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Urysohn Lemma and Tietze Extension Theorem in Fuzzy soft topological space

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ABSTRACT
In this paper fuzzy soft mapping, fuzzy soft continuity on family of soft sets are introduced. Equivalent conditions related these concepts are proved. The famous Urysohn lemma and Tietze Extension theorem are established in fuzzy soft setting.

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1. Introduction

In 1999, Molodtsov[5] proposed a new approach viz soft set theory for modeling vagueness and uncertainties inherent in the problems of physical science, biological science, engineering, economics, social science ,medical science, etc. After that in 2001 to 2003 Maji et al[3,4] worked on some mathematical aspects of soft sets and fuzzy soft sets. On the other hand, Biswas and Nanda[2] and Rosenfeld[7] worked on rough groups and fuzzy groups respectively. In 2007 Aktas and Cagman[1] introduced a basic version of soft groups [6] theory which further extended to fuzzy soft group[6] in 2011. Recently, in 2011, Shabir and Naz[9] introduced a notion of fuzzy soft topological spaces.

In this paper fuzzy soft mapping and fuzzy soft continuity on family of soft sets are defined and some basic theorems related to these concepts are established. Later the Urysohn Lemma and Tietze Extension theorem are proved in fuzzy soft topological space.

2. Preliminaries

In this section we present some basic definitions of fuzzy soft set. Throughout our discussion, U refers to an intial universe, E the set of all parameters for U and $P(\tilde{U})$ the set of all fuzzy sets of U. (U,E) means the universal set U and the parameter set E.

Definition 2.1 [5]

A pair (F, E) is called a soft set (over U) if and only if F i s a mapping of E into the set of all subsets of the set U.

Definition 2.2 [4]

A pair (F, A) is called a fuzzy soft set over U where $F : A \to P(\tilde{U})$ is a mapping from A into $P(\tilde{U})$.

Definition 2.3 [4]

For two fuzzy soft sets (F, A) and (G, B) in a fuzzy soft cl ass (U, E), we say that

(F, A) is a fuzzy soft subset of (G, B), if$\subseteq$

(i) $A \subseteq B$

(ii) For all $\varepsilon \in A$, $F(\varepsilon) \subseteq G(\varepsilon)$ and is written as

$(F, A) \tilde{\subseteq} (G, B)$.

Definition 2.4 [4]

Union of two fuzzy soft sets (F, A) and (G, B) in a soft class (U, E) is a fuzzy soft set (H,C) where $C = A \cup B$ and $\forall \varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), & \text{if } \varepsilon \in A - B \\ G(\varepsilon), & \text{if } \varepsilon \in B - A \\ F(\varepsilon) \cup G(\varepsilon), & \text{if } \varepsilon \in A \cap B \end{cases}$$

and is written as

and is written as $(F, A) \tilde{\cup} (G, B) = (H,C)$

Definition 2.5 [4]

Intersection of two fuzzy soft sets (F, A) and (G, B) in a soft class (U, E) is a fuzzysoft set (H, C) where $C = A \cap B$ and $\forall \varepsilon \in C$, $H(\varepsilon) = F(\varepsilon)$ or $G(\varepsilon)$ (as both are same fuzzy set) and is written as $(F, A) \tilde{\cap} (G, B) = (H,C)$

Definition 2.6 [8]

Let $A \subseteq E$ then the mapping $F_A : E \to \tilde{P}(U)$,defined by $F_A(e) = \mu^e F_A$ (a fuzzy subset of U), is called soft set over (U,E), where $\mu^e F_A = \tilde{0}$ if $e \in E - A$ and $\mu^e F_A \neq \tilde{0}$ if $e \in A$. The set of all fuzzy soft set over (U,E) is denoted by FS (U,E).

Definition 2.7 [8]

The fuzzy soft set $F_\varphi \in FS(U, E)$ is called null fuzzy s oft set and it is denoted by $\tilde{\Phi}$. Here $F_\phi(e) = \tilde{0}$ for every $e \in E$

Definition 2.8 [8]

Let $F_E \in FS(U,E)$ and $F_E(e) = \tilde{1}$ for all $e \in E$.

Then F_E is called absolute fuzzy soft set. It is denoted by $\tilde{E}$.

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Definition 2.9 [8]

Let $F_A, G_B \in FS(U,E)$. If $F_A(e) \subseteq G_B(e)$ for all $e \in E$

i.e., if $\mu^e F_A \subseteq \mu^e G_B$ for all $e \in E$, *i.e.*, if $\mu^e F_A(x) \le \mu^e G_B(x)$ for all $x \in U$ and for all $e \in E$, then F_A is said to be fuzzy soft subset of G_B, denoted by $F_A \tilde{\subseteq} G_B$.

Definition 2.10 [8]

Let $F_A, G_B \in FS(U,E)$. Then the union of F_A and G_B is also fuzzy softset H_C, defined by

$H_C(e) = \mu^e H_C = \mu^e F_A \cup \mu^e G_B$ for all $e \in E$ where $C = A \cup B$. Here we write $H_C = F_A \tilde{\cup} G_B$.

