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Modern method to compute the determinants of matrices of order 4

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ABSTRACT

In this paper we present a new method to compute the determinants of matrices of order 4. This method computes the determinant of matrices of order 4 by using the schemes.

1. Introduction

We know that the definition of determinant of a matrix is as follows:

Let A be $n \times n$ matrix,

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Determinant of n^{th} order matrix is the sum, which has $n!$ different terms $\in_{j_1 j_2 j_3 \cdots j_n} a_{1j_1} a_{2j_2} a_{3j_3} \ldots a_{nj_n}$ see [5],[6],[7],[9].

$$Det\ A = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \in_{j_1 j_2 j_3 \cdots j_n} a_{1j_1} a_{2j_2} a_{3j_3} \ldots a_{nj_n}$$

where

$$\in_{j_1,j_2,j_3,\ldots,j_n} = \begin{cases} +1 \text{ if } j_1 j_2 j_3 \ldots j_n \text{ is even permutation} \\ -1 \text{ if } j_1 j_2 j_3 \ldots j_n \text{ is odd permutation} \end{cases}$$

Thus for 4×4 order matrix, the above definition gives,

$$Det\ A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \in_{j_1 j_2 j_3 j_4} a_{1j_1} a_{2j_2} a_{3j_3} a_{4j_4}$$

$$\begin{aligned} &= \mathcal{E}_{1234}a_{11}a_{22}a_{33}a_{44} + \mathcal{E}_{1243}a_{11}a_{22}a_{34}a_{43} + \mathcal{E}_{1324}a_{11}a_{23}a_{32}a_{44} + \mathcal{E}_{1342}a_{11}a_{23}a_{34}a_{42} \\ &\quad + \mathcal{E}_{1432}a_{11}a_{24}a_{33}a_{42} + \mathcal{E}_{1423}a_{11}a_{24}a_{32}a_{43} + \mathcal{E}_{2134}a_{12}a_{21}a_{33}a_{44} + \mathcal{E}_{2143}a_{12}a_{21}a_{34}a_{43} \\ &\quad + \mathcal{E}_{2431}a_{12}a_{24}a_{33}a_{41} + \mathcal{E}_{2413}a_{12}a_{24}a_{31}a_{43} + \mathcal{E}_{2314}a_{12}a_{23}a_{31}a_{44} + \mathcal{E}_{2341}a_{12}a_{23}a_{34}a_{41} \\ &\quad + \mathcal{E}_{3124}a_{13}a_{21}a_{32}a_{44} + \mathcal{E}_{3142}a_{13}a_{21}a_{34}a_{42} + \mathcal{E}_{3214}a_{13}a_{22}a_{31}a_{44} + \mathcal{E}_{3241}a_{13}a_{22}a_{34}a_{41} \\ &\quad + \mathcal{E}_{3412}a_{13}a_{24}a_{31}a_{42} + \mathcal{E}_{3421}a_{13}a_{24}a_{32}a_{41} + \mathcal{E}_{4123}a_{14}a_{21}a_{32}a_{43} + \mathcal{E}_{4132}a_{14}a_{21}a_{33}a_{42} \\ &\quad + \mathcal{E}_{4213}a_{14}a_{22}a_{31}a_{43} + \mathcal{E}_{4231}a_{14}a_{22}a_{33}a_{41} + \mathcal{E}_{4312}a_{14}a_{23}a_{31}a_{42} + \mathcal{E}_{4321}a_{14}a_{23}a_{32}a_{41} \\ &= a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{23}a_{32}a_{44} + a_{11}a_{23}a_{34}a_{42} - a_{11}a_{24}a_{33}a_{42} + a_{11}a_{24}a_{32}a_{43} \\ &\quad - a_{12}a_{21}a_{33}a_{44} + a_{12}a_{21}a_{34}a_{43} + a_{12}a_{24}a_{33}a_{41} - a_{12}a_{24}a_{31}a_{43} + a_{12}a_{23}a_{31}a_{44} - a_{12}a_{23}a_{34}a_{41} \\ &\quad + a_{13}a_{21}a_{32}a_{44} - a_{13}a_{21}a_{34}a_{42} - a_{13}a_{22}a_{31}a_{44} + a_{13}a_{22}a_{34}a_{41} + a_{13}a_{24}a_{31}a_{42} - a_{13}a_{24}a_{32}a_{41} \\ &\quad - a_{14}a_{21}a_{32}a_{43} + a_{14}a_{21}a_{33}a_{42} + a_{14}a_{22}a_{31}a_{43} - a_{14}a_{22}a_{33}a_{41} - a_{14}a_{23}a_{31}a_{42} + a_{14}a_{23}a_{32}a_{41} \end{aligned}$$

