Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 93 (2016) 39758-39761

ISSN: 2229-712X

New Integral Transform "Tarig Transform" and System of Integro-Differential Equations

Tarig. M. Elzaki[1] and Salih M. Ezaki[2]

[1]Mathematics Department, Faculty of Sciences and Arts-Alkamil, King Abdulaziz University, Jeddah Saudi Arabia

[2]Mathematics Department, Sudan University of Science and Technology

ARTICLE INFO

Article history:

Received: 17 October 2012;

Received in revised form:

15 April 2016;

Accepted: 21 April 2016;

Keywords

Tarig transform,

Integro-differential equations.

ABSTRACT

In this work a new integral transform, namely Tarig transform was introduced and applied to solve linear systems of Integro-differential equations with constant coefficients. The brilliance of the method in obtaining analytical solution of some systems of volterra integral and Integro-differential equation.

Introduction

Many problems of physical interest are described by differential and integral equations with appropriate or boundary conditions. These problems are usually formulated as initial value problem, boundary value problems, or initial – boundary value problem that seem to be mathematically more vigorous and physically realistic in applied and engineering sciences. Tarig transform method is very effective for solution of the response of differential and integral equations and a linear system of differential and integral equations.

The technique that we used is Tarig transform method which is based on Fourier transform. It introduced by tarig M. Elzaki (2010) see [Tarig and Salih (2011)and (2012)].

In this study, Tarig transform is applied to integral and integrao-differential equations system which the solution of these equations have a major role in the fields of science and engineering. When a physical system is modeled under the differential sense, if finally gives a differential equation, an integral equation or an integro-differential equation systems.

Recently .Tarig M. ELzaki introduced a new transform and named as Tarig transform which is defined by:

$$T\left[f(t),u\right]=F(u)=\frac{1}{u}\int_0^\infty e^{-\frac{t}{u^2}}f(t)dt \quad , \quad u\neq 0 \qquad (1)$$

Or for a function $f(t)$ which is of exponential order,

$$|f(t)|<\begin{cases} Me^{-t/k_1} \quad , & t\leq 0 \\ Me^{t/k_2} \quad , & t\geq 0 \end{cases} \qquad (2)$$

Tarig transform, henceforth designated by the operator $T[\cdot]$, is defined by the integral equation.

$$T\left[f(t)\right]=F(u)=\int_0^\infty f(ut)\, e^{-\frac{t}{u}}dt \quad , \; u\neq 0 \qquad (3)$$

Where M is a real finite number and k_1, k_2 can be finite or infinite.

Theorem 1:

If $T\left[f(t)\right]=F(u)$ then,

$$(i)\; T\left[f'(t)\right]=\frac{F(u)}{u^2}-\frac{1}{u}f(0) \quad (ii)\; T\left[f''(t)\right]=\frac{F(u)}{u^4}-\frac{1}{u^3}f(0)-\frac{1}{u}f'(0)$$

$$(iii)\; T\left[f^{(n)}(t)\right]=\frac{F(u)}{u^{2n}}-\sum_{i=1}^{n}u^{2(i-n)-1}f^{(i-1)}(0)$$

Proof:

$(i)\; T\left[f'(t)\right]=\frac{1}{u}\int_0^\infty f'(t)e^{\frac{-t}{u^2}}dt$ Integrating by parts to find that:

$$T\left[f'(t)\right]=\frac{1}{u}\left\{-f(0)+\frac{1}{u}T\left[f(t)\right]\right\}$$ And

$$T\left[f'(t)\right]=\frac{F(u)}{u^2}-\frac{1}{u}f(0)$$

(ii) By (i) $T\left[G'(t)\right]=\frac{T\left[G(t)\right]}{u^2}-\frac{1}{u}G(0)$. Let $G(t)=f'(t)$.

then:

$$T\left[f''(t)\right]=\frac{T\left(f'(t)\right)}{u^2}-\frac{1}{u}f'(0)=\frac{1}{u^2}\left[\frac{F(u)}{u^2}-\frac{1}{u}f(0)\right]-\frac{1}{u}f'(0) \quad and$$

$$T\left[f''(t)\right]=\frac{F(u)}{u^4}-\frac{1}{u^3}f(0)-\frac{1}{u}f'(0)$$

The generalization to nth order derivatives in (iii) can be proved by using mathematical induction.