Definition 2.11 [8]

Let $F_A, G_B \in FS(U,E)$. Then the intersection of F_A and G_B is also a fuzzy soft set H_C, defined by $H_C(e) = \mu^e H_C = \mu^e F_A \cap \mu^e G_B$ for all $e \in E$ where $C = A \cap B$. Here we write $H_C = F_A \tilde{\cap} G_B$

Definition 2.12

Let $F_A \in FS(U,E)$. The complement of F_A is denoted by $F_A^{\,C}$ and is defined

By $F_A^{\,C} : E \to \tilde{P}(U)$ is a mapping given by $F_A^{\,C}(\varepsilon)$= $[F(\varepsilon)]^C$, $\forall \varepsilon \in E$.

3.Urysohn Lemma and Tietze Extension Theorem in Fuzzy soft topological space

Definition 3.1

Let FS(U,E) and FS(U',E') be families of fuzzy soft sets over U and U' respectively and E,E' be parameters for universe U and U' respectively. Let u:U→U', p:E→E' then the fuzzy soft mapping $h_{up} : FS(U,E) \to FS(U',E')$ is defined as

1) If F_A is a fuzzy soft set in FS(U,E) then the image of F_A under h_{up} is written as $(h_{up})F_A$ a fuzzy soft set in $FS(U',E')$ such that

$$[h_{up}(F_A)](e')(s) = \begin{cases} \sup\limits_{s \in U^{-1}(s')} \left[\sup\limits_{e \in p^{-1}(e')} F_A(e) \right](s) & if\ p^{-1}(e) \ne \tilde{\phi}\ \ and\ \ U^{-1}(s) \ne \tilde{\phi} \\ 0 & otherwise \end{cases}$$

for every $s \in S'$ and $e \in E'$

2) If $F_{A'}$ be a fuzzy soft set in FS(U',E'). The inverse image of $F_{A'}$ under h_{up} is written as $(h_{up})^{-1} F_{A'}$ a fuzzy soft set in $FS(U,E)$ such that

$$[h_{up}^{-1}(F_{A'})](e)(s) = \begin{cases} F_{A'}(p(e))(u(s) & for\ p(e) \tilde{\in} E' \\ 0 & otherwise \end{cases}$$

for every $s \in S'$ and $e \in E'$

Definition 3.2

Let $(U_1, E_1, \Im_1)$ and $(U_2, E_2, \Im_2)$ be two fuzzy soft topological spaces relative to parameters E_1 and E_2 respectively. Then a fuzzy soft mapping $h_{up} : FS(U_1, \Im_1) \to FS(U_2, \Im_2)$ is said to be fuzzy soft continuous if $(h_{up})^{-1} F_{A'} \in \Im_1$ for each $F_{A'} \in \Im_2$.

Theorem 3.3

Let $(U_1, E_1, \Im_1)$ and $(U_2, E_2, \Im_2)$ be two fuzzy soft topological spaces and $h_{up} : FS(U_1, \Im_1) \to FS(U_2, \Im_2)$ be fuzzy soft mapping. Then the following are equivalent

i) h_{up} is is continuous

ii) For every fuzzy soft set $F_A \in FS(U_1, E_1)$, $h_{up}(\overline{F}_A) \tilde{\subseteq} \overline{h_{up}(F_A)}$.

iii) For every fuzzy soft closed set $F_{A'}$ in $FS(U_2, E_2)$, $(h_{up})^{-1} F_{A'}$ is fuzzy soft closed in $FS(U_1, E_1)$.

iv) For each $F_{e_1} \in FS(U_1, E_1)$ and each fuzzy soft neighbourhood $F_{A'}$ of $(h_{up})(F_{e_1})$ there exists a fuzzy soft neighbourhood F_A of (F_{e_1}) such that $(h_{up})(F_A) \tilde{\subset} F_{A'}$.

Proof

(i) $\Rightarrow$ (ii) Let us assume that the fuzzy soft mapping h_{up} is fuzzy soft continuous. Let F_A be any fuzzy soft set in $FS(U_1, E_1)$. We show that if $F_{e_1} \in \overline{F}_A$ then $(h_{up})(F_{e_1}) \tilde{\in} \overline{(h_{up})F_A}$. Let $F_{e_1} \in \overline{F}_A$ and F_A' be a fuzzy soft neighbourhood of $(h_{up})(F_{e_1})$.

Then $(h_{up})^{-1}(F_{A'})$ is a fuzzy soft neighbourhood of F_{e_1} in $FS(U,E)$. Then $(h_{up})^{-1}(F_{A'})$ and F_A are disjoint and so $F_{A'}(h_{up})F_A$ are disjoint. ie, $F_{e_1} \in \overline{(h_{up})F_A}$, hence $(h_{up})\overline{F}_A \tilde{\subseteq} \overline{h_{up} F_A}$.

