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Boundary Digraph and Boundary Neighbour Digraph of a graph G

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ABSTRACT

A digraph D is a pair (V, A), where V is a non-empty set whose elements are called the vertices and A is the subset of the set of ordered pairs of distinct elements of V. The elements of A are called the arcs of D. A vertex v is a boundary vertex of u if $d(u, w) \leq d(u, v)$ for all $w \in N(v)$. The boundary digraph BD(G) of a graph(digraph) G is the digraph that has the same vertex set as G and an arc from u to v exists in BD(G) if and only if v is a boundary vertex of u in G. The boundary neighbor digraph BND(G) of a graph G is the graph that has the same vertex set as G and a directed edge (arc) from u to v exists in BND(G) if and only if v is a boundary neighbor of u in G.

1.Introduction

Interconnection networks are pervasive in today's society, including networks for the distribution of goods, communication networks, social networks and the Internet, to name just a few. The topology of an interconnection network is usually modeled by a graph, either directed or undirected, depending on the particular application. In all cases, there are some common fundamental characteristics of networks such as the number of nodes, number of connections at each node, total number of connections, clustering of nodes, etc. Many of the most important basic properties, under pinning the functionalities of a network, are related to the distance between the nodes in a network. Such properties include the eccentricities of the nodes, the radius of the network and the diameter of the network.

The notion of the eccentric digraph of a graph was introduced by Buckley [3]. This construction was refined and extended by others, including Boland and Miller [5]. This has led to the study of the boundary graph of a graph G and boundary digraph of a graph G by us in [1].

A digraph D is a pair (V, A), where V is a non-empty set whose elements are called the vertices and A is the subset of the set of ordered pairs of distinct elements of V. The elements of A are called the arcs of D. The order of G is the cardinality of V(G) and is denoted by $|G| = |V(G)|$. If (u, v) is an arc, it is said that u is adjacent to v and also that v is adjacent from u. A digraph D = (V, A) is said to be complete if both uv and vu $\in$ A, for all u, v $\in$ V. Obviously, this corresponds to K_n, where $|V| = n$, and is denoted by K_n^*. A complete anti-symmetric digraph or a complete oriented graph is called a tournament. Clearly, a tournament is an orientation of K_n. The set of vertices which are adjacent from [to] a given vertex v is denoted by $N^+(v)$ [$N^-(v)$] and its cardinality is the out-degree of v denoted by $d^+(v)$ [in-degree of v denoted by $d^-(v)$]. A vertex v for which $d^+(v) = d^-(v) = 0$ is called an isolate [2, 6].

A vertex v is called a transmitter or a receiver according as $d^+(v) > 0$, $d^-(v) = 0$ or $d^+(v) = 0$, $d^-(v) > 0$. A vertex v is called a carrier if $d^+(v) = d^-(v) = 1$. The total degree (or simply degree) of v is $d(v) = d^+(v) + d^-(v)$. If d(v) = k for every v $\in$ V, then D is said to be a k-regular digraph. If for every v $\in$ V, $d^+(v) = d^-(v)$, the digraph is said to be an isograph or balanced digraph. A walk of length h from a vertex u to a vertex v ($u \to v$ walk) in G is a sequence of vertices $u = u_0, u_1, . . . , u_{h-1}, u_h = v$ such that each pair (u_{i-1}, u_i) is an arc of G. An arc sequence in a digraph D is an alternating sequence of vertices and arcs of D. A digraph G is strongly connected if there is a $u \to v$ walk for any pair of vertices u and v of G. The length of a shortest $u \to v$ walk is the distance from u to v, denoted by dist(u, v). If there is no $u \to v$ walk in G then we define dist(u, v) = ∞. The eccentricity of a vertex u, denoted by e(u), is the maximum distance from u to any vertex in G. If dist(u, v) = e(u) ($v \neq u$) we say that v is an eccentric vertex of u. The radius of G, rad(G), is the minimum eccentricity of the vertices in G; the diameter, diam(G), is the maximum eccentricity of the vertices in G. If these two are equal in a graph, that graph is called self-centered graph with radius r. A graph G is unique eccentric vertex graph if each vertex in G has exactly one eccentric vertex [6, 7].

In digraph, * denotes the symmetric digraph.

The corona [10] G_1 o G_2 of two graphs G_1 and G_2 was defined as the graph G obtained by taking one copy of G_1(which has n vertices) and n copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

The Cocktail Party graph [11] $H_{m,n}$, m, n $\geq$ 2, is the graph with a vertex set V = $\{v_1, v_2, …, v_{mn}\}$ partitioned into n independent sets $I_1, I_2, …, I_n$ such that $V = I_1 \cup I_2 … \cup I_n$ each of size m such that $v_iv_j \in E$ for all i, j $\in$ {1, 2, …, mn} where $i \in I_p$, $j \in I_q$, $p \neq q$.

