# Acceleration of Newton-Raphson's Method Using Logarithmic Convexity for Solving Systems of Nonlinear Equations of Two Variables <br> Addisu Yitbarek, Genanew Gofe and Hailu Muleta <br> Department of Mathematics, Jimma University, Ethiopia. 

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#### Abstract

This paper extends the work of Hernandez [5] to functions of two variables in which the emphasis is given to the influence of convexity on Newton-Raphson's method using two functions with different degree but having the same solution. Upon the properties of logarithmic degree of convexity the third order convergent iterative method for the solutions of systems of nonlinear equations which avoids the computation of second order derivative of the function is obtained. The result shows the accelerated NewtonRaphson's method is faster than the other methods considered in this paper.


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## I. Introduction

Newton-Raphson's method is one of the most powerful and well known methods for solving algebraic and transcendental equations. As a result, it is the widely used algorithm for finding simple roots which starts with an initial approximation closer $x_{o}$ to the root and generates a sequence of successive iterates $\left\{x_{n}\right\}$ converging quadratically to simple roots. It is given by

$$
\begin{equation*}
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)} \tag{1}
\end{equation*}
$$

which is discussed in [5]. And also Newton's method is widely used in the calculation of the roots of nonlinear equations; likewise, it is used for solving nonlinear systems of equations where the formation of the matrix of derivatives is essential in the solution which is given by

$$
\begin{equation*}
x_{n+1}=x_{n}-r\left(x_{n}\right) F\left(x_{n}\right) \tag{2}
\end{equation*}
$$

, and discussed by Gutierrez [3]. Systems of nonlinear equations are difficult to solve in general. The best way to solve these equations is by iterative methods. One of the classical methods to solve the system of nonlinear equations is Newton-Raphson's method which has second order rate of convergence. For the solutions of systems of nonlinear equations Gutierrez [3] presented a new method for finding majorizing sequences for Newton's method by using the linear operator $L_{F}(x)$ in connection with this method. The majorizing function is a cubic polynomial of one variable that has different roots. The study of higher-order convergence iterative processes is becoming more and more important in recent years. Among these methods, the most famous are Chebyshev, Halley, and super- Halley methods and are given by

Chebyshev's method

$$
x_{n+1}=x_{n}-\left[I+1 / 2 L_{F}\left(x_{n}\right)\right] F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right), n \geq 0
$$

Halley's method (or method of tangent hyperbola

$$
x_{n+1}=x_{n}-\left[I+-1 / 2 L_{F}\left(x_{n}\right)\left(I-1 / 2 L_{F}\left(x_{n}\right)^{-1}\right] \times F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right), n \geq 0\right.
$$

Convex acceleration of Newton's method (or Super-Halley method)

$$
\begin{equation*}
x_{n+1}=x_{n}-\left[I+1 / 2 L_{F}\left(x_{n}\right)\left(I-L_{F}\left(x_{n}\right)^{-1}\right] F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)\right. \tag{5}
\end{equation*}
$$

,where $I$ is the identity matrix and the linear operator
$L_{F}(x)=F^{\prime}(x)^{-1} F^{\prime \prime}(x) F^{\prime}(x)^{-1} F(x$
is the degree of logarithmic convexity provided that $F^{\prime}(x)^{-1}$ exists.

Halley and Chebyshev's methods start with two parameters, and they constructed a system of recurrence relations consisting of four real sequences of positive numbers which yield an increasing convergent sequence that majorizes the sequence in Banach spaces. In Halley and Chebyshev's methods the sequence $\left\{x_{n}\right\}$ defined in a Banach space analyzed by the real sequences $\left\{a_{n}\right\},\left\{b_{n}\right\},\left\{c_{n}\right\}$, and $\left\{d_{n}\right\}$ .To establish the convergence of $\left\{x_{n}\right\}$, it needs to prove the convergence of the real sequence $\left\{d_{n}\right\}$ which shows the sequence that majorizes the convergence given by Gutierrez and Hernandez [8].

