Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 95 (2016) 41143-41147

ISSN: 2229-712X

The Split Majority Domatic Number of a Graph

J.Joseline Manora[1] and S.Veeramanikandan[2]
Department of Mathematics, T.B.M.L. College, Porayar-609 307, India.

ARTICLE INFO

Article history:
Received: 25 April 2016;
Received in revised form:
17 June 2016;
Accepted: 22 June 2016;

Keywords
Set,
Split Majority Dominating set.

ABSTRACT

Let G = (V,E) be any simple finite graph. A subset D of V (G) is said to be Split Majority Dominating set of G if $|N[D]| \geq \left\lceil \frac{p}{2} \right\rceil$ and the induced subgraph $\langle V - D \rangle$ is disconnected. A split majority dominating set D is said to be minimal if there exists a vertex v of V such that D-{v} is not a split majority dominating set of G. The Split Majority Domatic Number denoted by $d_{sm}(G)$ is the maximum number of disjoint minimal split majority dominating sets obtained for a graph G. In this article, we have initiated the study of this concept.

Introduction

Let G = (V (G), E (G)) be any simple finit graph with $|V(G)| = p$ and $|E(G)| = q$. With usual notations, the degree of a vertex v, the maximum and the minimum degree of a graph G are denoted by $d(v), \Delta(G)$ and $\delta(G)$ respectively.

A set $D \subseteq V(G)$ is said to be a dominating set [2] of G if for every vertex v in V-D there exists atleast one vertex u in D such that u and v are adjacent in G. A Dominating set D is said to be minimal if for some vertex v of G, $D - \{v\}$ is not a dominating set. The minimum cardinality of a minimal dominating set is called the domination number of G and it is denoted by $\gamma(G)$.

A set $D \subseteq V(G)$ is said to be a majority dominating set [3] of G if atleast half of the vertices of G are either in D or adjacent to the vertices of D. i:e) $|N[D]| \geq \left\lceil \frac{p}{2} \right\rceil$. A majority Dominating set D is said to be minimal if for some vertex v of G, $D - \{v\}$ is not a majority dominating set. The minimum cardinality of a minimal majority dominating set is called the ma-jority domination number of G and it is denoted by M (G).This parameter was defined by Swaminathan and Joseline Manora.

A Dominating set $D \subset V(G)$ is said to be a split dominating set[8] if the induced subgraph <V – D> is disconnected. with usual inferences, the minimum cardinality of minimal split dominating set is denoted by s (G).This parameter was intoduced by kulli and Janakiram.

A subset D of V (G) is said to be Split Majority Dominating set[5] of G if $|N[D]| \geq \left\lceil \frac{p}{2} \right\rceil$ and the induced subgraph $\langle V - D \rangle$ is disconnected. As usual, the minimum cardinaliy of minimal split majority dominating set is called split majority domination number of a graph denoted by $\gamma_{sm}(G)$. This parameter was defined and studied by Joseline Manora and Veeramanikandan.

A partition Δ of its vertex set V (G) is called a domatic partition of G if each class of Δ is a dominating set in G. The maximum number of classes of a domatic partition of G is called the domatic number of G and is dentoed by d (G). The domatic number was introduced by Cockayne and Hedetniemi. In a similar fashion, a majority domatic partition of a graph G was introduced and each class of it is a majority dominating set in G. The maximum number of classes of a majority domatic partition of G is called the majority domatic number [4] and is denoted by $d_M(G)$. This parameter was introduced by Swaminathan and Joseline Manora.

2 Split Majority Domatic Number of a Graph

In this section, we define Split Majority Domatic Number of a graph G and this number $d_{sm}(G)$ is determined for some families of graphs.

Definition 2.1

Let $\Re$ be the family of all disjoint minimal split majority dominating sets of G. The split majority domatic number of a graph G is defined to be the maximum number of disjoint minimal split majority dominating sets of G and it is denoted by $d_{sm}(G)$.

Remark 2.2

In this article, we consider only the family of disjoint minimal split majority dominating sets of G rather than the partition of vertices of G. The reason is that there are some vertices that are not the elements of any minimal split majority dominating set D of G since the definition of split majority dominating set is violated when these vertices are included in any set D.