The hypercube Q_n is the graph whose vertex set is $\{0, 1\}^n$ and where two vertices are adjacent if they differ in exactly one coordinate. Let $V(Q_n) = \{v_1, v_2, …, v_{2^n}\}$ and let $v_i \in V(Q_n)$ with $v_i = (v_{i1}, v_{i2}, …, v_{in})$, v_{ij} = 0 or 1. Then $\overline{v_i} = (\overline{v_{i1}}, \overline{v_{i2}}, ..., \overline{v_{in}})$ is a vertex of $V(Q_n)$ with $\overline{v_{ij}} \neq v_{ij}$, i, j = 0 or 1.

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A Spider is a tree with at most one vertex of degree more than two, called the center of Spider. A leg of a Spider is a path from the center to a vertex of degree one. Thus, a star with *k* legs is a Spider of *k* legs, each of length 1.

For disjoint graphs G and H, the join G + H has $V(G) \cup V(H)$ vertices and $E(G+H) = E(G) \cup E(H) \cup \{uv: u \in V(G)$ and $v \in V(H)\}$.

The eccentric digraph[5] of a digraph G, denoted by ED(G), is the digraph with the same vertex set as G with an arc from vertex u to vertex v in ED(G) if and only if v is an eccentric vertex of u in G.

Given a positive integer $k \geq 2$, the k^{th} iterated eccentric digraph of G [8] is written as $ED^k(G) = ED(ED^{k-1}(G))$ where $ED^0(G) = G$.

Let G be a graph. A vertex v is a boundary vertex [4] of u if $d(u, w) \leq d(u, v)$ for all $w \in N(v)$. A vertex v is called a boundary neighbor of u if v is a nearest boundary of u.

We introduced the boundary graph $G_b(G)$ [1] of a graph G and boundary digraph BD(G) of a graph G. Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G). The boundary graph of a graph G, denoted by $G_b(G)$ has the same set of vertices as G with two vertices u and v being adjacent in $G_b(G)$ if and only if either v is a boundary vertex of u in G or u is a boundary vertex of v in G.

We need the following results to study the boundary digraph of a graph G.

Theorem: 1.1 [9] A digraph D is unilateral if and only if it has a spanning arc sequence.

Theorem: 1.2 [9] A digraph is strong if and only if it has a spanning closed arc sequence.

2. Boundary Digraph and Boundary Neighbor Digraph of a graph G:

In [1], we have defined the boundary digraph BD(G) of a graph G as follows:

The boundary digraph BD(G) of a graph (digraph) G is the digraph that has the same vertex set as G and a directed edge (arc) from u to v exists in BD(G) if and only if v is a boundary vertex of u in G.

Example: 2.1

G:

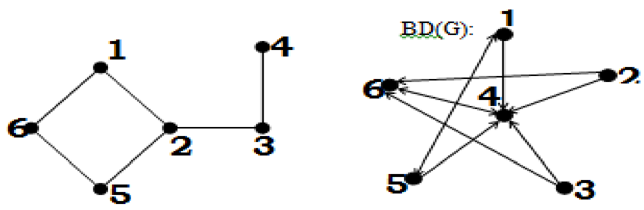

Fig. 2.1

Boundary digraph

A digraph G is said to be boundary digraph if there exists a graph H such that G = BD(H).

Example: 2.2

G: H:

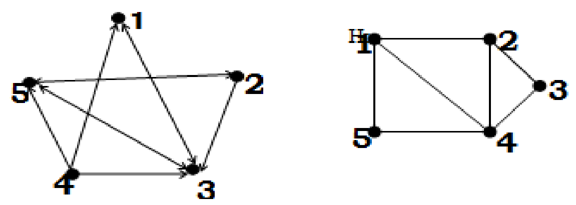

Fig. 2.2

Here G is a boundary digraph, since $BD(H) \cong G$.

Now, we define new graphs known as boundary neighbor graph and boundary neighbor digraph of a graph G as follows:

The boundary neighbor graph BN(G) of a graph G is the graph that has the same vertex set as G with two vertices u and v being adjacent in BN(G) if and only if either v is a boundary neighbor of u in G or u is a boundary neighbor of v in G.

The boundary neighbor digraph BND(G) of a graph G is the graph that has the same vertex set as G and a directed edge (arc) from u to v exists in BND(G) if and only if v is a boundary neighbor of u in G.

Example: 2.3

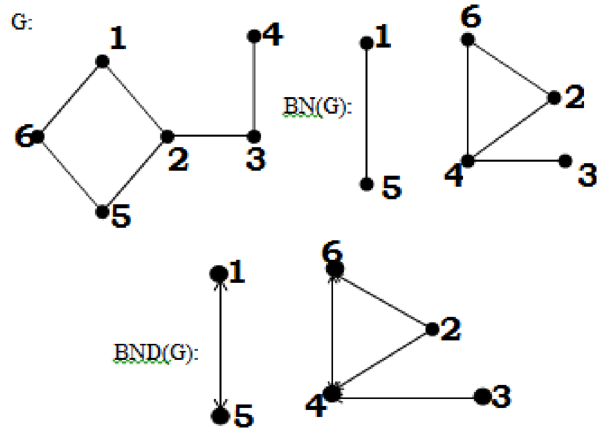

Fig 2.3

Observation: 2.1

(i) Since every vertex in a connected graph has at least one boundary vertex, in a boundary digraph, every vertex will have out degree at least one.