Later on Chio's [2,4] gives condensation method of finding determinant of order $n \times n$ in order $(n-1)\times(n-1)$ determinant in the year 1853. Similarly Dodgson's [3], given method of finding determinant of $n \times n$ matrix in terms of $(n-1) \times (n-1)$ determinant in year 1866.

Ali Ahmad and K.L. Bondar [1] developed schemes to obtain determinant of a matrix of order 3 by 3.

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2. New method to compute the determinant of matrices of order 4

This new method includes schemes to compute the determinant of a matrix of order 4. To describe and draw the first scheme we follow the following steps:

Let Det $A=\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$

1. Delete the first row, then there remains three rows and four columns.

$$\begin{matrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{matrix}$$

2. From the remaining rows and columns, delete the first column and write the remaining column.

$$\begin{matrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{matrix}$$

3. Now repeat the step 2 for the remaining columns of arrangement in step 1and write them as follows.

$$\begin{matrix} a_{22} & a_{23} & a_{24} & a_{21} & a_{23} & a_{24} & a_{21} & a_{22} & a_{24} & a_{21} & a_{22} & a_{23} \\ a_{32} & a_{33} & a_{34} & a_{31} & a_{33} & a_{34} & a_{31} & a_{32} & a_{34} & a_{31} & a_{32} & a_{33} \\ a_{42} & a_{43} & a_{44} & a_{41} & a_{43} & a_{44} & a_{41} & a_{42} & a_{44} & a_{41} & a_{42} & a_{43} \end{matrix}$$

4. After that; remove the element a_{34} from the ninth column and second row and place it before the first element in the same row and also remove the last element a_{33} from the twelfth column and place it in the position of a_{34} as following.

$$\begin{matrix} & a_{22} & a_{23} & a_{24} & a_{21} & a_{23} & a_{24} & a_{21} & a_{22} & a_{24} & a_{21} & a_{22} & a_{23} \\ a_{34} & a_{32} & a_{33} & a_{34} & a_{31} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & \\ & a_{42} & a_{43} & a_{44} & a_{41} & a_{43} & a_{44} & a_{41} & a_{42} & a_{44} & a_{41} & a_{42} & a_{43} \end{matrix}$$

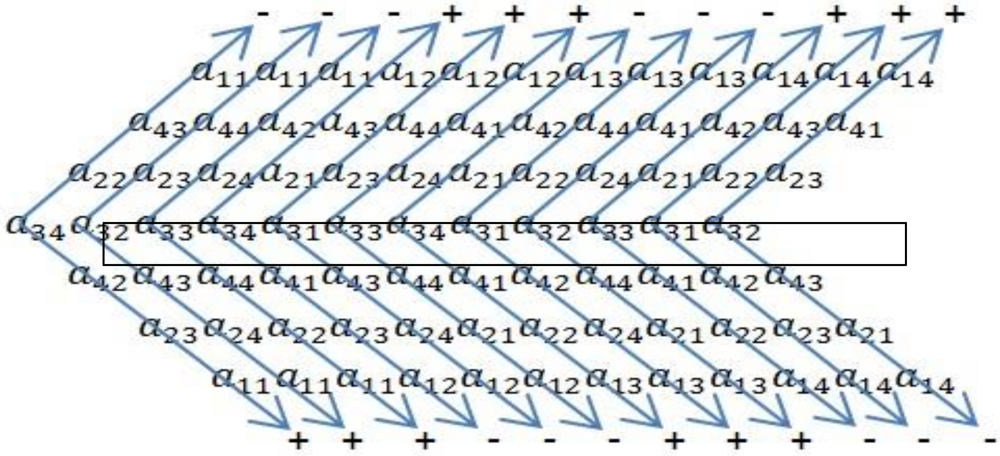
5. Now in above arrangement repeat the third row above the first row, after removing the first element a_{42} and place it in the position of the fourth element a_{41} and write a_{41} in the last of the same row. Also repeat the first row under the third row, after removing the first element a_{22} and place it in the position of fourth element a_{21} and write remove the forth element a_{21} in the last of the same row. Thus we obtain following arrangement.