The Super-Halley method improves the two methods by using majorizing sequence assuming the following conditions:

1. There exists a continuous linear operator

$$
r_{0}=\left[F^{\prime}\left(x_{0}\right)\right]^{-1}, x_{0} \in \Omega_{0}
$$

2. $\| r_{0}\left(F^{\prime \prime}(x)-F^{\prime \prime}(y)\|\leq k\| x-y \|\right.$,

$$
x, y \in \Omega_{0}, k \geq 0
$$

3. $\left\|r_{0} F\left(x_{0}\right)\right\| \leq a,\left\|r_{0} F^{\prime \prime}\left(x_{0}\right)\right\| \leq b$
4. $\quad P(t) \equiv \frac{k}{6} t^{3}+\frac{b}{2} t^{2}-t+a=0$
, where $P(t)$ has one negative root and two positive roots $r_{1}$ and $r_{2}\left(r_{1}<r_{2}\right)$ if $k>0$, or has two positive roots $r_{1}$ and $r_{2}\left(r_{1} \leq r_{2}\right)$ if $k=0$. Gutiérrez [3] showed that the convergence of $\left\{t_{n}\right\}$ implies the convergence of $\left\{x_{n}\right\}$ to a limit $x$. Using the majorizing sequence requires long process to obtain the convergence of the solution. To this end, this paper concentrates on solving the system of nonlinear equation $\boldsymbol{F}(\boldsymbol{x})=\mathbf{0}$ using the accelerated Newton-Raphson's method in Banach space by using the logarithmic degree of convexity without the computation of second order derivative of the given function and also shows the influence of convexity on Newton's method in order to obtain the solution of systems of nonlinear equations. The degree of logarithmic convexity is obtained by replacing the computation of second order derivative of the linear operator given in Eq. 6 with the difference of the first derivatives of $x$ and $H(x)$.
$H(x)=F^{\prime}(x)^{-1} F(x)$, and
$L_{F}(x)=(x-H(x))^{\prime}=I-H^{\prime}(x)$
The advantage of this expression is that it avoids the computation of second order derivatives in the process of finding the degree of logarithmic convexity. Ezquerro and Hernandez [4] found convex acceleration of Newton's method (or Super- Halley method) to approximate the solutions of nonlinear systems of equations. They provided sufficient convergence conditions for this method in three space settings: real line, complex plane and Banach space and several applications were used for the result [4].The NewtonRaphson's method is applicable to systems of multivariate functions by finding the majorizing sequence $\mathrm{P}(\mathrm{t})$ of a polynomial function to accelerate the solution of the system of nonlinear equations [1].

## 2. Preliminaries

Consider the equation
$F(x, y)=0$
Let $f$ and $g$ are $C^{2}$ functions on a given domain. Using Taylor's expansion for the two functions near $(x, y)$, we find

$$
\begin{array}{r}
f(x+h, y+k)=f(x, y)+h \frac{\partial f}{\partial x}+k \frac{\partial f}{\partial y}+O\left(h^{2}+k^{2}\right) \\
g(x+h, y+k)=g(x, y)+h \frac{\partial g}{\partial x}+k \frac{\partial g}{\partial y}+O\left(h^{2}+k^{2}\right)
\end{array}
$$

And if we keep the first order terms, we are looking for a couple $(h, k)$ such as
$f(x+h, y+k) \approx f(x, y)+h \frac{\partial f}{\partial x}+k \frac{\partial f}{\partial y}$
$g(x+h, y+k) \approx g(x, y)+h \frac{\partial g}{\partial x}+k \frac{\partial g}{\partial y}$
Hence, it is equivalent to the linear system

$$
\left[\begin{array}{ll}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{array}\right]\binom{h}{k}=-\binom{f(x, y)}{g(x, y)}
$$

,where $\left[\frac{\partial f}{\partial x} \frac{\partial f}{\partial y}\right]$ is a Jacobian matrix and is denoted

$$
\left[\begin{array}{ll}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{array}\right]
$$

by $F^{\prime}(x, y)$ and is given by $F^{\prime}(x, y)^{-1}=r(x, y)$, which is equivalent to

$$
\begin{equation*}
\left(x_{n+1}, y_{n+1}\right)=\left(x_{n}, y_{n}\right)-r\left(x_{n}, y_{n}\right) F\left(x_{n}, y_{n}\right) \tag{8}
\end{equation*}
$$

Eq. 8 is Newton-Rapson's formula for solving systems of nonlinear functions of two variables.