(ii) $\Rightarrow$ (iii) Let F_A' be any fuzzy soft closed set in $FS(U_2, E_2)$ and let $(h_{up})^{-1}(F_{A'}) = F_A$.

Let us prove that F_A is fuzzy soft closed. That is $\overline{F}_A = F_A$.

Hence $(h_{up})(F_A) = h_{up}[(h_{up})^{-1} F_{A'}] \tilde{\subseteq} F_{A'}$.

If $F_{e_1} \in \overline{F}_A$, then $(h_{up})(F_{e_1}) \tilde{\in} (h_{up})(\overline{F}_A)$

$\tilde{\subseteq} \overline{(h_{up})(F_A)}$ (by (ii)) $\tilde{\subseteq} \overline{F}_{A'} = F_{A'}$ ($\because$ $F_{A'}$ is fuzzy soft closed)

So , $F_{e_1} \in (h_{up})^{-1}(F_{A'}) = F_A$. Thus $F_{e_1} \in \overline{F}_A$ implies $F_{e_1} \in F_A$. Hence $\overline{F}_A = F_A$. Therefore F_A is fuzzy soft closed.

(iii) $\Rightarrow$ (iv) Let $F_{A'}$ be any fuzzy soft compact open set in $FS(U_2, E_2)$, then $F_{A'}{}^C$ is fuzzy soft closed in $FS(U_2, E_2)$. (By (iii)). $(h_{up})^{-1} F_{A'}{}^C$ is fuzzy soft closed in $FS(U_1, E_1)$. And $(h_{up})^{-1}(F_{A'}{}^C) = [(h_{up})^{-1} F_{A'}]^C$. $\therefore (h_{up})^{-1} F_{A'}$ is fuzzy soft open in $FS(U_1, E_1)$ and (h_{up}) is fuzzy soft continuous.

(iv) $\Rightarrow$ (i) Proof is similar.

Theorem 3.4

Let $FS(U_1, E_1)$ and $FS(U_2, E_2)$ be families of all fuzzy soft sets over U_1 and U_2 respectively. For a function $h_{up} : FS(U_1, E_1) \to FS(U_2, E_2)$ the following statement are true.

i) $(h_{up})^{-1} F_{A'}{}^C = [(h_{up})^{-1} F_{A'}{}^C$ for any fuzzy soft set $F_{A'}$ in $FS(U_2, E_2)$

ii) $h_{up}[(h_{up})^{-1} F_{A'}] \tilde{\subseteq} F_{A'}$ if (h_{up}) is surjective.

iii) $F_A \tilde{\subseteq} (h_{up})^{-1}[(h_{up})\ F_A]$ for any fuzzy soft set F_A in $FS(U_1, E_1)$.

Proof

i) Consider

$([(h_{up})^{-1}(F_A')^C](e_1)(s) = F_{A'}{}^C(p(e_1)(U(s))$

= 1- $F_{A'}(p(e_1)U(s))$

$= 1_Y - [(h_{up})^{-1}(F_{A'})](e_1)(s)$

$= ([(h_{up})^{-1}(F_{A'})](e_1)(s))^C$

Hence $(h_{up})^{-1}(F_{A'})^C = ((h_{up})^{-1} F_{A'})^C$

ii) $[(hup)^{-1} F_{A'}](e_1)(s) = F_{A'}(p(e_1)(U(s))$

$(h_{up})[F_{A'} p(e_1)U(s)](e_2)(S')$

$$= \begin{cases} \sup_{S \in U^{-1}(s')} \left[\sup_{e_1 \in p^{-1}(e_2)} F_{A'}(p(e_1)U(s)) \right](e_1)(s) & if \quad U^{-1}(s') \neq \phi, p^{-1}(e_2) \neq \phi \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} \sup_{S \in U^{-1}(s')} \left[\sup_{e_1 \in p^{-1}(e_2)} [(h_{up})^{-1} F_{A'}] \right](e_2)(s) & if \quad U^{-1}(s') \neq \phi, p^{-1}(e_2) \neq \phi \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} \left\{ \left[\sup_{S = U^{-1}(s')} \left[\sup_{e_1 = p^{-1}(e_2)} F_{A'} \right](e_2)(s') \right](e_1)(s) \right\} \\ 0 \quad otherwise \end{cases}$$

$$= \left\{ \sup[\sup\ F_{A'}](e_1)(s) \right\}$$

if (h_{up})is surjective

$= F_{A'}$

iii) $(hup)^{-1}[(hup) F_A](e_2)(S')$

$$= (hup)^{-1} \begin{cases} \sup_{S \in U^{-1}(s')} \left[\sup_{e_1 \in p^{-1}(e_2)} (hup)^{-1} F_A \right](e_1)(s) & if \quad U^{-1}(s') \neq \phi, p^{-1}(e_2) \neq \phi \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} \sup_{S \in U^{-1}(s')} \left[\sup_{e_1 \in p^{-1}(e_2)} (hup)^{-1} F_A \right](e_1)(s) & if \quad U^{-1}(s') \neq \phi, p^{-1}(e_2) \neq \phi \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} \sup_{S \in U^{-1}(s')} \left[\sup_{e_1 \in p^{-1}(e_2)} F_A \quad p(e_1)U(s) \right] & if \quad U^{-1}(s') \neq \phi, p^{-1}(e_2) \neq \phi \\ 0 & otherwise \end{cases}$$

$= [F_A](e_2)(s)$

$if \quad s = U^{-1}(s') \neq \phi, e_1 = p^{-1}(e_2) \neq \phi$

ie., (h_{up}) is surjective.