Tele:
E-mail address: joseline_manora@yahoo.co.in

2.1 $d_{sm}(G)$ for some families of graphs

1. For $G = \overline{K_p}$, a totally disconnected graph.

$$d_{sm}(\overline{K_p}) = \begin{cases} 1 \text{ if } p \text{ is odd} \\ 2 \text{ if } p \text{ is even} \end{cases}$$

3. If G is a star $K_{1,p-1}, p \geq 3$. Then $d_{sm}(G) = 1$.
4. Suppose G being double star $D_{r,s}, p = r + s + 2$. Then $d_{sm}(G) = 2$.
5. For $G = C_p, p \geq 3$. Then $d_{sm}(G) = 3$.
6. If G is a complete biparpite graph $K_{m,n}, m \leq n, d_{sm}(G) = 2$.
7. Let G be a petersen graph. Then $d_{sm}(G) = 2$.
8. Suppose G is a fan $F_p, p \geq 4$. Then $d_{sm}(G) = 1$.

3 Main Results on $d_{sm}(G)$.

Theorem 3.1

If G has a full degree vertex, $d_{sm}(G) = 1$.

Proof

Suppose G has a full degree vertex v. If v is a cut vertex then $D = \{v\}$ is a split majority dominating set of G. Assume that there exists another split majority dominating set S of G. Then S must contain v. If not, $\langle V - S\rangle$ is connected, a contradiction. Therefore $d_{sm}(G) = 1$. if v is not a cut vertex, $\gamma_{sm}(G) \geq 2$ and v is in every split majority dominating set of G. Applying the same argument as above, we get a contradiction. Therefore $d_{sm}(G) = 1$.

Theorem 3.2

If every vertex of a graph is such that $d(v) > \left\lceil \frac{p}{2} \right\rceil$ then $d_{sm}(G) = 1$.

Proof

Suppose $\delta(G) > \left\lceil \frac{p}{2} \right\rceil$. Then every vertex is a majority dominating vertex. Let D be a minimum split majority dominating set of G. Then $\gamma_{sm}(G) \geq \delta(G)$. That is $|D| > \left\lceil \frac{p}{2} \right\rceil$. This implies that D contains atleast one vertex more than $\left\lceil \frac{p}{2} \right\rceil$ vertices. Then $|V - D| < \left\lceil \frac{p}{2} \right\rceil$ implying that V-D is not a majority dominating set. Therefore there exists only one split majority dominating set fo G and hence $d_{sm}(G) = 1$.

Theorem 3.3

For any graph G, $1 \leq d_{sm}(G) \leq \left\lceil \frac{p}{2} \right\rceil + 1$

Proof

If G has a full degree vertex then the lower bound is attained. When $\delta(G) > \left\lceil \frac{p}{2} \right\rceil$ then $d_{sm}(G) = 1$. Consider a minimally connected graph G, namely a tree T. If T has exactly two end vertices then it is a path P_p. When $p \leq 6$, then every intermediate vertex is a split majority dominating set of G. Therefore $d_{sm}(G) = 4 \leq \left\lceil \frac{p}{2} + 1 \right.$. Suppose p > 7. Then $\gamma_{sm}(G) \geq 2$ but $d_{sm}(G) < \left\lceil \frac{p}{2} \right\rceil$ then only intermediate vertices constitute split majority dominating sets of G and $d_{sm}(G) < \left\lceil \frac{p}{[} 2 \right\rceil$. Thus $1 \leq d_{sm}(G) \leq \left\lceil \frac{p}{2} + 1 \right.$.

Proposition 3.4

If G is any graph with $diam(G) = 2$ then $d_{sm}(G) = 1$.

Proof

Suppose G is a graph with $diam(G) = 2$. Let v be the center of the graph G. If D is the minimal split majority dominating set of a graph G and containing the vertex v then no other minimal split majority dominating set is obtained without v. Therefore there exists only one split majority dominating set of G. Thus $d_{sm}(G) = 1$.

Proposition 3.5

For a tree T with $diam(T) = 3, d_{sm}(G) = 2$.

Proof

Suppose T is a tree with $diam(T) = 3$. Since every tree has atleast two end vertices. If $diam(T) = 3$ then T is a double star $D_{r,s}$ or P_4 or pendants adjacent to intermediate vertices. If T is $D_{r,s}$ or P_4, $d_{sm}(G) = 2$. If pendants are adjacent to intermediate vertices, $d_{sm}(G) = 2$.

Theorem 3.6

Let G = P_p be a path on p vertices, p > 4, $d_{sm}(P_p) = \frac{p}{\gamma_{sm}(G)}$ if and only if $p = 8,10,15,20,25.+$

Proof

Let $P_p = \{u_1, u_2, \ldots, u_p\}$ be a path on p vertices and $\gamma_{sm}(P_p) = \left\lceil \frac{p}{6} \right\rceil$.