On the other hand, to accelerate Newton's method the logarithmic convexity of a function was used by Hernandez [5]. The convergence of this method is analyzed by means of the convexity of the functions $f$ and $f^{\prime}$. Provided that $f$ satisfies the conditions that if
$f:[a, b] \leq R \rightarrow R$ be sufficiently differentiable and $f^{\prime}(t)>0, f^{\prime \prime}(t) \geq 0$ on $[a, b]$ and $f(a)<0<f(b)$ and the derivative of the logarithmic convexity is given as discussed in [2\&5].

$$
\begin{equation*}
x_{n+1}=g\left(x_{n}\right)=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)} \tag{9}
\end{equation*}
$$

Consider Eq. 1 and derivate Eq. 9 to get the log-degree of convexity

$$
\begin{equation*}
g^{\prime}\left(x_{n}\right)=\frac{f^{\prime \prime}\left(x_{n}\right) f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)^{2}}=L_{f}\left(x_{n}\right) \tag{10}
\end{equation*}
$$

Eq. 10 is the logarithmic degree of convexity of the function $\boldsymbol{f}$. Hernandez [5] presented the accelerated NewtonRaphson's method for function of one variable which is given by

$$
\begin{equation*}
y_{n}=f\left(x_{n}\right)=x_{n-1}-\frac{f\left(x_{n+1}\right)}{2 f^{\prime \prime}\left(x_{n-1}\right)} \frac{2-L_{f}\left(x_{n-1}\right)}{1-L_{f}\left(x_{n-1}\right)} \tag{11}
\end{equation*}
$$

## 3. The Accelerated Newton-Raphson's <br> Method for Functions of two Variables <br> Definition1

Let F be a function in the Banach space $\mathrm{B}(\mathrm{X}, \mathrm{Y})$ and $\mathrm{x}_{\boldsymbol{\epsilon}} \mathrm{X}$ such that $\mathrm{r}(\mathrm{x})$ exists.

Then the degree of logarithmic convexity of $\boldsymbol{F}$ in $\boldsymbol{X}$ is defined by the following linear operator

$$
L_{F}: X \rightarrow X
$$

Depending on the Newton's method given in Eq. 1 and the degree of logarithmic convexity in Eq. 10, we can write the sequence

$$
X_{n+1}=G\left(X_{n}\right)
$$

where $G\left(X_{n}\right)=X_{n}-r\left(X_{n}\right) F\left(X_{n}\right)$,

$$
\begin{aligned}
& X_{n}=\left(x_{n}, y_{n}\right) \\
& r(X)=F^{\prime}(X)^{-1}
\end{aligned}
$$

$G(X)=X-r(X) F(X)$
Let $H(X)=r(X) F(X)$
Such that $G(X)=X-H(X)$ consequently,
we get $H^{\prime}(X)=r^{\prime}(X) F(X)+r(X) F^{\prime}(X)=r^{\prime}(X) F(X)+I$ $G^{\prime}(X)=-r^{\prime}(X) F(X)$

Therefore, $r^{\prime}(X)=-r(X) F^{\prime \prime}(X) r(X) \in k(Y, X)$
Thus,
$G^{\prime}(X)=r(X) F^{\prime \prime}(X) r(X) F(X)=L_{F}(X)$
The degree of logarithmic convexity for functions of two variables is given by

$$
\begin{equation*}
L_{F}(x, y)=r(x, y) F^{\prime \prime}(x, y) r(x, y) F(x, y) \tag{12}
\end{equation*}
$$