(ii) Odd cycles are a class of graphs for which $BD(G) \cong G$.

(iii) Two isomorphic graphs have their boundary digraph isomorphic but the converse need not be true always.

Example: 2.4

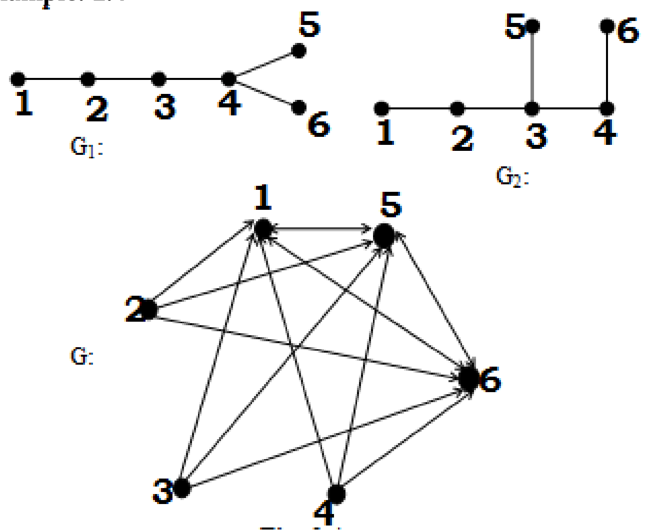

Fig: 2.4

G, G_2 are non isomorphic graphs, but $BD(G_1) = BD(G_2)$.

(iv) All the eccentric vertices of a connected graph G are boundary vertices also, but not conversely. Hence, always eccentric digraph of a connected graph G is a spanning sub graph of the boundary digraph of G.

(v) If the eccentric vertices are the only boundary vertices, then ED(G) = BD(G).

(vi) $K_{1,n}$ is not a boundary digraph of any other graph.

(vii) If G is disconnected, eccentric vertices are not boundary vertices.

(viii) If G is the complete graph K_n, then $BD(G) = G^*$.
(ix) The boundary digraph of a directed cycle is $BD(C_n) = C_n$.
(x) A non-trivial boundary digraph has no vertex of out degree zero.
(xi) If boundary vertices are at least two in G, then $d^-(v) > 0$ for at least two vertices in BD(G).
(xii) There exists at least one bidirectional edge in BD(G).
(xiii) BND(G) is a sub graph of BD(G) and BN(G) is a sub graph of $G_b(G)$.
(xiv) BD(G) = BND(G) if and only if there exists only one boundary vertex for each vertex in G.
(xv) BND(G) may be connected or disconnected.
(xvi) If $P = v_1, v_2, \ldots, v_n$ is a diametral path in G, then v_1 and v_n are boundary neighbors for some vertices of G. Thus G has at least two boundary neighbors.

Theorem: 2.1

$BD(K_n) = BND(K_n) = K_n^*$.

Proof

When $G = K_n$. Any vertex $u \in V(G)$ is a boundary vertex of other vertices. Hence, $BD(K_n) = K_n^*$ and $BD(K_n)$ is symmetric and its underlying graph is $G_b(G) = K_n$.

Theorem: 2.2

Let W_n, $n \geq 4$ be a wheel graph. Then the underlying graph of $BD(W_n)$ is $K_1+\overline{C_n}$. Also, $BD(W_n) = BND(W_n)$.

Proof

Let $G = W_n$, $n \geq 4$ be a wheel graph with $V(G) = \{v, v_1, v_2, \ldots, v_n\}$. Let v be a central vertex of G. $v_1, v_2, \ldots, v_n$ are the boundary vertices of v and $v_1, v_2, \ldots, v_{i-2}, v_{i+2}, \ldots, v_n$ are the boundary vertices of v_i. This proves the theorem.

Theorem: 2.3

If $G = C_n$, then $BD(G) = BND(G) = \begin{cases} C_n^* & \text{if } n \text{ is odd} \\ \dfrac{n}{2}K_2^* & \text{if } n \text{ is even} \end{cases}$

Proof

Let G be a cycle with n vertices.

Case (i) n is odd ($n = 2r+1$, where r is the radius of C_n)

Let $v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_n$ be the vertices of C_n. Every vertex v_i of C_n has two boundary vertices. Thus, the out-degree of every vertex in BD(G) is two. Each vertex of C_n is a boundary vertex of two vertices in G. Thus, in-degree of every vertex v_i of BD(G) is two in BD(G). Thus, we get a directed cycle C_n^*. Hence, $BD(G) = C_n^*$.

Case (ii) n is even ($n = 2r$, where r is the radius of C_n)

Every vertex of C_n has exactly one boundary vertex. Also, if u is a boundary vertex of v then v is a boundary vertex of u. Therefore, we get $BD(G) = (n/2)K_2^*$.