Definition 3.5

Let $(U, E, \Im)$ be a fuzzy soft topological space. Then a subfamily **B** of $\Im$ is called a base for $\Im$ if every member of $\Im$ can be written as a union of members of **B**.

Definition 3.6

Let $(U, E, \Im)$ be a fuzzy soft topological space. Then a subfamily **S** of $\Im$ is called a subbase for $\Im$ if the family of finite intersection of its members forms a base for $\Im$.

Definition 3.7

A fuzzy soft topological space over U is said to be generated by a subfamily **S** of fuzzy soft set over U if every member of $\Im$ is a union of finite intersection of members of S.

Lemma 3.8

Let $(U, E, \Im)$ be a fuzzy soft topological space and F_A be fuzzy soft closed in U, if G_A is a fuzzy soft open set containing F_A then there exists a fuzzy soft open set H_A containing F_A such that $F_A \tilde{\subseteq} H_A \tilde{\subseteq} \overline{H}_A \tilde{\subseteq} G_A{}^C$.

Proof

Let $(U,E,\Im)$ be a fuzzy soft normal space . let F_A and G_A be two disjoint fuzzy soft closed sets in $(U,E,\Im)$. Then $(G_A)^C$ is fuzzy soft open and contains F_A. By the hypothesis there exists fuzzy soft open set H_A containing F_A such that $F_A \tilde{\subseteq} H_A \tilde{\subseteq} \overline{H}_A \tilde{\subseteq} G_A{}^C$.

Let $\overline{H}_A{}^C = W_A$ ie.,

$\mu_{W_A}^{e(a)} = 1 - \mu_{\overline{H}_A}^{e(a)}, \ forall \quad e \in E, a \in A$.

Then $\overline{H}_A \tilde{\subseteq} G_A{}^C$

$$\Rightarrow \mu_{\overline{H}_A}^{e(a)} = 1 - \mu_{G_A}^{e(a)}, \quad for \quad all \quad a \in A, e \in E$$

$$\Rightarrow \mu_{G_A}^{e(a)} = 1 - \mu_{\overline{H}_A}^{e(a)}, \quad for \quad all \quad a \in A, e \in E$$

=1- $\inf\left\{ \mu_{S_A}^{e(a)} \right.$: S_A is a fuzzy soft closed set containing H_A.

$\therefore G_A \tilde{\subseteq} \overline{H}_A{}^C$

Hence H_A and W_A are two fuzzy soft open set containing F_A and G_A respectively with H_A and W_A being disjoint.

Conversly suppose $(U,E,\Im)$ is fuzzy soft normal. Let G_A be a fuzzy soft open set containing the fuzzy soft closed set F_A. F_A is fuzzy soft closed implies $F_A{}^C$ is fuzzy soft open.

$\therefore F_A$ and $G_A{}^C$ are disjoint fuzzy soft closed sets in $(U,E,\Im)$.

Using fuzzy soft normality we can find a pair of disjoint fuzzy soft open sets H_A and W_A such that $F_A \tilde{\subseteq} H_A$ and $G_A{}^C \tilde{\subseteq} W_A$

where $H_A \tilde{\cap} W_A = \phi$

$$\Rightarrow H_A \tilde{\subseteq} W_A{}^C$$

$$\Rightarrow \overline{H}_A \tilde{\subseteq} \overline{W}_A{}^C$$

$$\Rightarrow \overline{H}_A \tilde{\subseteq} G_A$$

Since H_A is fuzzy soft open, $H_A \tilde{\subseteq} \overline{H}_A$

$$\therefore F_A \tilde{\subseteq} H_A \tilde{\subseteq} \overline{H}_A \tilde{\subseteq} G_A{}^C.$$

Definition 3.9

Let $a,b \in \Re$, let $F_{A_{(a,b)\tilde{\cap}[0,1]}}$ be a fuzzy soft set $F_{A_{(a,b)\tilde{\cap}[0,1]}} : E \to I^I$ Where I=[0,1] and I^I represents the set of all fuzzy sets on [0,1] defined by $F_{A_{(a,b)\cap[0,1]}}(e) = \mu_{F_A}^{e((a,b)\tilde{\cap}[0,1])}$ for every $e \in E$. Then the collection **B**= $\left\{ F_{A_{(a,b)\tilde{\cap}[0,1]}} : a,b \in \Re \right\}$ forms a base for the fuzzy soft topology on [0,1].