## Lemma1:-

Let L be an $\boldsymbol{n x n}$ matrix and $\boldsymbol{\alpha}$ be a real number and $\mathbf{I}$ be the identity matrix of order $\boldsymbol{n}$. Assume that $I-\alpha L$ is invertible, then

$$
L(I-\alpha L)^{-1}=(I-\alpha L)^{-1} L
$$

Proof:- $\quad L(I-\alpha L)=L-\alpha L^{2}=(I-\alpha L) L$.
So, we have

$$
L(I-\alpha L)=(I-\alpha L) L
$$

By pre- and post-multiplying the expression

$$
\begin{aligned}
& (I-\alpha L) \text { by }(I-\alpha L)^{-1} \text {, we get } \\
& (I-\alpha L)^{-1} L(I-\alpha L)(I-\alpha L)^{-1} \\
& =(I-\alpha L)^{-1}(I-\alpha L) L(I-\alpha L)^{-1} .
\end{aligned}
$$

The expression $\quad(I-\alpha L)(I-\alpha L)^{-1}$ gives the identity matrix.

Thus $\quad(I-\alpha L)^{-1} L=L(I-\alpha L)^{-1}$

## Lemma2:-

Let $L \in R^{n \times n}$, I be the identity matrix and $\alpha$ be a real number.
Assume that $I-\alpha L$ is invertible, then

$$
\begin{equation*}
(I-\alpha L)^{-1}\left(I+\left(\frac{1}{2}-\alpha\right) L\right)=I+1 / 2 L(I-\alpha L)^{-1} \tag{13}
\end{equation*}
$$

Proof:- Starting from the left hand side of Eq. 13 and arranging gives

$$
\begin{aligned}
& (I-\alpha L)^{-1}(I+(1 / 2-\alpha) L)= \\
& (I-\alpha L)^{-1}(I+1 / 2 L-\alpha L)= \\
& I+1 / 2(I-\alpha L)^{-1} L=I+1 / 2 L(I-\alpha L)^{-1}
\end{aligned}
$$

This complete the proof of the Lemma.
Let consider the class of iteration which Schewatlick defined as a class for a real scalar $\alpha$ and $i$ is the number of iteration, and is called the Schewatlick's class [6]. It was stated that the class is obtained by solving

$$
\begin{align*}
& F\left(x_{n}\right)+F^{\prime}\left(x_{n}\right)\left(y_{n}^{(i+1)}-x_{n}\right)+\alpha F^{\prime \prime}\left(x_{n}\right)\left(y_{n}^{(i)}-x_{n}\right)\left(y_{n}^{(i+1)}-x_{n}\right) \\
& +(1 / 2-\alpha) F^{\prime \prime}\left(x_{n}\right)\left(y_{n}^{(i)}-x_{n}\right) \times\left(y_{n}^{(i)}-x_{n}\right) \tag{14}
\end{align*}
$$

for

$$
y_{n}^{(i+1)}, i=0,1,2,3, \ldots
$$

Based on the above discussion and Lemma1, we derive the accelerated Newton-Raphson's method for functions of two variables as follows

Let $y_{n}^{(0)}=x_{n}$ and $y_{n}^{(2)}=x_{n+1}$, and from Eq.14, we get

$$
\begin{align*}
& F\left(x_{n}\right)+F^{\prime}\left(x_{n}\right)\left(y_{n}^{(2)}-x_{n}\right) \\
& +\alpha F^{\prime \prime}\left(x_{n}\right)\left(y_{n}^{(1)}-x_{n}\right)\left(y_{n}^{(2)}-x_{n}\right) \\
& +(1 / 2-\alpha) F^{\prime \prime}\left(x_{n}\right)\left(y_{n}^{(1)}-x_{n}\right)\left(y_{n}^{(1)}-x_{n}\right)=0 \tag{15}
\end{align*}
$$

First we solve for $y_{n}^{(1)}$, that means for $i=0$. Since $y_{n}^{(0)}=x_{n}$, we get

$$
\begin{equation*}
F\left(x_{n}\right)+F^{\prime}\left(x_{n}\right)\left(y_{n}^{(1)}-x_{n}\right)=0 \tag{16}
\end{equation*}
$$