Theorem: 2.4

If $G = P_n$, then BD(G) has $(n-2)$ transmitters.

Proof

Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n. The vertices v_1 and v_n are the pendant vertices. The non pendant vertices of P_n have two boundary vertices v_1 and v_n, and the non pendant vertices are not boundary vertices. Thus, the $(n-2)$ non pendant vertices have indegree zero and outdegree 2. Hence, BD(G) has $(n-2)$ transmitters.

Theorem: 2.5

Let G be a graph $K_{2n}-M$, where M is a 1-factor, then $BD(G) = nK_2$.

Proof

Let $G = K_{2n}-M$, where M is a 1-factor. G is a unique eccentric point graph. Thus in this graph, eccentric vertices are as same as the boundary vertices. Then $BD(G) = nK_2$.

Theorem: 2.6

Let G be a graph $K_{m,n}$, then $BD(K_{m,n}) = K_m^* \cup K_n^*$.

Proof

Let G be a graph $K_{m,n}$ with vertex set $V(G) = V_1 \cup V_2$. $|V_1(G)| =$ m vertices form a complete digraph, since every vertex $u \in V_1(G)$ have their boundary vertices in $V_1(G)$ and vise versa. By the definition of BD(G), we have two components K_m^* and K_n^* of BD(G).

Corollary: 2.6

The boundary digraph of a complete multipartite digraph is a disjoint union of complete digraphs more precisely, $BD(K_{n1, n2, \ldots, nk}) = K_{n1} \cup K_{n2} \cup \ldots \cup K_{nk}$.

Theorem: 2.7

(i) If $G = \overline{C_n}$, then $BD(G) = C_n^*$ and BD(G) is symmetric.

(ii) If $G = \overline{P_n}$, then $BD(G) = P_n^*$ and BD(G) is symmetric.

Proof

(i) Let $G = \overline{C_n}$. In $\overline{C_n}$, every vertex $v_i \in V(BD(G))$ has two boundary vertices v_{i-1}, v_{i+1}. $d^+(v_i) = 2$ in BD(G). Also, every vertex v is a boundary vertex of exactly two vertices of G. Therefore, $d^-(v_i) = 2$ in BD(G). Hence, $BD(G) = C_n^*$.

(ii) Let $G = \overline{P_n}$ with $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$. The non adjacent vertices of v_i in G are the boundary vertices of v_i in BD(G). In BD(G), for every vertex v_i, $d^+(v_i) = d^-(v_i) = 2$ for $i = 2, 3, \ldots, n-1$ and $d^+(v_i) = d^-(v_i) = 2$ for $i = 1, n$. Hence, $BD(G) = P_n^*$.

Remark: 2.1

(i) When n is even, $BD(C_n) = (n/2)K_2^*$ and $BD(\overline{C_n}) = C_n^*$.

(ii) When n is odd, $BD(C_n) = BD(\overline{C_n}) = C_n^*$.

Theorem: 2.8

If $G = S_{m,n}$, then BD(G) has 2 transmitters and m+n vertices of BD(G) form a complete symmetric sub graph, where $S_{m,n} = \overline{K_m} + K_1 + K_1 + \overline{K_n}$.

Proof

Let G be a bistar $S_{m,n}$ with m+n pendant vertices, u and v are the central vertices. The m+n pendent vertices form a complete symmetric sub graph in BD(G) and the central vertices u and v have out degree m+n in BD(G) and $d^+(v) = d^+(u) = 0$. Hence, u and v are transmitters in BD(G).

Theorem: 2.9

If $G = K_{1,n}$, then $BD(K_{1,n})$ is unilateral.

Proof

Consider $G = K_{1,n}$ with vertex set $V(G) = \{v, v_1, v_2, \ldots, v_n\}$. Let v be a central vertex of G and v has n boundary vertices in G. Thus, the out-degree of a vertex v is n and in-degree is zero in BD(G). The pendant vertices of G form a complete digraph in BD(G). Thus, $v, v_1, v_2, \ldots, v_n$ is a spanning arc sequence in BD(G). By Theorem 1.1, BD(G) is unilateral. The underlying graph of $BD(K_{1,n})$ is $K_1 + K_n = K_{n+1}$.

Theorem: 2.10

A digraph BD(G) is strong if $G = \overline{C_n}$ or $G = C_n$, n is odd.

Proof

(i) Assume $G = C_n$, n is odd. By Theorem 2.3, we get $BD(G) = C_n^*$. Let u and v be two distinct vertices of C_n. If v follows u in any spanning closed arc sequence, say Q of BD(G), then there exists a sequence of the arcs of Q forming an arc sequence from u to v. If u follows v in Q, then there is an arc sequence from u to the last vertex of Q and an arc sequence from that vertex to v. By Theorem 1.2 BD(G) is strong.