3.10 Urysohn's Lemma

Let $(U,E,\Im)$ be a fuzzy soft topological space and consider [0,1] with fuzzy soft topology. Then $(U,E,\Im)$ is fuzzy soft normal iff for any two disjoint fuzzy soft closed subsets F_A and G_A in $(U,E,\Im)$ there exists a fuzzy soft continuous map $h_{up} : FS(U_1,E_1) \to FS([0,1],E_2)$ such that

$$(h_{up})(F_A) = [(h_{up})F_A](e_2)(d) = F_{e_0} = (h_{up})(F_{e_a})$$

$$(h_{up})(G_A) = [(h_{up})G_A](e_2)(d) = F_{e_1} = (h_{up})(G_{e_a})$$

where $F_{e_a} \in F_A$, $G_{e_a} \in G_A$, $F_{e_0}(x) = \tilde{0}$ and $F_{e_1}(x) = \tilde{1}$, $\tilde{0},\tilde{1}$ are zero and unit fuzzy sets.

Proof

Let D be the set of all rational numbers in [0,1]. Arrange D in some order so that $d_0 = \tilde{0}$ and $d_1 = \tilde{1}$. Let the elements of D be listed as $\{d_0, d_1, ..., d_n\}$. Define for each $d \in D$ a fuzzy soft open set F_{A_d} in $(U,E,\Im)$ in such a way that for in $d,h \in D$ with $d < h$ then $\overline{F}_{A_d} \tilde{\subseteq} F_{A_h}$.

Construct a sequence of fuzzy soft open sets in $(U,E,\Im)$ as follows. First define $F_{A_{d_1}} = G_A{}^C$; a fuzzy soft closed set contained the fuzzy soft open set $F_{A_{d_0}} = F_A$ using fuzzy soft normality of $(U,E,\Im)$ and by lemma,

We get $F_{A_{d_0}} \tilde{\subseteq} \overline{F}_{A_{d_0}} \tilde{\subseteq} F_{A_{d_1}}$

In general let D_n denote the set consisting of all first 'n' rational numbers in the sequence. Suppose $F_{A_{d_0}}, F_{A_{d_1}}, ..., F_{A_{d_n}}$ be fuzzy soft sets satisfying the property. $\overline{F}_{A_d} \tilde{\subseteq} F_{A_h}$ for $d < h$ where $d,h \in \{d_0, d_1, ..., d_n\}$ consider the set $D_{n+1} = D \tilde{\cup} \{d_{n+1}\}$ which is a finite subset of [0,1]. In a finite simply ordered set every element has an immediate predecessor and an immediate successer. Let the immediate predecessor of d_{n+1} be d and the immediate successor by h. Where $d,h \in D_n$. The set F_{A_d} and F_{A_h} are already defined and let $d_r, d_s \in D_n$ such that $d_r < d$ or $h < d_s$. By induction hypothesis, $\overline{F}_{A_d} \tilde{\subseteq} F_{A_h}$.

Therefore by normality $(U,E,\Im)$, there exists a fuzzy soft open set H_A in $(U,E,\Im)$ such that

$$F_{A_d} \tilde{\subseteq} H_A \tilde{\subseteq} \overline{H}_A \tilde{\subseteq} F_{A_h}$$

Take $F_{A_{d_{n+1}}} = H_A$. It can be concluded by lemma that

$\overline{F}_{A_d} \tilde{\subseteq} F_{A_{d_r}} \tilde{\subseteq} \overline{F}_{A_{d_r}} \tilde{\subseteq} F_{A_h}$

If both the elements lie in D_n then * holds by induction hypothesis.

Let d_s and d_r be elements in D such that either $d_s \le d$ (or) $d_r \ge h$ then

$$\overline{F}_{A_{d_s}} \tilde{\subseteq} F_{A_d} \tilde{\subseteq} F_{A_{d_r}}$$

and $\overline{F}_{A_{d_r}} \tilde{\subseteq} F_{d_h} \tilde{\subseteq} F_{d_s}$ respectively

Thus for every pair of elements of D_{n+1} * holds.

Extend this definition for all $d_t \in D$ by defining

$$F_{d_t} = \phi_A \quad ; d_t < 0$$

$$= X_A \quad ; d_t > 1 \qquad \text{----------------- (1)}$$

The relation (*) is still free for any pair of rational numbers $d_r < d_s$.

Define a fuzzy soft mapping $(h_{up}) : FS(U,E) \to FS([0,1],E\ )$ by $(h_{up})F_A = F_A = (h_{up})F_{e_a}$ where $(h_{up})(x_e) = \inf\{d / d \in F_{A_d}(x_e)\}$

$(h_{up})(x_e) = F_{e_a(h_{up})(x_e)}$ for every $e \in E$ and $x \in X$.

