \qquad (4)$$

In incidence matrix $N^{'}$, $J_{v\times b}$ denotes a $v\times b$ matrix whose all elements are one and $A \otimes B$ denotes the Kronecker product.

Theorem 3.1

The existence of symmetric balanced incomplete block design D with parameters v = b, r = k, λ implies the existence of 3-resolvable group divisible type partially balanced incomplete block design D$^{'}$ with unequal block sizes within each resolution set and parameters $v^{'} = 2v$, $b^{'} = 4b$, $r^{'} = 3b$, $k_1^{'} = v + k$, $k_2^{'} = 2v - k$, $\lambda_1^{'} = 3b + 2(\lambda - r)$, $\lambda_2^{'} = 2b$, m = 2 and n = v.

Proof

Let N be the $v \times b$ incidence matrix of a symmetric balanced incomplete block design D with parameters v = b, r = k, λ. In the present method of construction, the design D$^{'}$ yields the parameters $v^{'} = 2v$, $b^{'} = 4b$, $r^{'} = 3b$, $k_1^{'} = v + k$, $k_2^{'} = 2v - k$, which are obvious. Here $v^{'} = 2v$ treatments are divided into m = 2 groups of n = v treatments each, such that any two treatments in the same group are first associates and any two treatments from different groups are second associates. Further, the parameters $\lambda_1^{'}$ and $\lambda_2^{'}$ can be determined as: In the present construction method any (θ, Ø) pair occurs as given below

$$\dfrac{\overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^{\lambda}}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}} \;\; \dfrac{\overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^{(r-\lambda)}}{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} \;\; \dfrac{\overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}^{(r-\lambda)}}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}} \;\; \dfrac{\overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}^{(b-2r+\lambda)}}{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} \;\vdots\; \dfrac{\overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}^{\lambda}}{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} \;\; \dfrac{\overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}^{(r-\lambda)}}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}} \;\; \dfrac{\overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^{(r-\lambda)}}{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} \;\; \dfrac{\overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^{(b-2r+\lambda)}}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}} \qquad (5)$$

I-part II-part

From the structure given in (5), corresponding to (1), we can see that

(a) The inner product of first row with $(3,5,\ldots,2v-1)^{th}$ row in first part of above structure is 2(λ) + (r–λ) + (r–λ) + b–2r+λ = b+λ and in second part of above structure is λ + (r–λ) +(r–λ) + 2(b–2r+λ) = 2b–2r+λ. So the value of $\lambda_1^{'}$ can be calculated as $\lambda_1^{'} = (b+\lambda) + (2b - 2r + \lambda) = 3b + 2(\lambda - r)$.

(b) The inner product of first row with $(2,4,\ldots 2v)^{th}$ row in first part and in second part of above structure is λ + (r–λ) + (r–λ) + b–2r+λ = b. So the value of $\lambda_2^{'}$ can be calculated as $\lambda_2^{'} = b + b = 2b$.

Here 2v treatments in N$^{'}$ can be divided into two groups of size v each and these two groups containing even and odd numbered treatments, respectively, which form a GD association scheme of the resultant design. In the present construction method the resolution sets can be formed by pairing

$$\text{(i)}\, \dfrac{\overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^{\lambda}}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}} \text{ with } \dfrac{\overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}^{\lambda}}{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}},\ \text{(ii)}\, \dfrac{\overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^{(r-\lambda)}}{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} \text{ with } \dfrac{\overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}^{(r-\lambda)}}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}},\ \text{(iii)}\, \dfrac{\overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}^{(r-\lambda)}}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}} \text{ with } \dfrac{\overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^{(r-\lambda)}}{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} \text{ and (iv)}\, \dfrac{\overbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}^{(b-2r+\lambda)}}{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}} \text{ with } \dfrac{\overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^{(b-2r+\lambda)}}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}} \qquad (6)$$

From the structure given in (6), we have λ resolution sets from (i), (r–λ) resolution sets from (ii) and (iii) and from (iv) we have (b–2r+λ) resolution sets. Thus the blocks are grouped into λ + 2(r–λ) + (b–2r+λ) = b resolution sets of 4 blocks each such that in each resolution set any of the $v^{'}$ treatments is replicating 3 times and $b^{'} \geq v^{'} + b - 1$. Hence the resultant design is 3-resolvable group divisible type partially balanced incomplete block design with unequal block sizes in each resolution set. This completes the proof.

Corollary 3.2

If v (=4t–1) is a prime or prime power then the symmetric balanced incomplete block design with parameters v = 4t–1 = b, r = 2t–1 = k, λ = t–1 and it's complementary design with parameters v = 4t–1 = b, r = 2t = k, λ = t always exists and both of these symmetric balanced incomplete block designs implies the existence of a 3-resolvable group divisible type partially balanced incomplete block design with parameters $v^{'}$ = 8t–2, b$^{'}$=16t–4, $r^{'}$=12t–3, $k_1^{'} = 6t - 1$, $k_2^{'} = 6t - 2$, $\lambda_1^{'} = 10t - 3$, $\lambda_2^{'} = 8t - 2$, m = 2 and n = 4t–1.

Corollary 3.3

When $4t$+3 is a prime or prime power, then the symmetric balanced incomplete block design with parameters $v = 4t+3 = b$, $r = 2t+1 = k$, $\lambda = t$ always exists and this implies the existence of a 3-resolvable group divisible type partially balanced incomplete block design with parameters $v' = 8t+6$, $b' = 16t+12$, $r' = 12t+9$, $k_1' = 6t+4$, $k_2' = 6t+5$, $\lambda_1' = 10t+7$, $\lambda_2' = 8t+6$, $m = 2$ and $n = 4t+3$.