(ii) Assume G = $\overline{C_n}$, by theorem 2.7, BD($\overline{C_n}$) = C_n*. The proof is similar to proof of case(i).

Theorem: 2.11

If G is a graph with radius 1, then radius of BD(G) is one.

Proof

Let G be a graph with radius one. The center vertex u has p–1 boundary vertices. Thus the out degree of u is p–1 in BD(G). Hence, the eccentricity of u is one in BD(G).

Theorem: 2.12

If a connected graph G has a pendant vertex, then in degree of that vertex is p–1 in BD(G).

Proof

Let G be a connected graph with pendant vertex u. u is a boundary vertex of all other vertices of G. By the definition of BD(G), in degree of u is p–1.

Theorem: 2.13

If T is a tree with m pendant vertices, then BD(T) has p–m transmitters and m vertices having in-degree p–1 and out-degree m–1.

Proof

Let T be a tree with m pendant vertices. These pendant vertices are only the boundary vertices. Thus the pendant vertices are boundary vertex of other vertices of T. Hence, the in-degree of pendent vertex of T is p–1 and the out-degree of pendant vertex is m–1 in BD(T). A non pendant vertex of T has in degree zero and out-degree m > 0 in BD(T). Therefore, BD(T) has p–m transmitters.

Theorem: 2.14

If G = C_{2n+1}, then BD(G) is Hamiltonian and Eulerian.

Proof

Let G = C_{2n+1}. By Theorem 2.10, BD(G) has a spanning closed arc sequence. Thus, BD(G) is Hamiltonian. By Theorem 2.3, in BD(G), $d^-(v_i)=d^+(v_i)$ = 2 for all $v_i \in V(BD(G))$, i = 1, 2, 3, …, 2n+1. Thus, BD(G) is Eulerian.

Theorem: 2.15

Let G = H^+, then BD(G) has n transmitters and n vertices form a complete sub graph.

Proof

Consider G = H^+. $|V(H)|$ = n and $|V(G)|$ = 2n. By the definition of boundary digraph, the non pendant vertices of G are transmitters. The pendant vertices of G form a complete symmetric sub graph on n vertices.

Theorem: 2.16

Let G be a graph with r(G) > 1. If G has a pendant vertex, then ED(G) ≠ BD(G).

Proof

Suppose v is a pendant vertex of G and u is its support vertex. Then there is an arc from u to v in BD(G), but it is not in ED(G). Hence, ED(G) ≠ BD(G).

Theorem: 2.17

If there is a triangle with a vertex of degree two or there exist a clique in G with r vertices, having a vertex of degree r–1, then ED(G) ≠ BD(G).

Proof

Let G be a connected graph. Consider G has a triangle with a vertex of degree two. Let u, v, w form a triangle with deg w = 2. Then w is a boundary vertex of u and v. But w is not an eccentric vertex of u and v and hence ED(G) ≠ BD(G). Similarly, if there exist a clique in G with r vertices and having a vertex of degree r−1 in that clique, then BD(G) ≠ ED(G).

Remark: 2.2

The converse of the theorem 2.16 is not true.

For example,

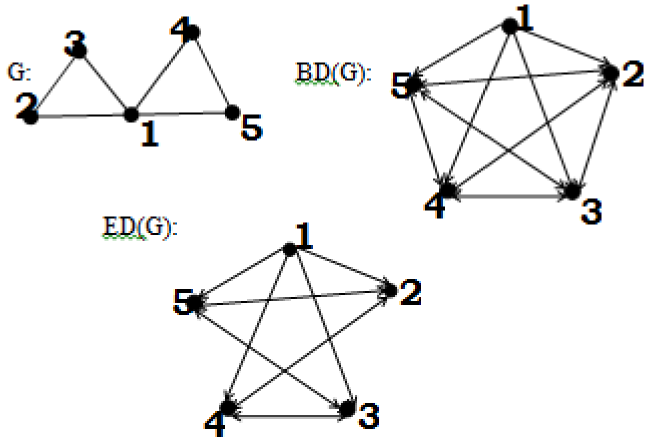

Fig: 2.5

Here ED(G) ≠ BD(G). But G has no pendant vertex.

Theorem: 2.18

If G is a self-centered unique eccentric point graph, then BD(G) = $(n/2)K_2^*$.

Proof

Assume G is a self-centered unique eccentric point graph with n vertices. Every vertex is an eccentric vertex in G. Each vertex has exactly only one boundary vertex which is also the eccentric vertex. Hence, BD(G) = $(p/2)K_2^*$ and its underlying graph is $G_b(G) = (p/2)K_2$.

Corollary 2.18:

If G is a Hypercube graph Q_n, then BD(G) = $2^{n-1}K_2^*$.

Proof

Let G be a Hypercube graph Q_n. Q_n is a self centered unique eccentric point graph. In this graph, the boundary vertices are the eccentric vertices. By Theorem 2.18, BD(G) = ED(G) = $(2^n/2)\ K_2^* = 2^{n-1}K_2^*$.