Then by above definition $(h_{up})F_A = F_{e_0} = \tilde{0}$ and $(h_{up})G_A = F_{e_1} = \tilde{1}$. To prove the continuity of the fuzzy soft mapping (h_{up}), we show that inverse image of fuzzy soft open set in $([0,1],E,\Im')$ are fuzzy soft open in $(U,E,\Im)$.

For $t \in [0,1]$, we show that $(h_{up})^{-1}F_{A_{[0,h]}} = U\{F_{A_d} : d < h\}$ if $F_{e_a} \in (h_{up})^{-1}F_{A_{[0,h]}}$

$$\Leftrightarrow (h_{up})(F_{e_a}) \tilde{\in} F_{A_{[0,h]}}$$

$$\Leftrightarrow (F_{e_a})(h_{up})(x_e) \tilde{\in} F_{A_{[0,h]}}$$

$$\Leftrightarrow (h_{up})(x_e) \tilde{\in} [0,h]$$

$$\Leftrightarrow (h_{up})(x_e) < h$$

$$\Leftrightarrow (h_{up})(x_e) < d < h \text{ for some } d(<h) \in D \tilde{\cap} [0,1]$$

$$\Leftrightarrow (F_{e_a}) \tilde{\in} F_{A_d} \text{ for some } d(<h) \in D \tilde{\cap} [0,1]$$

$$\Leftrightarrow (F_{e_a}) \tilde{\in} U \quad \{F_{A_d} : d \in D \tilde{\cap} [0,1] and \quad d < h\}$$

Therefore, $(h_{up})^{-1}F_{A_{[0,h]}} = U\{F_{A_d} : d < h\}$

Again if $F_{e_a} \in (h_{up})^{-1}F_{A_{[0,h]}}$

$$\Leftrightarrow (h_{up})F_{e_a} \in F_{A_{[0,h]}}$$

$$\Leftrightarrow F_{e_{a(h_{up})(x_e)}} \in F_{A_{[0,h]}}$$

$$\Leftrightarrow (h_{up})(x_e) \in [0,t]$$

$$\Leftrightarrow (F_{e_a}) \tilde{\in} F_{A_d} \text{ for any } h(>d) \in D.$$

Also for any $d \in D$ with $d > h$ there exists $d_R \in Q$ with $d > d_R > h$ and consequently,

$\overline{F}_{A_{d_R}} \tilde{\in} F_{A_d}$. Thus $(F_{e_a}) \tilde{\in} F_{A_d}$ for any $d(>h) \in D$

iff $(F_{e_a}) \tilde{\in} \overline{F}_{A_d}$

Hence $(h_{up})^{-1}F_{A_{[0,h]}} = \tilde{\cap}\{\overline{F}_{A_d} : d \in D, d > h\}$

Then $(h_{up})^{-1}F_{A_{[0,h]}}$ is fuzzy soft closed in $(U,E,\Im)$.

Consider

$$[\ (h_{up})^{-1}F_{A_{[0,h]}}\ ]^C(e)(s)$$

$$= 1_X - [\ (h_{up})^{-1}F_{A_{[0,h]}}](e)(s)$$

$$= 1_X - F_{A_{[0,h]}}((p(e))U(s))$$

$$= F_{A_{[0,h]}}{}^C((p(e))U(s))$$

$= (h_{up})^{-1}[F_{A_{[0,h]}}{}^C](e)(s)$ is fuzzy soft open in $(U,E,\Im)$.

Hence inverse image of fuzzy soft open set.

$F_{A_{[0,h]}}{}^C$ is a fuzzy soft open set in $(U,E,\Im)$ and so (h_{up}) is a fuzzy soft continuous function.

Define $D(F_e) = \{d_t / F_e \in F_d\}$

From (1) $D(F_e) = \phi_A \quad d_t < 0$;

$$dt > 1$$

Definition 3.11

Define a fuzzy soft mapping $(\phi,\psi) : F_A \to ([a,b],E',\Im')$ is defined as

$$(\phi,\psi)F_A(e')(t) = \begin{cases} \sup\limits_{S\in\phi^{-1}(t)\, e\in\psi^{-1}(e)} [\sup F_A](e)(s) & if \quad \phi^{-1}(t) \neq \phi, \psi^{-1}(e') \neq \phi \quad and \quad a = e \\ 0 & otherwise \end{cases}$$

Tietze's Extension Theorem 3.12

Statement

If $(U,E,\Im)$ is fuzzy soft topological space and $([a,b],E,\Im')$ be a fuzzy soft topological space with topology as in definition (3.11) then $(U,E,\Im)$ is fuzzy soft normal iff for any fuzzy soft closed F_A in $(U,E,\Im)$ and a fuzzy soft continuous function $(\phi,\psi) : F_A \to ([a,b],E',\Im')$ there exists a fuzzy soft continuous function $(\phi',\psi') : (U,E,\Im) \to ([a,b],E',\Im')$ such that $(\phi',\psi')(F_e) = (\phi,\psi)(F_e)$ for every $F_e \in F_A$.