$$\begin{matrix} a_{43} & a_{44} & a_{42} & a_{43} & a_{44} & a_{41} & a_{42} & a_{44} & a_{41} & a_{42} & a_{43} & a_{41} \\ a_{22} & a_{23} & a_{24} & a_{21} & a_{23} & a_{24} & a_{21} & a_{22} & a_{24} & a_{21} & a_{22} & a_{23} \\ a_{34} \; a_{32} & a_{33} & a_{34} & a_{31} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & \\ a_{42} & a_{43} & a_{44} & a_{41} & a_{43} & a_{44} & a_{41} & a_{42} & a_{44} & a_{41} & a_{42} & a_{43} \\ a_{23} & a_{24} & a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{24} & a_{21} & a_{22} & a_{23} & a_{21} \end{matrix}$$

6. Write the elements of a first row in original determinant as first and last row with repeat every element three times in above arrangement as follows.

$$\begin{matrix} a_{11} & a_{11} & a_{11} & a_{12} & a_{12} & a_{12} & a_{13} & a_{13} & a_{13} & a_{14} & a_{14} & a_{14} \\ a_{43} & a_{44} & a_{42} & a_{43} & a_{44} & a_{41} & a_{42} & a_{44} & a_{41} & a_{42} & a_{43} & a_{41} \\ a_{22} & a_{23} & a_{24} & a_{21} & a_{23} & a_{24} & a_{21} & a_{22} & a_{24} & a_{21} & a_{22} & a_{23} \\ a_{34} \; a_{32} & a_{33} & a_{34} & a_{31} & a_{33} & a_{34} & a_{31} & a_{32} & a_{33} & a_{31} & a_{32} & \\ a_{42} & a_{43} & a_{44} & a_{41} & a_{43} & a_{44} & a_{41} & a_{42} & a_{44} & a_{41} & a_{42} & a_{43} \\ a_{23} & a_{24} & a_{22} & a_{23} & a_{24} & a_{21} & a_{22} & a_{24} & a_{21} & a_{22} & a_{23} & a_{21} \\ a_{11} & a_{11} & a_{11} & a_{12} & a_{12} & a_{12} & a_{13} & a_{13} & a_{13} & a_{14} & a_{14} & a_{14} \end{matrix}$$

7. Consider the middle row as a common row, and draw the diagonals below and above. The sign of above diagonals are alternative for every three diagonals and starting with "-" and similarly for the below diagonals starting with "+".



Since, any process works on the rows, it works on the column. Then similarly we can get another scheme as the following:

Let $DetA=\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$

1. Delete the first column, then there remains three columns and four rows.

$$\begin{array}{ccc} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{array}$$

2. From the remaining rows and columns, delete the first rows and write the remaining rows.

$$\begin{array}{ccc} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{array}$$

3. Now repeat the step 2 for the remaining rows of arrangement in step 1and write them as follows.

$$\begin{array}{ccc} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \\ a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \\ a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \\ a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{array}$$

4. After that; remove the element a_{43} from the ninth row and second column and place it above the first one in the same column and also remove the last element a_{33} from the twelfth row and third column and place it in the position of a_{43}.

$$\begin{array}{ccc} & a_{43} & \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \\ a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \\ a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{33} & a_{44} \\ a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & & a_{34} \end{array}$$