Corollary 3.4

The existence of a symmetric balanced incomplete block design with parameters $v = 8t-1 = b$, $r = 4t-1 = k$, $\lambda = 2t-1$ implies the existence of a 3-resolvable group divisible type partially balanced incomplete block design with parameters $v' = 16t-2$, $b' = 4(8t-1)$, $r' = 3(8t-1)$, $k_1' = 12t-2$, $k_2' = 12t-1$, $\lambda_1' = 20t-3$, $\lambda_2' = 16t-2$, $m = 2$ and $n = 8t-1$.

Example 3.5

Let us consider a symmetric balanced incomplete block design with parameters $v = b = 3$, $r = k = 2$, $\lambda = 1$; whose blocks are given by (1,2), (1,3), (2,3). Then Theorem 3.1 yields a 3-resolvable group divisible type partially balanced incomplete block design with unequal block sizes and parameters $v' = 6$, $b' = 12$, $r' = 9$, $k_1' = 5$, $k_2' = 4$, $\lambda_1' = 7$ and $\lambda_2' = 6$. The incidence matrix of the design is given as

$$N' = \left[\begin{array}{cccccc|cccccc}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1
\end{array}\right]$$

On pairing the blocks of above design as given in (6), to form the resolution sets, we will get 3 resolution sets. The incidence matrix of the resultant design is given as

$$N' = \left[\begin{array}{cccc|cccc|cccc}
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}\right]$$

The GD association scheme of the above design can be written as

$$\begin{array}{llll} G_1: & 1 & 3 & 5 \\ G_2: & 2 & 4 & 6 \end{array}$$

Here, $b' \geq v' + b - 1$ and every treatment is replicated 3 times in each resolution set. Hence the design constructed above is 3-resolvable group divisible type partially balanced incomplete block design with unequal block sizes.

Example 3.6

Let us consider a symmetric balanced incomplete block design with parameters $v = b = 7$, $r = k = 4$, $\lambda = 2$; whose blocks are given by (3,5,6,7), (1,4,6,7), (1,2,5,7). (1,2,3,6), (2,3,4,7), (1,3,4,5), (2,4,5,6). Then Theorem 3.1 yields a 3-resolvable group divisible type partially balanced incomplete block design with unequal block sizes and parameters $v' = 14$, $b' = 28$, $r' = 21$, $k_1' = 11$, $k_2' = 10$, $\lambda_1' = 17$ and $\lambda_2' = 14$. The incidence matrix of the design is given as

$$N' = \left[\begin{array}{cccccccccccccc|cccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0
\end{array}\right]$$

On pairing the blocks of above design as given in (6), to form the resolution sets, we will get 7 resolution sets. The incidence matrix of the resultant design is given as

$$N' = \left[\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc|cccc}
0&1&1&1&1&1&0&1&1&1&0&1&1&1&0&1&0&1&1&1&1&1&0&1&0&1&1&1\\
1&1&1&0&1&0&1&1&1&0&1&1&1&0&1&1&1&1&1&0&1&0&1&1&1&1&1&0\\
0&1&1&1&0&1&1&1&1&1&0&1&1&1&0&1&1&1&0&1&0&1&1&1&1&1&0&1\\
1&1&1&0&1&1&1&0&1&0&1&1&1&0&1&1&1&0&1&1&1&1&1&0&1&0&1&1\\
1&1&0&1&0&1&1&1&0&1&1&1&1&1&0&1&1&1&0&1&1&1&0&1&0&1&1&1\\
1&0&1&1&1&1&1&0&1&1&1&0&1&0&1&1&1&0&1&1&1&0&1&1&1&1&1&0\\
0&1&1&1&1&1&0&1&0&1&1&1&0&1&1&1&1&1&0&1&1&1&0&1&1&1&0&1\\
1&1&1&0&1&0&1&1&1&1&1&0&1&1&1&0&1&0&1&1&1&0&1&1&1&0&1&1\\
1&1&0&1&0&1&1&1&1&1&0&1&0&1&1&1&0&1&1&1&1&1&0&1&1&1&0&1\\
1&0&1&1&1&1&1&0&1&0&1&1&1&1&1&0&1&1&1&0&1&0&1&1&1&0&1&1\\
1&1&0&1&1&1&0&1&0&1&1&1&1&1&0&1&0&1&1&1&0&1&1&1&1&1&0&1\\
1&0&1&1&1&0&1&1&1&1&1&0&1&0&1&1&1&1&1&0&1&1&1&0&1&0&1&1\\
1&1&0&1&1&1&0&1&1&1&0&1&0&1&1&1&1&1&0&1&0&1&1&1&0&1&1&1\\
1&0&1&1&1&0&1&1&1&0&1&1&1&1&1&0&1&0&1&1&1&1&1&0&1&1&1&0
\end{array}\right]$$

The GD association scheme of the above design can be written as

$$G_1: \quad 1 \quad 3 \quad 5 \quad 7 \quad 9 \quad 11 \quad 13$$

$$G_2: \quad 2 \quad 4 \quad 6 \quad 8 \quad 10 \quad 12 \quad 14$$

Here, $b' \geq v' + b - 1$ and every treatment is replicated 3 times in each resolution set. Hence the design constructed above is 3-resolvable group divisible type partially balanced incomplete block design with unequal block sizes.

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