Theorem: 2.19

If G is a Cocktail party graph $H_{m,\,n}$, then BD(G) = nK_m^*.

Proof

Let G be a Cocktail party graph $H_{m,\ n}$. The boundary vertices are the eccentric vertices. BD(G) = ED(G) = nK_m^*.

Theorem: 2.20

If G is a disconnected graph with k components, then

(i) BD(G) is also a disconnected digraph.

(ii) ED(G) is connected and ED(G) is a k-partite graph.

Proof:

(i) Let G be a disconnected graph with two components G_1 and G_2. Here the boundary vertices of G_1 are in G_1 and vice versa. Thus $BD(G_1)$ and $BD(G_2)$ are the components of BD(G). Hence BD(G) is a disconnected graph with two components. Similarly if G has k components, then BD(G) also has k components.

(ii) Consider a connected graph G has k components. Let G_1, G_2, …, G_k be the components of G. For each vertex $v \in V(G_i)$, $e(v) = V(G_1) \cup V(G_2) \cup \ldots \cup V(G_{i-1}) \cup V(G_{i+1}) \cup \ldots \cup V(G_k)$. Thus all the vertices of G_i are adjacent to all the vertices of other components of G in ED(G). There is no arcs between any two vertices of G_i in ED(G). Therefore, ED(G) is a k-partite graph.

Theorem: 2.21

For a path P_m,

(i) $BN(P_m) = \overline{K}_{n-1} + K_1 + K_1 + \overline{K}_{n-1}$ if m is even and m = 2n.

(ii) $BN(P_m) = G'$ if n is odd and m = 2n+1, where G′ is given in figure 2.6.

G′:

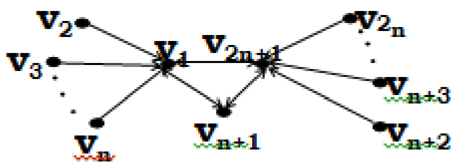

Fig: 2.6

BND(G) is a digraph whose underlying graph is $BN(P_m)$.

Proof

Let $G = P_m$ be a path with n vertices $v_1, v_2, v_3, \ldots, v_m$.

Case(i): m is even (m = 2n)

$V(G) = \{v_1, v_2, v_3, \ldots, v_n, \ldots, v_{2n}\}$. v_1 and v_{2n} are the boundary vertices of P_n. v_1 is a boundary neighbour of $v_2, v_3, \ldots, v_n, v_{2n}$ and vertex v_{2n} is a boundary neighbour of $v_{n+1}, v_{n+2}, v_{n+3}, \ldots, v_{2n-1}, v_1$. There exists a bidirectional edge between the vertices v_1 and v_{2n} in BND(G). The vertices v_1 and v_{2n} have in-degree n and out-degree 1 in BND(G).

Hence, $BN(G) = \overline{K}_{n-1} + K_1 + K_1 + \overline{K}_{n-1}$ and BND(G) is a digraph whose underlying graph is $\overline{K}_{n-1} + K_1 + K_1 + \overline{K}_{n-1}$.

Case(ii): m is odd (m = 2n+1)

$V(G) = \{v_1, v_2, v_3, \ldots, v_n, v_{n+1}, \ldots, v_{2n}, v_{2n+1}\}$. v_1 and v_{2n} are the boundary vertices of P_n. v_1 is a boundary neighbor of $v_2, v_3, v_4, \ldots, v_n, v_{n+1}, v_{2n+1}$ and vertex v_{2n+1} is a boundary neighbor of $v_{n+1}, v_{n+2}, v_{n+3}, \ldots, v_{2n}, v_1$. There exists a bidirectional edge between the vertices (v_1, v_{2n+1}), (v_1, v_{n+1}), (v_{2n+1}, v_{n+1}) in BND(G). The vertices v_1 and v_{2n+1} have in-degree n+1 and out-degree 2 in BND(G). Hence BND(G) = G′ is given in Fig 2.6.

Theorem: 2.22

Let H be a path of length n. If G is obtained from H by attaching a path of length m, then $BN(G) = P_n \circ (m-1)K_1$ and BND(G) is a digraph whose underlying graph is $P_n \circ (m-1)K_1$.

Proof

Let H be a path P_n and G has n boundary vertices $u_{1m}, u_{2m}, u_{3m}, \ldots, u_{nm}$. These vertices are the boundary vertices of G. u_{im} is the boundary neighbor of $u_{(i-1)m}$ and $u_{(i+1)m}$ in G. It follows that, the pendant $v_1, v_2, v_3, \ldots, v_n$ represent the vertices of H. Consider a path P_m with vertex set $\{u_1, u_2, u_3, \ldots, u_m\}$. A graph G obtained from H by attaching a path P_m at each vertex v_i of H. Thus vertices of G form a path P of length n in BN(G). Pendant vertex u_{1m} is the boundary neighbor of $u_{11}, u_{12}, u_{13}, \ldots, u_{1(m-1)}$. In BN(G), $u_{11}, u_{12}, \ldots, u_{1(m-1)}$ becomes the pendant vertices of BN(G). Hence degree of u_{1m} is m in BN(G). Similarly, degree of u_{nm} is also m in BN(G). The vertices $u_{2m}, u_{3m}, \ldots, u_{(n-1)m}$ have the degree m+1 in BN(G). Hence $BN(G) = P_n \circ (m-1)K_1$ and BND(G) is a digraph whose underlying graph is $P_n \circ (m-1)K_1$.