Then
$y_{n}^{(1)}=x_{n}-F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)$
By substituting Eq. 16 into Eq. 15, we obtain

$$
\begin{align*}
& F\left(x_{n}\right)+F^{\prime}\left(x_{n}\right)\left(y_{n}^{(2)}-x_{n}\right)-\alpha F^{\prime \prime}\left(x_{n}\right)\left(F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)\right)\left(y_{n}^{(2)}-x_{n}\right) \\
& +(1 / 2-\alpha) F^{\prime \prime}\left(x_{n}\right) \times\left(F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)\right)\left(F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)\right)=0 \tag{17}
\end{align*}
$$

Collecting the terms with $\left(y_{n}^{(2)}-x_{n}\right)$ and rearranging the equation yields

$$
\begin{align*}
& \left(F^{\prime}\left(x_{n}\right)-\alpha F^{\prime \prime}\left(x_{n}\right)\left(F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)\right) \times\left(y_{n}^{(2)}-x_{n}\right)=\right. \\
& -F\left(x_{n}\right)-(1 / 2-\alpha) F^{\prime \prime}\left(x_{n}\right)\left(F^{\prime}\left(x_{n}\right)^{-1}\right. \\
& \left.\times F\left(x_{n}\right)\right)\left(F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)\right) \tag{18}
\end{align*}
$$

Substituting $\quad x_{n+1}=y_{n}^{(2)}$ into Eq.17, we have $\left(F^{\prime}\left(x_{n}\right)\left(I-\alpha L_{F}\left(x_{n}\right)\right)\left(x_{n+1}-x_{n}\right)\right.$ $=-F\left(x_{n}\right)-(1 / 2-\alpha) F^{\prime \prime}\left(x_{n}\right)\left(F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)\right)$ $\times\left(F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)\right)$
Multiplying by $\left(I-\alpha L_{F}\left(x_{n}\right)\right)^{-1} F^{\prime}\left(x_{n}\right)^{-1}$ and rearranging the last equation, we have

$$
\begin{aligned}
& x_{n+1} \\
& =x_{n}-\left(I-\alpha L_{F}\left(x_{n}\right)\right)^{-1}\left[I+(1 / 2-\alpha) L_{F}\left(x_{n}\right)\right] \\
& \times F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)
\end{aligned}
$$

By using Lemma 2, then we get
$x_{n+1}=x_{n}-\left[I+1 / 2 L_{F}\left(x_{n}\right)\left(I-\alpha L_{F}\left(x_{n}\right)^{-1}\right]\right.$
$\times F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)$
Substituting $\alpha=1$, We get

$$
\begin{aligned}
& x_{n+1}= \\
& x_{n}-\left[I+1 / 2 L_{F}\left(x_{n}\right)\left(I-L_{F}\left(x_{n}\right)\right)^{-1}\right] F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)
\end{aligned}
$$

Keeping $X_{n}=\left(x_{n}, y_{n}\right)$, so that the accelerated NewtonRaphson's formula for functions of two variables becomes

$$
\begin{align*}
& \left(x_{n+1}, y_{n+1}\right)= \\
& \left(x_{n}, y_{n}\right)-\left[I+1 / 2 L_{F}\left(x_{n}, y_{n}\right)\left(I-L_{F}\left(x_{n}, y_{n}\right)\right)^{-1}\right] \\
& \times F^{\prime}\left(x_{n}, y_{n}\right)^{-1} F\left(x_{n}, y_{n}\right) \tag{19}
\end{align*}
$$

The next issue is to determine the order of convergence of this method. To this end, let us consider the following remark.