Suppose $p = 8,10,15,20,25$. Then $\gamma_{sm}(G) = 2,2,3,4,5$. It is clear that $\gamma_{sm}(G)$ divides p. When $p = 8, d_{sm}(P_8) = 4$. When p = 6k + 2. Then $\frac{p}{\gamma_{sm}(G)} = 4$ if k = 1. Hence $d_M(P_8) = 4 = \frac{p}{\gamma_{sm}(G)}$. Therefore split majority domatic partition of P_8 is $\{\{u_1,u_5\},\{u_2,u_6\},\{u_3,u_7\},\{u_4,u_8\}\}$.

Let $p = 10,15,20,25$.Then $d_{sm}(P_p) = 5$.

When p = 10, (i. e.) $p = 6k + 4$, then $\left\lfloor\frac{p}{\gamma_{sm}(G)}\right\rfloor = 5$ if k = 1.

When p = 15, (i. e.) $p = 6k + 3$, then $\left\lfloor\frac{p}{\gamma_{sm}(G)}\right\rfloor = 5$ if k = 2.

When p = 20, (i. e.) $p = 6k + 2$, then $\left\lfloor\frac{p}{\gamma_{sm}(G)}\right\rfloor = 5$ if k = 3.

When p = 25, (i. e.) $p = 6k + 1$, then $\left\lfloor\frac{p}{\gamma_{sm}(G)}\right\rfloor = 5$ if k = 4.

Therefore the split majority domatic partitions of $V(P_p)$ are

$D_1 = \left\{u_1,u_6,\dots,u_{(\gamma_{sm}(G)-1)\left\lfloor\frac{p}{\gamma_{sm}(G)}\right\rfloor+1}\right\}$, $D_2 = \left\{u_2,u_7,\dots,u_{(\gamma_{sm}(G)-1)\left\lfloor\frac{p}{\gamma_{sm}(G)}\right\rfloor+2}\right\}$,

$D_3 = \left\{u_3,u_8,\dots,u_{(\gamma_{sm}(G)-1)\left\lfloor\frac{p}{\gamma_{sm}(G)}\right\rfloor+3}\right\}$, $D_4 = \left\{u_4,u_9,\dots,u_{(\gamma_{sm}(G)-1)\left\lfloor\frac{p}{\gamma_{sm}(G)}\right\rfloor+4}\right\}$,

$D_5 = \left\{u_5,u_{10},\dots,u_{(\gamma_{sm}(G)-1)\left\lfloor\frac{p}{\gamma_{sm}(G)}\right\rfloor+5}\right\}$, In all cases, $\frac{p}{\gamma_{sm}(G)} = 5 = d_{sm}(P_p)$ if $p = 8,10,15,20,25$.

Conversely let $d_{sm}(P_p) = \frac{p}{\gamma_{sm}(G)}$. Suppose $p \equiv 0(mod\ 6)$. Then $d_{sm}(P_p) = 5$. But $d_{sm}(P_p) = \frac{p}{\gamma_{sm}(G)}$, implies that $d_{sm}(P_p) = 6$ which is a contradic-tion. Hence $p \not\equiv 0\ (mod\ 6)$.

Suppose $p \equiv 1,2,3,4,5(mod\ 6)$. Let $p = 6k + 1, 1 \le l \le 5$. Then $\gamma_{sm}(G) = \left\lceil\frac{p}{6}\right\rceil = k + 1$ and $\frac{p}{\gamma_{sm}(G)} = \frac{6k+1}{k+1} = m(say), m \neq 0$. It implies that $k = \frac{m-1}{6-m}$. If $m - 1 > 0$ and $6 - m > 0$ then $l < m < 6$.

Take $l = 1$. Then $m = 2,3,4,5$. $k = \frac{m-1}{6-m}$. Then

$$k = \begin{cases} \frac{1}{4}\ if\ m = 2 \\ \frac{2}{3}\ if\ m = 3 \\ \frac{3}{2}\ if\ m = 4 \\ \ if\ m = 5. \end{cases}$$

Hence $k = 4$ is an integer if $l = 1$. Therefore for $k = 4$ and $l = 1$ implies $p = 6k + 1 = 25$. In a similar way, take $l = 2$. Then $m = 3,4,5$. $k = \frac{m-1}{6-m} = 1$ is an integer if $m = 4$ and $k = 3$ if $m = 5$. Therefore for $k = 1$ and $l = 2$ implies $p = 6k + 1 = 8$ and for $k = 3$ and $l = 2$ implies $p = 6k + 1 = 20$.