Theorem 2:

If $T\left[f(t)\right]=F(u)$. Then: $T\left[\int_0^t f(\omega)d\omega\right]=u^2F(u)$

Proof:

Let $G(t)=\int_0^t f(\omega)d\omega$. Then $G'(t)=f(t)$ and $G(0)=0$.

Tele:

E-mail address: Tarig.alzaki@gmail.com

Taking Tarig transform of both sides, we have: $\frac{T(G(t))}{u^2}-\frac{1}{u}G(0)=F(u)$

Then: $T\left[G(t)\right]=u^2F(u)$, and $T\left[\int_0^T f(\omega)d\omega\right]=u^2F(u)$

Theorem 3:

If $T\left[f(t)\right]=G(u)$ and $L\left[f(t)\right]=F(s)$ then $G(u)=\frac{F\left(\frac{1}{u^2}\right)}{u}$ where $F(s)$ is Laplace transform of $f(t)$.

Proof:

$T\left[f(t)\right]=\int_0^\infty f(ut)e^{\frac{-t}{u}}dt=G(u)$ Let $w=ut$, then we have:

$$G(u)=\int_0^\infty f(w)e^{\frac{-w}{u^2}}\frac{dw}{u}=\frac{F\left(\frac{1}{u^2}\right)}{u}$$

Theorem 4:

If $T\left[f(t)\right]=F(u)$, then: $T\left[t f(t)\right]=\frac{1}{2}\left[u^3\frac{d}{du}F(u)+u^2F(u)\right]$,

Proof:

The definition of Tarig transform is:

$$F(u)=\frac{1}{u}\int_0^\infty f(t)e^{\frac{-t}{u^2}}dt,\quad then: \frac{d}{du}F(u)=\frac{2}{u^4}\int_0^\infty t f(t)e^{\frac{-t}{u^2}}-\frac{1}{u^2}\int_0^\infty f(t)e^{\frac{-t}{u^2}}dt$$

$$\frac{1}{u}\int_0^\infty t f(t)e^{\frac{-t}{u^2}}dt=\frac{1}{2}\left[u^3\frac{d}{du}F(u)+u^2F(u)\right]\Rightarrow T\left[t f(t)\right]=\frac{1}{2}\left[u^3\frac{d}{du}F(u)+u^2F(u)\right]$$

Theorem 5 (convolution):

Let $f(t)$ and $g(t)$ having Laplace transform $F(s)$ and $G(s)$ and Tarig transform $M(u)$ and $N(u)$. respectively then: $T\left[(f*g)(t)\right]=u\,M(u)N(u)$

Proof:

First recall that the Laplace transforms of $(f*g)$ is given by,

$$L\left[(f*g)(t)\right]=F(s)G(s)$$

Now, since, by the duality relation in **Theorem 3**.we has:

$$T\left[(f*g)(t)\right]=\frac{1}{u}L\left[(f*g)(t)\right]$$

And since,

$$M(u)=\frac{F\left(\frac{1}{u^2}\right)}{u},\ N(u)=\frac{G\left(\frac{1}{u^2}\right)}{u}$$

Tarig transform of $(f*g)$ is obtained as follows:

$$T\left[(f*g)(t)\right]=\frac{F\left(\frac{1}{u^2}\right)\times G\left(\frac{1}{u^2}\right)}{u}=uM(u)N(u)$$

Application to System of Integro-Differential Equations

Let us consider the general first order system of integro-differential equation,

$$\begin{cases} y'_1=f(t)+\int_0^t\left[y_1(x)+y_2(x)\right]dx \\ y'_2=g(t)+\int_0^t\left[y_2(x)-y_1(x)\right]dx \end{cases} \quad (4)$$

With the initial conditions,

$$y_1(0)=\alpha,\ y_2(0)=\beta \quad (5)$$

By using Tarig transform into eq (4) we have:

$$\begin{cases} \frac{\bar{y}_1}{u^2}-\frac{1}{u}y_1(0)=F(u)+u^2\bar{y}_1+u^2\bar{y}_2 \\ \frac{\bar{y}_2}{u^2}-\frac{1}{u}y_2(0)=G(u)+u^2\bar{y}_2-u^2\bar{y}_1 \end{cases} \quad (6)$$

Where $\bar{y}_1$, $\bar{y}_2$ are Tarig transform of y_1, y_2. respectively.