Remark: 2.3

(i) A symmetric path P of length n is an induced sub graph of BND(G).

(ii) BND(G) has n(m−1) transmitters.

Theorem: 2.23

Let G be a connected graph with m vertices. Then $BN(G \circ K_n) = mK_{n+1}$ and $BND(G \circ K_n) = mK_{n+1}^*$.

Proof

Consider a connected graph G with vertex set |V(G)| = m. In $G \circ K_n$, every vertex of G has n boundary neighbors. Thus, every vertex of G is adjacent to all the vertices of K_n in $BN(G \circ K_n)$. A vertex $v \in V(K_n)$ has n boundary neighbours in $BN(G \circ K_n)$. The vertices of G are disjoint in $BN(G \circ K_n)$. Hence $BN(G \circ K_n) = mK_{n+1}$ and $BND(G \circ K_n) = mK_{n+1}^*$.

Theorem: 2.24

Let G be a connected graph with n vertices. Then $BN(G \circ 2K_1)$ has n triangles and also disconnected.

Proof:

Consider a connected graph G with vertex set $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$. Let v_i' and v_i'' be the pendant vertices adjacent to v_i in $G \circ 2K_1$ for i = 1, 2, …, n. The vertices v_i' and v_i'' are the boundary neighbor of v_i in $G \circ 2K_1$ and v_i' is the boundary neighbour of v_i'' and vice versa. Thus the vertices of G are disjoint in $BN(G \circ 2K_1)$. Hence, $BN(G \circ 2K_1)$ is disconnected and has n triangles and its digraph is $BND(G \circ 2K_1)$. nK_2^* is an induced sub graph of $BND(G \circ 2K_1)$.

Corollary: 2.24

For any connected graph G,

(i) $BN(G \circ mK_1) = P_m + K_1$.

(ii) n symmetric arc sequence with length m in $BND(G \circ mK_1)$

Proof:

The proof follows from Theorem 2.24.

Theorem: 2. 25

If G is a spider with 2n+1 vertices, then $BN(G) = (K_{n+1} \circ K_1) - u'$, where u′ is a vertex which is attached with u in $K_{n+1} \circ K_1$, where u is the central vertex of G.

Proof:

Consider a spider G with 2n+1vertices. Let u be a central vertex of G. G has n pendant vertices. By the definition of BN(G), the pendant vertices of G form a complete sub graph in BN(G). Central vertex u of G has n boundary neighbors in G. Thus u is adjacent to all the pendant vertices of G in BN(G). Hence it follows that, K_{n+1} is an induced sub graph of BN(G). Consider a leaf $e_i = v_i'v_{i+1}'$ in G. v_{i+1}' is a boundary neighbor of v_i' in G. Thus the pendant vertices are the boundary neighbors of the corresponding support vertices. Hence, the support vertices of G are adjacent to their corresponding pendant vertex in BN(G). Hence, $BN(G) = (K_{n+1} \circ K_1) - u'$.

Corollary: 2.25

If G is a spider with km+1 vertices, then $BN(G) = (K_{n+1} \circ K_1) - \{v_1', v_2', v_3', \ldots, v_{k-1}'\}$, where$\{v_1', v_2', v_3', \ldots, v_{k-1}'\}$ are the vertices which are attached with u in $K_{n+1} \circ K_1$, where u is the central vertex of G.

Proof:

The proof follows from the Theorem.

Theorem: 2.26

If G is a spider with 2n+1 vertices, then BND(G) has n+1 transmitters and K_n^* is an induced sub graph of BND(G).

Proof

Let G be a spider with 2n+1 vertices. The non pendant vertex u of G has $d^-(u) = 0$ and $d^+(u) > 0$ in BND(G). Thus BND(G) has n+1 transmitters. By the definition of BND(G), the pendant vertices are adjacent to each other. Hence K_n^* is an induced sub graph of BND(G).

Theorem: 2.27

If G is a wounded spider with one wounded leg, then BN(G) = G.

Proof

When G is a wounded spider having one wounded leg e = uv, Let u be a central vertex. By the definition of BN(G), vertex v is a boundary neighbor of the pendant vertices of G and the central vertex u. The boundary vertex of other support vertices is their corresponding pendant vertices in G. Thus the support vertex of G other than u become a pendant vertex in BN(G) and the pendant vertex v of wounded leg e in G become a central vertex in BN(G). The central vertex of G is adjacent to v only in BN(G). Hence, BN(G) is a wounded spider.

Corollary: 2.27

If G is a wounded spider, then BN(G) is a graph of radius 2 and diameter 4.