Remark:- Let X, Y be Banach spaces and consider a nonlinear operator $F: \Omega \subseteq X \rightarrow Y$ which is twice Frechet differentiable on an open convex set $\Omega o \subseteq \Omega$. Let us assume that $F^{\prime}\left(x_{o}\right)^{-1} \in L(Y, X)$ exists at some $x_{o} \in \Omega_{o}$, where $L(Y, X)$ is the set of bounded linear operators from $Y$ into $X$. The convex acceleration of Newton's method for approximating a solution $x \in \Omega_{o}$ of the equation $\boldsymbol{F}(\boldsymbol{x})=\mathbf{0}$ in the form of

$$
\begin{aligned}
& x_{n+1}= \\
& x_{n}-\left[I+\frac{1}{2} L_{F}\left(x_{n}\right)\left(I-L_{F}\left(x_{n}\right)\right)^{-1}\right] F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)
\end{aligned}
$$

for $\boldsymbol{n} \geq \mathbf{0}$,
, where the identity operator is denoted by $I$ on $\boldsymbol{x}$ and $L_{F}(x)$ is the linear operator defined by

$$
L_{F}(x)=F^{\prime}(x)^{-1} F^{\prime \prime}(x) F^{\prime}(x)^{-1} F(x), x \in X
$$

Theorem1: Let $F: \Omega \subseteq X \rightarrow Y$, is k-times Frechet differentiable on an open convexset $\Omega_{o} \in \Omega$ containing the root $\boldsymbol{\alpha}$ of $F(x)=0$. Then the accelerated NewtonRaphson's method is third order convergent.

Proof:- For any $\mathrm{X}, X_{n} \in \Omega$ we write the Taylor's expansion for $\boldsymbol{F}$ and keeping $X_{n}=\left(x_{n}, y_{n}\right)$ $F(X)=$
$F\left(X_{n}\right)+F^{\prime}\left(X_{n}\right)\left(X-X_{n}\right)+\frac{1}{2!} F^{\prime \prime}\left(X_{n}\right)\left(X-X_{n}\right)^{2}$
$+\frac{1}{3!} F^{(3)}\left(X_{n}\right)\left(X-X_{n}\right)^{3}+\ldots+\frac{1}{k!} F^{(k)}\left(X_{n}\right)\left(X-X_{n}\right)^{k .}$
,where

$$
F^{\prime}(x)\left(x-x_{n}\right)=\frac{\partial F}{\partial x}\left(x-x_{n}\right)+\frac{\partial F}{\partial y}\left(y-y_{n}\right)
$$

Let $\boldsymbol{\alpha}$ be the root of the systems of equation $\boldsymbol{F}(\boldsymbol{x})=\mathbf{0}$, we have
$F(x)=F\left(x_{n}\right)+F^{\prime}\left(x_{n}\right)\left(\alpha-x_{n}\right)+\frac{1}{2!} F^{\prime \prime}\left(x_{n}\right)\left(\alpha-x_{n}\right)^{2}$
$+\frac{1}{3!} F^{(3)}\left(x_{n}\right)\left(\alpha-x_{n}\right)^{3}+\ldots+\frac{1}{k!} F^{(k)}\left(x_{n}\right)\left(\alpha-x_{n}\right)^{k .}$
Let
$e_{n}=x_{n}-\alpha$,
We have
$F(x)=$
$F\left(x_{n}\right)+F^{\prime}\left(x_{n}\right) e_{n}+\frac{1}{2!} F^{\prime \prime}\left(x_{n}\right) e_{n}^{2}+\frac{1}{3!} F^{(3)}\left(x_{n}\right) e_{n}^{3}$
$+\ldots+\frac{1}{k!} F^{(k)}\left(x_{n}\right) e_{n}^{k}$
$F(x)=0 \approx F\left(x_{n}\right)+F^{\prime}\left(x_{n}\right) e_{n}+\frac{1}{2!} F^{\prime \prime}\left(x_{n}\right) e_{n}^{2}$
$+\frac{1}{3!} F^{(3)}\left(x_{n}\right) e_{n}^{3}+O\left(\left\|e_{n}^{4}\right\|\right)$
$F\left(x_{n}\right)=F^{\prime}\left(x_{n}\right) e_{n}-\frac{1}{2!} F^{\prime \prime}\left(x_{n}\right) e_{n}^{2}$
$+F^{(3)}\left(x_{n}\right) e_{n}^{3}+O\left(\left\|e_{n}^{4}\right\|\right)$
Multiplying both sides by $F^{\prime}\left(x_{n}\right)^{-1}$