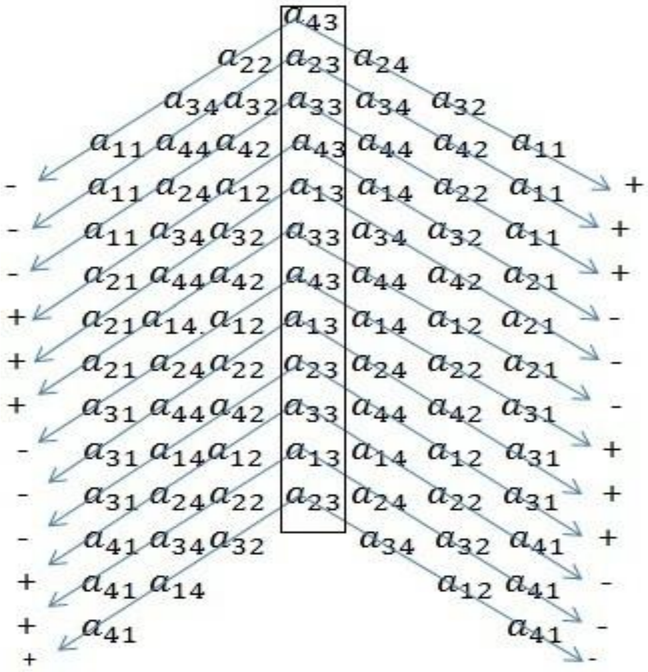
5. Now in above arrangement repeat the third column on the left of the first column, after removing the first element a_{24} and place it at the position of the fourth element a_{14} and write a_{14} in the last of the same column. Also repeat the first column on the right of the third column, after removing the first element a_{22} and write it in the position of the fourth element a_{12} and write a_{12} in the last of the same column. Thus we obtain following arrangement.

$$\begin{array}{ccccc} & & a_{43} & & \\ & a_{22} & a_{23} & a_{24} & \\ a_{34} & a_{32} & a_{33} & a_{34} & a_{32} \\ a_{44} & a_{42} & a_{43} & a_{44} & a_{42} \\ a_{24} & a_{12} & a_{13} & a_{14} & a_{22} \\ a_{34} & a_{32} & a_{33} & a_{34} & a_{32} \\ a_{44} & a_{42} & a_{43} & a_{44} & a_{42} \\ a_{14} & a_{12} & a_{13} & a_{14} & a_{12} \\ a_{24} & a_{22} & a_{23} & a_{24} & a_{22} \\ a_{44} & a_{42} & a_{33} & a_{44} & a_{42} \\ a_{14} & a_{12} & a_{13} & a_{14} & a_{12} \\ a_{24} & a_{22} & a_{23} & a_{24} & a_{22} \\ a_{34} & a_{32} & & a_{34} & a_{32} \\ a_{14} & & & & a_{12} \end{array}$$

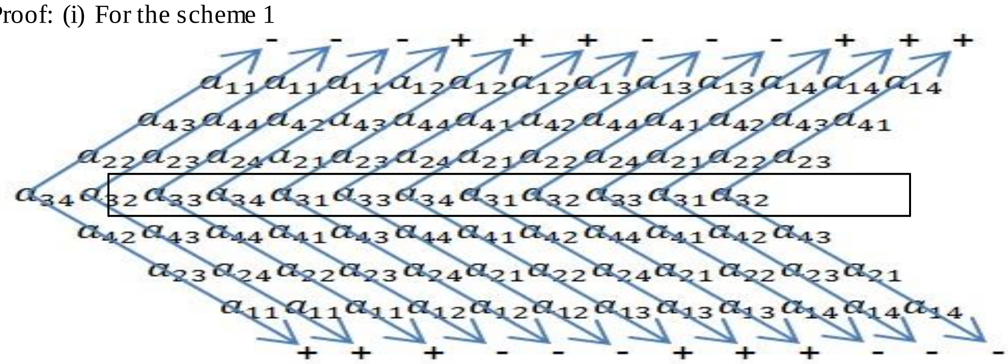
6. Write the elements of first column in original determinant as first and last column with repeat every element three times in above, arrangement as follows.