Proof

The proof follows from the Theorem 2.27.

Remark: 2.4

If G is a wounded spider with m wounded legs and n non wounded legs, then

(i) $K_{m,n}$ is an induced sub graph of BN(G).

(ii) $K_{1,m}$ is an induced sub graph of BN(G).

(iii) BN(G) has n pendant vertices.

Theorem: 2.28

If G is a wounded spider with n non wounded legs, then BND(G) has n+1 transmitters.

Proof

Assume G has m wounded legs and n non wounded legs. The central vertex of G and m pendant vertices have n boundary neighbors. The support vertices of G has only one boundary neighbor. The support vertex v_i and central vertex u of G are not a boundary neighbor of any other vertex. Therefore, $d^+(v_i) = d^+(u) = 0$ and $d^-(v_i) > 0$, $d^-(u) > 0$. This implies, BND(G) has n+1 transmitters.

Iterated Boundary Digraph

The Boundary Digraph BD(G) of a digraph G is the digraph that has the same vertex set as G and the arc set defined as follows there is an arc from u to v if and only if v is a boundary vertex of u.

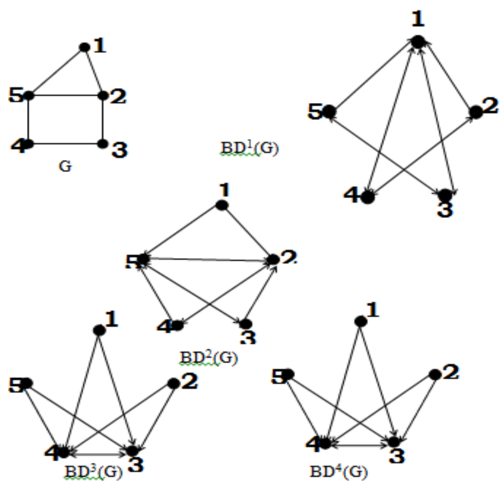

Fig: 2.7

The boundary digraph of a graph and boundary graph of a graph was introduced by us in [1]. An example of a graph and its boundary digraph is given in Example 2.1. Note that arcs of graphs are drawn as directed edges with arrows.

Given a positive integer $k \geq 2$, the k^{th} iterated boundary digraph of G is written as $BD^k(G) = BD(BD^{k-1}(G))$ where $BD^0(G) = G$. The following example illustrates these definitions showing graph G and its iterated boundary digraphs BD(G), $BD^2(G)$, $BD^3(G)$ and $BD^4(G)$. Note that in this case, $BD^3(G) = BD^4(G)$.

An interesting line of investigation concerns the iterated sequence of boundary digraphs. For every digraph G there exist smallest integer numbers $p > 0$ and $t \geq 0$ such that $BD^t(G) \cong BD^{p+t}(G)$, where $\cong$ denotes graph isomorphism.

Iterated Boundary Neighbor Digraph

The Boundary Neighbor Digraph BND(G) of a digraph G is the digraph that has the same vertex set as G and the arc set defined as follows: there is an arc from u to v if and only if v is a boundary neighbor of u.

An example of a graph and its boundary neighbor digraph is given in Example 2.3. Note that arcs of graphs are drawn as directed edges with arrows.

Given a positive integer $k \geq 2$, the k^{th} iterated boundary neighbor digraph of G is written as $BND^k(G) = BND(BND^{k-1}(G))$ where $BND^0(G) = G$. The following example illustrates these definitions showing graph G and its iterated boundary neighbor digraphs BND(G), $BND^2(G)$ and $BND^3(G)$. Note that in this case, $BND^2(G) = BND^3(G)$.

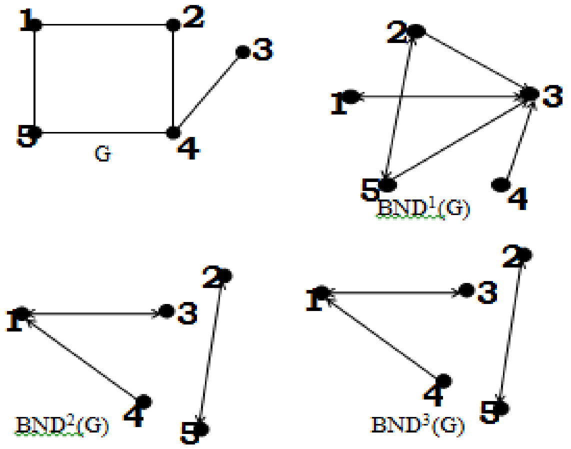

Fig: 2.8

For every digraph G there exist smallest integer numbers $p > 0$ and $t \geq 0$ such that $BND^t(G) \cong BND^{p+t}(G)$, where $\cong$ denotes graph isomorphism.

Conclusion

In this paper, some properties of boundary digraph of a graph G, Boundary neighbor graph of a graph G and Boundary neighbor digraph of a graph G are discussed. Iterated boundary digraph and Iterated boundary neighbor digraph are studied.

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