Proof

Assume that $(U,E,\Im)$ is fuzzy soft normal. Let $(\phi,\psi) : F_A \to ([a,b],E',\Im')$ be a fuzzy soft continuous map, F_A being a fuzzy soft closed subset of $(U,E,\Im)$. Take a=-1,b=1.

Define a fuzzy soft map $(\phi_0,\psi_0):F_A\to([-1,1],E',\Im')$ as

$[(\phi_0,\psi_0)F_A](e_1)(t)=[F_a](e)(s)$ where $a=e'$

= 0 otherwise

For every $F_a\in F_A$. Divided the closed interval [-1,1] into three parts namely [-1,-1/3] [-1/3,1/3] and [1/3,1]. F_A is a fuzzy soft closed set means that it is a function from $F:A\to[-1,1]^I$ where $A\tilde{\subset}E$. Similarly define $G_{[-1,-1/3]}$ a fuzzy soft closed in $([-1,1],E')$ as a function from $G:A'\to[-1,-1/3]^I$ where $A'\tilde{\subset}E'$ and define $H_{[1/3,1]}$ as $H:A'\to[1/3,1]^I$ with $A'\tilde{\subset}E'$ a fuzzy soft closed set in $([-1,1],E',\Im')$.

Let $[(\phi_0,\psi_0)^{-1}G_{[-1,-1/3]}=G_{A_0}$ and $[(\phi_0,\psi_0)^{-1}H_{[1/3,1]}=H_{A_0}$. Since $G_{[-1,-1/3]}$ and $H_{[1/3,1]}$ are fuzzy soft closed in $([-1,1],E',\Im')$ G_{A_0} and H_{A_0} are disjoint fuzzy soft closed set in $(U,E,\Im)$ because (ϕ_0,ψ_0) is continuous as it is the restricted map of (ϕ,ψ) on the range.

By Urysohn Lemma, there exists a fuzzy soft continuous mapping $(\phi_1,\psi_1):(U,E,\Im)\to([-1,1],E',\Im')$ such that $(\phi_1,\psi_1)G_{A_0}=-\tilde{1}/3$ and $(\phi_1,\psi_1)H_{A_0}=\tilde{1}/3$

ie, $[(\phi_1,\psi_1)G_{a_0}](e')(t)=[G_{a_0}](e)(s)=-\tilde{1}/3$ for all $e'\in E,e\in E,s\in S,t\in[-1,1]$

$[(\phi_1,\psi_1)H_{a_0}](e')(t)=[H_{a_0}](e)(s)=\tilde{1}/3$

Construct a fuzzy soft mapping $(u_1,p_1):(U,E,\Im)\to([-1,1],E',\Im')$ as $(u_1,p_1)F_a=[(\phi_0,\psi_0)-(\phi_1,\psi_1)F_a$

Then

$[(u_1,p_1)F_a](e')(t)=[(\phi_0,\psi_0)G_a](e)(s)-[(\phi_1,\psi_1)F_a](e)(s)$

and

$[(\phi_0,\psi_0)G_a](e)(s)\in[-1,1],[[(\phi_1,\psi_1)G_a](e)(s)\in[-1/3,1/3]$

implies that $[(u_1,p_1)G_a](e')(t)\in[-2/3,2/3]$

Hence $(u_1,p_1)G_a\in G_{[-2/3,2/3]}$ for all $G_a\in G_A$

So $(u_1,p_1):(U,E,\Im)\to([-2/3,2/3],E',\Im')$ is a fuzzy soft mapping, define $G_{A_1}=(u_1,p_1)^{-1}G_{[-2/3,2/9]}$ and $H_{A_1}=(u_1,p_1)^{-1}H_{[2/9,2/3]}$. By similar argument G_{A_1} and H_{A_1} are disjoint fuzzy soft closed sets in $(U,E,\Im)$. Since $(U,E,\Im)$ is fuzzy soft normal, by Urysohn lemma there exists a fuzzy soft continuous mapping

$(\phi_2,\psi_2):(U,E,\Im)\to([-2/9,2/9],E',\Im')$

such that $(\phi_2,\psi_2)G_{A_1}=-\tilde{2}/9$ and $(\phi_2,\psi_2)H_{A_1}=\tilde{2}/9$

ie, $[(\phi_2,\psi_2)G_{A_1}](e_2)t)=-\tilde{2}/9$

$[(\phi_2,\psi_2)H_{A_1}](e_2)(t)=\tilde{2}/9$

$\Rightarrow$ $[G_{A_1}](e_1)(s)=-\tilde{2}/9$, $[H_{A_1}](e_1)(s)=\tilde{2}/9$

Define $(u_2,p_2)F_a=[(u_1,p_1)-(\phi_2,\psi_2)F_a$

$=[(\phi_0,\psi_0)-(\phi_1,\psi_1)-(\phi_2,\psi_2)]F_a$ for all $F_a\in F_A$

Then

$(u_2,p_2):(U,E,\Im)\to([-4/9,4/9,E',\Im')$ is a continuous fuzzy soft mapping continuous this process, we obtain a continuous fuzzy soft mapping.