$$\begin{array}{c}
a_{43}\\
a_{22}\ a_{23}\ a_{24}\\
a_{34}\ a_{32}\ a_{33}\ a_{34}\ a_{32}\\
a_{11}\ a_{44}\ a_{42}\ a_{43}\ a_{44}\ a_{42}\ a_{11}\\
a_{11}\ a_{24}\ a_{12}\ a_{13}\ a_{14}\ a_{22}\ a_{11}\\
a_{11}\ a_{34}\ a_{32}\ a_{33}\ a_{34}\ a_{32}\ a_{11}\\
a_{21}\ a_{44}\ a_{42}\ a_{43}\ a_{44}\ a_{42}\ a_{21}\\
a_{21}\ a_{14}\ a_{12}\ a_{13}\ a_{14}\ a_{12}\ a_{21}\\
a_{21}\ a_{24}\ a_{22}\ a_{23}\ a_{24}\ a_{22}\ a_{21}\\
a_{31}\ a_{44}\ a_{42}\ a_{33}\ a_{44}\ a_{42}\ a_{31}\\
a_{31}\ a_{14}\ a_{12}\ a_{13}\ a_{14}\ a_{12}\ a_{31}\\
a_{31}\ a_{24}\ a_{22}\ a_{23}\ a_{24}\ a_{22}\ a_{31}\\
a_{41}\ a_{34}\ a_{32}\qquad a_{34}\ a_{32}\ a_{41}\\
a_{41}\ a_{14}\qquad\qquad a_{12}\ a_{41}\\
a_{41}\qquad\qquad\qquad a_{41}
\end{array}$$

7. Consider the middle column as a common, and draw the right and left diagonals. The sign of left diagonals are alternative for every three diagonals and start with "-" and similarly for the right diagonals starting with"+"