$$
\begin{align*}
& F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)=F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right) e_{n}-\frac{1}{2} F^{\prime}\left(x_{n}\right)^{-1} F^{\prime \prime}\left(x_{n}\right) e_{n}^{2} \\
& +\frac{1}{6} F^{\prime}\left(x_{n}\right)^{-1} F^{(3)}\left(x_{n}\right) e_{n}^{3}+O\left(\left\|e_{n}^{4}\right\|\right. \tag{20}
\end{align*}
$$

From the iteration of Modified Newton's method of Weerakoon and Fernando [7] , we have
$x_{n+1}=x_{n}-F^{\prime}\left(x_{n}\right)^{-1}\left(F\left(x_{n}\right)+F\left(x_{n}\right)^{-1} F\left(x_{n}\right)\right)$. Then $e_{n+1}=$
$e_{n}-F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)-F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}-F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)\right)$

By substituting Eq. 20 into Eq. 21 and applying the Taylor's expansion for
$F\left(x_{n}-F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)\right)$ around $x_{n}$, we have
$F\left(x_{n}-F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)\right)=$
$1 / 2 F^{\prime \prime}\left(x_{n}\right)\left(F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)\right)^{2}$
$-1 / 6 F^{(3)}\left(x_{n}\right)\left(\left(F^{\prime}\left(x_{n}\right)^{-1} F\left(x_{n}\right)\right)+O\left(\left\|e_{n}^{4}\right\|\right)\right.$
$e_{n+1}=1 / 2 F^{\prime}\left(x_{n}\right)^{-1} F^{\prime \prime}\left(x_{n}\right) e_{n}^{2}$
$-1 / 6 F^{\prime}\left(x_{n}\right)^{-1} F^{(3)}\left(x_{n}\right) e_{n}^{3}+O\left(\left\|e_{n}^{4}\right\|\right)$
$-F^{\prime}\left(x_{n}\right)^{-1}\left[1 / 2 F^{\prime \prime}\left(x_{n}\right)\left(e_{n}-1 / 2 F^{\prime}\left(x_{n}\right)^{-1} F^{\prime \prime}\left(x_{n}\right) e_{n}^{2} O\left(\left\|e_{n}^{3}\right\|\right)\right)^{2}\right.$
$\left.+1 / 6 F^{(3)}\left(x_{n}\right)\left(e_{n}-1 / 2 F^{\prime}\left(x_{n}\right)^{-1} F^{\prime \prime}\left(x_{n}\right) e_{n}^{2}+O\left(\left\|e_{n}^{3}\right\|\right)\right)^{3}\right]+O\left(\left\|e_{n}^{4}\right\|\right)$
We obtain,
$e_{n+1}=1 / 2\left[F^{\prime}\left(x_{n}\right)^{-1} F^{\prime \prime}\left(x_{n}\right)\right]^{2} e_{n}^{3}+O\left(\left\|e_{n}^{4}\right\|\right)$
This shows the method is third-order convergent.
4. The Influence of convexity on Newton-Raphson's
Method

Consider functions of two variables $F$ and $G$ having the same solutions but the degree of $G$ is less than the degree of $F$. Let $G$ be a function having the same conditions with that of $F$ in a Banach Space and $G(t)=0$ where $t$ is the solution of $F$.
Assume the sequence $y_{n}=G\left(y_{n-1}\right) \quad$ and $G(x)=x-r(x) F(x)$ and $y_{0}=x_{0}$. Then by means of the log-degree of convexity we are going to compare the sequences $\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$, where the sequence $\left\{x_{n}\right\}$ is the iteration of the function $F\left(x_{n}, y_{n}\right)$ and the sequence $\left\{y_{n}\right\}$ is the iteration of the function $G\left(x_{n}, y_{n}\right)$.