$(\phi_n,\psi_n):(U,E,\Im)\to([-2^{n-1}/3^n,2^{n-1}/3^n],E',\Im')$

where $(\phi_n,\psi_n)G_{A_n}=-2^{n-1}/3^n$ and

$(\phi_n,\psi_n)H_{A_n}=2^{n-1}/3^n$ and

$(u_n,p_n):(U,E,\Im)\to([-2^n/3^n,2^n/3^n],E',\Im')$

defined by

$(u_n,p_n)F_a=[(\phi_0,\psi_0)-(\phi_1,\psi_1)+(\phi_2,\psi_2)+......+(\phi_n,\psi_n)(F_a)$

for all $F_a\in F_A$

Suppose $\tilde{A}_n(F_a)=\sum_{i=1}^{n}(\phi_i,\psi_i)F_a$

$\tilde{A}_n[(F_a)]_{e'}(t)=\sum_{i=1}^{n}[(\phi_i,\psi_i)F_a]_{e'}(t)$ for all $e'\in E,t\in[-1,1]$. As each (ϕ_i,ψ_i) is continuous, $\tilde{A}_n$ is also fuzzy soft continuous.

Also $\left|\ \tilde{A}_n[(F_a)](e')(t)\ \right|=\left|\ \sum_{i=1}^{n}[(\phi_i,\psi_i)F_a](e')(t)\ \right|$

$\leq\sum_{i=1}^{n}2^{i-1}/3^i$

$\leq 1/2\sum_{i=1}^{\infty}2^i/3^i$ ------------- (1)

By comparison test $\tilde{A}_n$ is uniformly fuzzy soft continuous. So the sum function $\sum_{i=1}^{\infty}[(\phi_i,\psi_i)F_a](e')(t)$ is fuzzy soft continuous and let

$(\phi',\psi')=\sum_{i=1}^{\infty}[(\phi_i,\psi_i)F_a](e')(t)$ for all $e'\in E,t\in[-1,1]$. Thus $(\phi',\psi'):(U,E,\Im)\to([-1,1],E',\Im')$ is a fuzzy soft continuous mapping.

Again

$(u_{n,},p_n)[F_a](e')(t)\ \left|=[(\phi_0,\psi_0)-\sum_{n=1}^{n}[(\phi_i,\psi_i)F_a](e')(t)\ \right|$

$\leq (2/3)^{n,}$ for all $e' \in E, t \in [-1,1]$

As $n \to \infty$

$$(\phi_0, \psi_0) = \sum_{n=1}^{\infty} [(\phi_i, \psi_i)$$

Which implies

$[(\phi_0, \psi_0) F_a](e')(t) = [(\phi', \psi') F_a](e')(t)$ for all $e' \in E, t \in [-1,1]$

Conversly suppose the given hypothesis holds. Let G_A and H_A be two disjoint fuzzy soft closed sets in $(U, E, \Im)$. Let $F_A = (G_A \cup H_A)$.

Let $(\phi, \psi) : (F_A, E, \Im) \to ([-1,1], E', \Im')$ be a fuzzy soft mapping defined by $[(\phi, \psi) G_a](e')(t) = 0$ and $[(\phi, \psi) H_a](e')(t) = 1$. Let $C_{[-1,1]}$ be any closed set in $([-1,1], E', \Im')$ then

$$[(\phi, \psi)^{-1} C_{[-1,1]}](e)(s) = C_{[-1,1]}(\varphi(e), \psi(s))$$

$$= \begin{cases} [G_A](e)s) & if \quad 0 \in C_{[-1,1]}, 1 \notin C_{[-1,1]} \\ [H_A](e)(s) & if \quad 1 \in C_{[-1,1]}, 0 \notin C_{[-1,1]} \\ [F_A](e)(s) & if \quad 0 \in C_{[-1,1]} \\ \phi & if \quad 0,1 \in C_{[-1,1]} \end{cases}$$

Then $(\phi, \psi)^{-1} C_{[-1,1]}$ is fuzzy soft closed $(F_A, E', \Im_{F_A})$. Hence (ϕ, ψ) fuzzy soft continuous. By the given hypothesis there is a fuzzy soft continuous $(\phi', \psi') : (U, E, \Im) \to ([-1,1], E', \Im')$ such that $[(\phi', \psi') F_a](e')(t) = [(\phi, \psi) F_a](e')(t)$ for every $e' \in E', t \in [-1,1]$. Then $[(\phi', \psi')^{-1}[B_{[-1,1/2]}]$ and $[(\phi', \psi')^{-1}[B_{[1/2,1]}]$ are disjoint fuzzy soft open sets and $G_A \tilde{\subseteq} [(\phi', \psi')^{-1}[B_{[-1,1/2]}]$ and $H_A \tilde{\subseteq} [(\phi', \psi')^{-1}[B_{[1/2,1]}]$.

Hence $(U, E, \Im)$ is fuzzy soft normal.

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