Take $l = 3$. Then $m = 4,5$. $k = \frac{m-1}{6-m} = 2$ is an integer if $m = 5$. For $k = 2$ and $l = 3$ implies $p = 6k + 1 = 15$.

Take $l = 4$. Then $m = 5$. $k = \frac{m-1}{6-m} = 1$ is an integer if $m = 5$. For $k = 1$ and $l = 4$ implies $p = 6k + 1 = 10$.

Take $l = 5$. Then $m = 5$. Then there is no integer value for k. Hence, $p = 8,10,15,20,25$ if $d_{sm}(P_p) = \frac{p}{\gamma_{sm}(G)}$.

Theorem 3.7

Let $G = C_p$ be a cycle on p vertices, p > 4. Then $d_{sm}(C_p) = \frac{p}{\gamma_{sm}(G)}$ if and only if $p = 8,10,15,20,25$. or $p \equiv 0(mod\ 6)$.

Proof

Let $C_p = \{u_1,u_2,\dots,u_p\}$ be a cycle on p vertices. Then $\gamma_{sm}(C_p) = \left\lceil\frac{p}{6}\right\rceil$. Suppose $p = 8,10,15,20,25$, then $\gamma_{sm}(G) = 2,2,3,4,5$ and suppose $p \equiv 0(mod\ 6)$ then $\gamma_{sm}(C_p) = \frac{6k}{6} = k$. It is clear that $\gamma_{sm}(G)$ divides p. When $p = 8, d_{sm}(P_8) = 4$. When $p = 6k + 2$, then $\frac{p}{\gamma_{sm}(G)} = 4$.

If $k = 1$. Hence $d_{sm}(C_8) = 4 = \frac{p}{\gamma_{sm}(G)}$. Therefore a split majority domatic partition of C_8 is

$\{\{u_1,u_5\},\{u_2,u_6\},\{u_3,u_7\},\{u_4,u_8\}\}$.

Let $p = 10,15,20,25$. Then $d_{sm}(C_p) = 5$.

When $p = 10$, (i. e.) $p = 6k + 4$, then $\left\lfloor \frac{p}{\gamma_{sm}(G)} \right\rfloor = 5$ if $k = 1$.

When $p = 15$, (i. e.) $p = 6k + 3$, then $\left\lfloor \frac{p}{\gamma_{sm}(G)} \right\rfloor = 5$ if $k = 2$.

When $p = 20$, (i. e.) $p = 6k + 2$, then $\left\lfloor \frac{p}{\gamma_{sm}(G)} \right\rfloor = 5$ if $k = 3$.

When $p = 25$, (i. e.) $p = 6k + 1$,then $\left\lfloor \frac{p}{\gamma_{sm}(G)} \right\rfloor = 5$ if $k = 4$. Therefore the split majority domatic partitions of $V(C_p)$ are

$D_1 = \left\{u_1,u_6,\dots,u_{(\gamma_{sm}(G)-1)\left\lfloor \frac{p}{\gamma_{sm}(G)} \right\rfloor+1}\right\}$, $D_2 = \left\{u_2,u_7,\dots,u_{(\gamma_{sm}(G)-1)\left\lfloor \frac{p}{\gamma_{sm}(G)} \right\rfloor+2}\right\}$,

$D_3 = \left\{u_3,u_8,\dots,u_{(\gamma_{sm}(G)-1)\left\lfloor \frac{p}{\gamma_{sm}(G)} \right\rfloor+3}\right\}$, $D_4 = \left\{u_4,u_9,\dots,u_{(\gamma_{sm}(G)-1)\left\lfloor \frac{p}{\gamma_{sm}(G)} \right\rfloor+4}\right\}$,

$D_5 = \left\{u_5,u_{10},\dots,u_{(\gamma_{sm}(G)-1)\left\lfloor \frac{p}{\gamma_{sm}(G)} \right\rfloor+5}\right\}$.