Substituting eq(5) into eq(6) we get:

$$\begin{cases} \bar{y}_1-u\alpha=u^2F(u)+u^4\bar{y}_1+u^4\bar{y}_2 \\ \bar{y}_2-u\beta=u^2G(u)+u^4\bar{y}_1-u^4\bar{y}_1 \end{cases}$$

Or

$$\begin{cases} \left(1-u^4\right)\bar{y}_1-u^4\bar{y}_2=u^2F(u)+u\alpha \\ \left(1-u^4\right)\bar{y}_2+u^4\bar{y}_1=u^2G(u)+u\beta \end{cases}$$

Solve these equations to find: $\bar{y}_1(u)=\frac{k(u)}{h(u)}=w(u)$

Where

$k(u)=\left(u^2-u^6\right)F(u)+\alpha u-u^5\alpha+u^6G(u)+u^5\beta$ and $h(u)=2u^8-2u^4+1$

Then: $y_1(t)=F^{-1}\left[w(u)\right]=H(t)$. Substituting $y_1(t)$ into eq (4) to find $y_2(t)$.

Example 1:

Consider the following system,

$$\begin{cases} y'_1=t+\int_0^t\left[y_1(x)+y_2(x)\right]dx \\ y'_2=-\frac{1}{12}t^4-2t+\int_0^t\left[(t-x)y_1(x)\right]dx \end{cases} \quad (7)$$

With the initial conditions,

$$y_1(0)=0 \quad , \quad y_2(0)=1 \quad (8)$$

$$\begin{cases} \frac{\bar{y}_1}{u^2}-\frac{1}{u}y_1(0)=u^3+u^2\bar{y}_1+u^2\bar{y}_2 \\ \frac{\bar{y}_2}{u^2}-\frac{1}{u}y_2(0)=-2u^9-2u^3+u\left[u^3\bar{y}_1\right] \end{cases} \quad (9)$$

Substituting eq (8) into eq (9) we get:

$$\begin{cases} \left(1-u^4\right)\bar{y}_1-u^4\bar{y}_2=u^5 \\ \bar{y}_2-u^6\bar{y}_1=u-2u^{11}-2u^5 \end{cases}$$

The solution of these equations is,

$$\left(1-u^3-u^{10}\right)\bar{y}_1=u^5-2u^{15}-2u^8+u^5 \Rightarrow \bar{y}_1(u)=2u^5 \ and \ y_1(t)=t^2$$

From the first equation of (7) we have:

$$y'_1=t+\int_0^t\left[y_1(x)+y_2(x)\right]dx \quad Or \quad \int_0^t y_2(x)dx=t-\frac{1}{3}t^3$$

Applying Tarig transform to the last equation we get:

$$u^2\bar{y}_2(u)=u^3-2u^7\Rightarrow \bar{y}_2(u)=u-2u^5 \ and \ y_2(t)=1-t^2$$

Example 2:

Consider the following system,

$$\begin{cases} y''_1=-1-y_1+\cos t+\int_0^t y_2(x)dx \\ y''_2=-y_2+\sin t-\int_0^t y_1(x)dx \end{cases} \quad (10)$$

With the initial conditions,

$$y_1(0)=1 \quad , \quad y_1'(0)=0 \qquad (11)$$
$$y_2(0)=0 \quad , \quad y_2'(0)=1$$

Solution:

Applying Tarig transform of eq(10) we get:

$$\begin{cases} \dfrac{\bar{y}_1}{u^4}-\dfrac{1}{u^3}y_1(0)-\dfrac{1}{u}y_1'(0)=-u-\bar{y}_1+\dfrac{u}{1+u^4}+u^2\bar{y}_2 \\ \dfrac{\bar{y}_2}{u^4}-\dfrac{1}{u^3}y_2(0)-\dfrac{1}{u}y_2'(0)=-\bar{y}_2+\dfrac{u^3}{1+u^4}-u^2\bar{y}_1 \end{cases} \qquad (12)$$

Substituting eq (11) into eq (12) we have:

$$\begin{cases} \left(1+u^4\right)\bar{y}_1-u^6\bar{y}_2=u+\dfrac{u^5}{1+u^4}-u^5 \\ u^6\bar{y}_1+\left(1+u^4\right)\bar{y}_2=u^3+\dfrac{u^7}{1+u^4} \end{cases}$$

Solve this equation to find:

$$\begin{cases} \left[u^{12}+u^8+2u^4+1\right]\bar{y}_1=\dfrac{u^{13}+u^9+2u^5+u}{1+u^4} \\ \bar{y}_1=\dfrac{u}{1+u^4} \\ y_1(t)=F^{-1}\left[\dfrac{u}{1+u^4}\right]=\cos t \end{cases}$$

Substituting $y_1(t)$ into eq(10) we get: $\int_0^t y_2(x)dx=1-\cos t$

Take Tarig transform of two side of this equation, we have:

$$u^2\bar{y}_2=u-\frac{u}{1+u^4}=\frac{u^5}{1+u^4} \Rightarrow \bar{y}_2(u)=\frac{u^3}{1+u^4} \; and \quad y_2(t)=\sin t$$

Example 3:

Consider the following linear Voltera type integro-differential equation system,

$$\begin{cases} y'_1=1+t+t^2-y_2(t)-\int_0^t\left[y_1(x)+y_2(x)\right]dx \\ y'_2=-1-t+y_1(t)-\int_0^t\left[y_1(x)-y_2(x)\right]dx \end{cases} \qquad (13)$$

With the initial conditions:

$$y_1(0)=1 \quad , \quad y_2(0)=-1 \qquad (14)$$

Solution:

By taking Tarig transform of eq (13) and making use of the conditions (14) we have:

$$\begin{cases} (1+u^4)\bar{y}_1+u^2\left(1+u^2\right)\bar{y}_2=u+u^3+u^5+2u^7 \\ u^2\left(u^2-1\right)\bar{y}_1+\left(1-u^4\right)\bar{y}_2=-u-u^3-u^5 \end{cases}$$

Solve these equations to find:

$$\bar{y}_1(u)=u^3+\frac{u^3+u}{1-u^4}=u^3+\frac{u}{1-u^2} \Rightarrow y_1(t)=F^{-1}\left[u^3+\frac{u}{1-u^2}\right]=t+e^t$$

Where that F^{-1} is the inverse Tarig transform. Substituting $y_1(t)$ into equation (13) we get: $\bar{y}_2=-\frac{1}{2}t^2+\int_0^t y_2(x)dx$

Applying Tarig transform to this equation we get:

$$\frac{\bar{y}_2}{u^2}+\frac{1}{u}=-u^5+u^2\bar{y}_2 \quad \Rightarrow \bar{y}_2(u)=\frac{-u^7-u}{1-u^4}=u^3-\frac{u^3+u}{1-u^4}=u^3-\frac{u}{1-u^2}$$
$$and \quad y_2(t)=F^{-1}\left[u^3-\frac{u}{1-u^2}\right]=t-e^t$$

Conclusion

In this paper, Tarig transform method for the solution of volterra integral and Integro-differential equation systems is successfully expanded. In the first example, we introduce the general system of the first order Integro-differential equation, and in the last three examples, Integro-differential equation systems are considered. In observed Tarig transform method is robust and is applicable to various types of system of Integro-differential and system integral equation.

References

[1] Tarig M. Elzaki, (2011), The New Integral Transform “Elzaki Transform” Global Journal of Pure and Applied Mathematics, ISSN 0973-1768,Number 1, pp. 57-64.