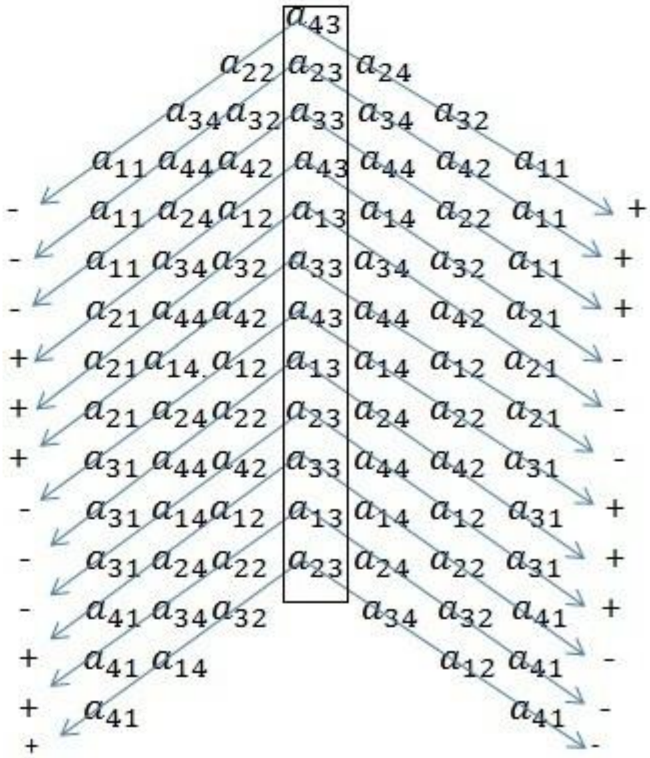


Proof: (i) For the scheme 1



$$= a_{34}a_{42}a_{23}a_{11} + a_{32}a_{43}a_{24}a_{11} + a_{33}a_{44}a_{22}a_{11} - a_{34}a_{41}a_{23}a_{12} - a_{31}a_{43}a_{24}a_{12} - a_{33}a_{44}a_{21}a_{1}$$
$$+ a_{34}a_{41}a_{22}a_{13} - a_{31}a_{42}a_{24}a_{13} + a_{32}a_{44}a_{21}a_{13} - a_{33}a_{41}a_{22}a_{14} - a_{31}a_{42}a_{23}a_{14} - a_{32}a_{43}a_{21}a_{14}$$
$$- a_{34}a_{22}a_{43}a_{11} - a_{32}a_{23}a_{44}a_{11} - a_{33}a_{24}a_{42}a_{11} + a_{34}a_{21}a_{43}a_{12} + a_{31}a_{23}a_{44}a_{12} + a_{33}a_{24}a_{41}a_{12}$$
$$+ a_{34}a_{21}a_{42}a_{13} - a_{31}a_{22}a_{44}a_{13} - a_{32}a_{24}a_{41}a_{13} + a_{33}a_{21}a_{42}a_{14} + a_{31}a_{22}a_{43}a_{14} + a_{32}a_{23}a_{41}a_{14}$$
$$= a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{23}a_{32}a_{44} + a_{11}a_{23}a_{34}a_{42} - a_{11}a_{24}a_{33}a_{42} + a_{11}a_{24}a_{32}a_{43}$$
$$- a_{12}a_{21}a_{33}a_{44} + a_{12}a_{21}a_{34}a_{43} + a_{12}a_{24}a_{33}a_{41} - a_{12}a_{24}a_{31}a_{43} + a_{12}a_{23}a_{31}a_{44} - a_{12}a_{23}a_{34}a_{41}$$
$$+ a_{13}a_{21}a_{32}a_{44} - a_{13}a_{21}a_{34}a_{42} - a_{13}a_{22}a_{31}a_{44} + a_{13}a_{22}a_{34}a_{41} + a_{13}a_{24}a_{31}a_{42} - a_{13}a_{24}a_{32}a_{41}$$
$$- a_{14}a_{21}a_{32}a_{43} + a_{14}a_{21}a_{33}a_{42} + a_{14}a_{22}a_{31}a_{43} - a_{14}a_{22}a_{33}a_{41} - a_{14}a_{23}a_{31}a_{42} + a_{14}a_{23}a_{32}a_{41}$$

(ii) For the scheme 2



$$= a_{34}a_{42}a_{23}a_{11} + a_{32}a_{43}a_{24}a_{11} + a_{33}a_{44}a_{22}a_{11} - a_{34}a_{41}a_{23}a_{12} - a_{31}a_{43}a_{24}a_{12} - a_{33}a_{44}a_{21}a_{1}$$
$$+ a_{34}a_{41}a_{22}a_{13} - a_{31}a_{42}a_{24}a_{13} + a_{32}a_{44}a_{21}a_{13} - a_{33}a_{41}a_{22}a_{14} - a_{31}a_{42}a_{23}a_{14} - a_{32}a_{43}a_{21}a_{14}$$
$$- a_{34}a_{22}a_{43}a_{11} - a_{32}a_{23}a_{44}a_{11} - a_{33}a_{24}a_{42}a_{11} + a_{34}a_{21}a_{43}a_{12} + a_{31}a_{23}a_{44}a_{12} + a_{33}a_{24}a_{41}a_{12}$$
$$+ a_{34}a_{21}a_{42}a_{13} - a_{31}a_{22}a_{44}a_{13} - a_{32}a_{24}a_{41}a_{13} + a_{33}a_{21}a_{42}a_{14} + a_{31}a_{22}a_{43}a_{14} + a_{32}a_{23}a_{41}a_{14}$$

$$= a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{23}a_{32}a_{44} + a_{11}a_{23}a_{34}a_{42} - a_{11}a_{24}a_{33}a_{42} + a_{11}a_{24}a_{32}a_{43}$$
$$- a_{12}a_{21}a_{33}a_{44} + a_{12}a_{21}a_{34}a_{43} + a_{12}a_{24}a_{33}a_{41} - a_{12}a_{24}a_{31}a_{43} + a_{12}a_{23}a_{31}a_{44} - a_{12}a_{23}a_{34}a_{41}$$
$$+ a_{13}a_{21}a_{32}a_{44} - a_{13}a_{21}a_{34}a_{42} - a_{13}a_{22}a_{31}a_{44} + a_{13}a_{22}a_{34}a_{41} + a_{13}a_{24}a_{31}a_{42} - a_{13}a_{24}a_{32}a_{41}$$
$$- a_{14}a_{21}a_{32}a_{43} + a_{14}a_{21}a_{33}a_{42} + a_{14}a_{22}a_{31}a_{43} - a_{14}a_{22}a_{33}a_{41} - a_{14}a_{23}a_{31}a_{42} + a_{14}a_{23}a_{32}a_{41}$$

3. Conclusion

This new method, comparing with other known methods, is one of the most usable ones, based on quickness and eas iness of computing the fourth order determinant. Furthermore, this new method enables the further research in computing methods of higher than fourth order determinants.

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