In all cases, $\frac{p}{\gamma_{sm}(G)} = 5 = d_{sm}(C_p)$ if $p = 8,10,15,20,25$. Let $p \equiv 0(mod\ 6)$. Then $d_{sm}(C_p) = 6$. Let $p = 6k$, Then $\frac{p}{\gamma_{sm}(G)} = 6$ since $\gamma_{sm}(G) = \frac{p}{6} = k$. Therefore $\{D_1,D_2,D_3,D_4,D_5,D_6\}$ are the split majority domatic partitions of V(G) Hence $\frac{p}{\gamma_{sm}(G)} = 6 = d_{sm}(C_p)$ if $p \equiv 0(mod\ 6)$.

Conversely, let $d_{sm}(C_p) = \frac{p}{\gamma_{sm}(G)} = \frac{p}{\left\lceil \frac{p}{6} \right\rceil}$. Therefore $p = d_{sm}(C_p)\left\lceil \frac{p}{6} \right\rceil$. (i.e.) $\left\lceil \frac{p}{6} \right\rceil$ divides p. If $p \equiv 0(mod\ 6)$, then $p = 6k$ and $\left\lceil \frac{p}{6} \right\rceil = k$. Thus $\left\lceil \frac{p}{6} \right\rceil$ divides p.

Suppose $p = 6k + 1,\ i \le l \le 5$. Applying the same argument in the converse part of the theorem, we obtain the values as $p = 8,10,15,20,25$.

Next, We discuss the split majority domatic number for complement of a graph G and Nordhaus-Gauddum type results.

Proposition 3.8

If G has a full degree vertex and all other vertices are of degree less than $\left\lceil \frac{p}{2} \right\rceil$ then $d_{sm}(\bar{G}) = p - 1$.

Proof

Suppose G has a full degree vertex and all other vertices are of degree less than $\left\lceil \frac{p}{2} \right.$. Then G has an isolate and all other vertices are of degree greater than or equal to $\left\lceil \frac{p}{[} 2 \right\rceil$.

Then every vertex except the isolate constitutes a majority dominating set of $\bar{G}$. Since $\bar{G}$ has an isolate v, every majority dominating set of $\bar{G}$ is split majority dominating set of $\bar{G}$. Thus $d_{sm} = p - 1$.

Theorem 3.9

For any graph G, $d_{sm}(G) + d_{sm}(\bar{G}) \le p + 2$ and $d_{sm}(G).d_{sm}(\bar{G}) \le 2p$.

Proof

If G has a full degree vertex v, then $d_{sm}(G) = 1$ and $\bar{G}$ has an isolate v. Suppose $\delta(G) \ge \left\lceil \frac{p}{2} \right\rceil - 1$. Then there exists atleast one vertex v in $\bar{G}$ such that $d(v) < \left\lceil \frac{p}{2} \right\rceil - 1$. In this case, there exists a minimal split majority dominating set of $\bar{G}$ with cardinality greater than or equal to two. Therefore $d_{sm}(\bar{G}) \le p - 1$ and $d_{sm}(G) + d_{sm}(\bar{G}) \le p$. suppose G is a complete bipartite graph with m = n. Then $\bar{G}$ has two components and each vertex v of $\bar{G}$ constitutes a split majority dominating set of $\bar{G}$. Therefore $d_{sm}(\bar{G}) = p$ and $d - sm(G) = 2$. In this case, $d - sm(G) + d_{sm}(\bar{G}) \le p + 2$. We prove the another result in the similar fashion.

References

[1] Cockayne, E. J. and Hedetniemi, S. J., Towards a theory of domination in graphs, Networks 7 (1977), 247 - 261.9

[2] Haynes.T.W., Hedetniemi S.T, Peter J. Slater - Fundamentals of Domination in Graphs, 1998 by Marcel Dekker, Inc., New york.

[3] Joseline Manora, J.and Swaminathan, V. - Majority Dominating Sets - published in J A R J: vol. 3, No. 2, (75 - 82) 2006.

[4] Joseline Manora, J.and Swaminathan, V. - Majority Domatic number of a Graph - published in GJPAM: vol. 6, No. 3, (275 - 283) 2010.

[5] Joseline Manora, J and Veeramanikandan, S. - The split majority domination number of a graph-Annals of Pure and Applied Mathematics, vol.9,No.1, 2015, 13-22.

[6] Joseline Manora, J and Veeramanikandan, S. - Some Results on Split Majority Dominating Set of a Graph-IJAER, Vol.10, No.15, 2015, 940-944.
[7] Kulli. V.R: Theory of Domination in Graphs Vishwa International Publications, ISBN: 81-900205-1-X (2010).
[8] Kulli, V.R and Janakiram, B -The split domination number of a graph-Graph Theory. Notes of New York., XXII, 16-19, 1997.