[2] Tarig M. Elzaki & Salih M. Elzaki, (2011), Application of New Transform “Elzaki Transform” to Partial Differential Equations, Global Journal of Pure and Applied Mathematics, ISSN 0973-1768,Number 1, pp. 65-70.

[3] Tarig M. Elzaki & Salih M. Elzaki, (2011), On the Connections Between Laplace and Elzaki transforms, Advances in Theoretical and Applied Mathematics, ISSN 0973-4554 Volume 6, Number 1,pp. 1-11.

[4] Tarig M. Elzaki & Salih M. Elzaki, (2011), On the Elzaki Transform and Ordinary Differential Equation With Variable Coefficients, Advances in Theoretical and Applied Mathematics. ISSN 0973-4554 Volume 6, Number 1,pp. 13-18.

[5] Tarig M. Elzaki, Adem Kilicman, Hassan Eltayeb. (2010), On Existence and Uniqueness of Generalized Solutions for a Mixed-Type Differential Equation, Journal of Mathematics Research, Vol. 2, No. 4, pp. 88-92.

[6] Tarig M. Elzaki, (2009), Existence and Uniqueness of Solutions for Composite Type Equation, Journal of Science and Technology,. pp. 214-219.

[7] Lokenath Debnath and D. Bhatta. (2006), Integral transform and their Application second Edition, Chapman & Hall /CRC.

[8] A.Kilicman and H.E.Gadain. (2009), An application of double Laplace transform and Sumudu transform, Lobachevskii J. Math.30 (3), pp.214-223.

[9] J. Zhang, (2007), Asumudu based algorithm m for solving differential equations, Comp. Sci. J. Moldova 15(3), pp – 303-313.

[10] Hassan Eltayeb and Adem kilicman, (2010), A Note on the Sumudu Transforms and differential Equations, Applied Mathematical Sciences, VOL, 4, , no.22,1089-1098.

[11] Kilicman A. & H. ELtayeb. (2010), A note on Integral transform and Partial Differential Equation, Applied Mathematical Sciences, 4(3), PP.109-118.

[12] Hassan ELtayeh and Adem kilicman, (2010), on Some Applications of a new Integral Transform, Int. Journal of Math. Analysis, Vol, 4, no.3, 123-132.

[13] Tarig M. Elzaki, and Salih M. Elzaki, (2011), on the new integral Transform 'Tarig Transform' and Systems of Ordinary Differential Equations. Applied Mathematics, Elixir Appl. Math. 36, 3226-3229.

[14] Tarig M. Elzaki, and Salih M. Elzaki, (2011), on the relationship between Laplace Transform and new integral transform Tarig Transform'. Applied Mathematics, Elixir Appl. Math. 36 , 3230-3233.

[15] Tarig M. Elzaki, and Salih M. Elzaki, (2011), On the Tarig Transform and ordinary differential equation with variable coefficients, Applied Mathematics, Elixir Appl. Math. 38, 4250-4252.

[16] Tarig M. Elzaki, and Salih M. Elzaki, (2011), the new integral transform "Tarig Transform" Properties and applications to differential equations, Applied Mathematics, Elixir Appl. Math. 38 , 4239-4242.
[17] Tarig M. Elzaki, and Salih M. Elzaki, (2012), Application of new transform "tarig transform" to partial differential equations, Applied Mathematics, Elixir Appl. Math. 42, 6369-6372.
[18] Tarig M. Elzaki, and Salih M. Elzaki, (2012), On the Tarig transform and system of partial differential equations, Applied Mathematics, Elixir Appl. Math. 42, 6373-6376.

Appendix

Tarig Transform of Simple Functions

S.N0.	$f(t)$	$F(u)$
1	1	u
2	t	u^3
3	e^{at}	$\frac{u}{1-au^2}$
4	t^n	$n!\,u^{2n+1}$
5	t^a	$\Gamma(a+1)u^{2a+1}$
6	$\sin at$	$\frac{au^3}{1+a^2u^4}$
7	$\cos at$	$\frac{u}{1+a^2u^4}$
8	$\sinh at$	$\frac{au^3}{1-a^2u^4}$
9	$\cosh at$	$\frac{u}{1-a^2u^4}$