## Theorem2:-

Let $\boldsymbol{F}$ and $\boldsymbol{G}$ be two functions with the degree of $\boldsymbol{G}$ less than the degree of $\boldsymbol{F}$ and have the same solution $t$ of $\left(\begin{array}{ll}t_{1} & t_{2}\end{array}\right)^{T}$.
If $\left\|L_{F}(x)\right\|>\left\|L_{G}(x)\right\|$ for $\|F(x)\|>0$, then the sequence $\left\{y_{n}\right\}$ converges faster to the solution $t$ than the sequence $\left\{x_{n}\right\}$ 。
Proof: - Let the sequences $\left\{y_{n}\right\}$ and $\left\{x_{n}\right\}$ be a decreasing sequences and converges to $t$.
Let us apply proof by induction to show that $y_{n}<x_{n}$ for all $n \in N$, such that $x_{n} \neq t$.
Besides, $y_{n}=t$ if $x_{n-1} \neq t$ and $x_{n}=t$, taking into account that $\|F(t)\|=\|G(t)\|=t$ and $x_{0} \neq t$
$\left\|x_{1}-y_{1}\right\| \leq\|(F-G)(x)\|-\|(F-G)(t)\|$, and therefore there exists $\xi \in X$, (i.e. a sequence of iterates $x_{n}$ is said to
converge with order $P \geq 1$ to a point $\xi$ $\left|\xi-x_{n+1}\right| \leq c\left|\xi-x_{n}\right|^{p}, n \geq 0 \quad$ for some $c>0$ such that $\left\|x_{1}-y_{1}\right\| \leq\left\|(F-G)^{\prime}(\xi)\right\|\left\|x_{0}-t\right\|$.
On
$\left\|(F-G)^{\prime}(x)\right\|=\left\|L_{F}(x)-L_{G}(x)\right\|$ and $\|F(\xi)\|>0$. we obtain, $\left\|x_{1}-y_{1}\right\|>0$
Now we assume that $x_{k}>y_{k}$ for $\mathrm{k}=1,2,3 \ldots n-1$. If $\boldsymbol{x}_{\boldsymbol{n}} \neq$ t , then $x_{n+1} \neq t$ and
$\left\|x_{n}-y_{n}\right\| \leq\left\|(F-G)\left(x_{n-1}\right)\right\| \leq\left\|F\left(x_{n-1}\right)-G\left(x_{n-1}\right)\right\|$
Since $\boldsymbol{G}$ is an increasing function and for $\mathrm{k}=1$, we obtain $\left\|x_{n}-y_{n}\right\|>0$
Thus, the conditions given in Theorem2 are satisfied.

## 5. Numerical Examples

Example1. Consider the following equations

$$
F(x, y)=0
$$

,where

$$
\begin{gathered}
\left\{\begin{array}{l}
F_{1}(x, y)=x^{3}+y^{2}-17=0 \\
F_{2}(x, y)=x^{4}+y^{2}-25=0
\end{array}\right. \\
\text { where }\left\{\begin{array}{c}
G(x, y)=0 \\
G_{1}(x, y)=x^{2}+y-7=0 \\
G_{2}(x, y)=x+3 y^{2}-29=0
\end{array}\right.
\end{gathered}
$$

Table1. Comparison of the solution of $F(x)$ and $G(x)$

| $n$ | $\mathbf{F}(\mathbf{x})$ |  |  | $\mathbf{G}(\mathbf{x})$ |
| :--- | :--- | :--- | :--- | :--- |
|  | $x_{n}$ | $y_{n}$ | $x_{n}$ | $y_{n}$ |
| 0 | 4.000000000 | 6.000000000 | 4.000000000 | 6.000000000 |
|  | 00000 | 000000 | 000000 | 000000 |
| 1 | 3.115384615 | 2.621794871 | 2.407665505 | 3.738675958 |
|  | 384615 | 794872 | 226481 | 188154 |
| 2 | 2.505972930 | 2.170502022 | 2.019539585 | 3.072101591 |
|  | 803136 | 904809 | 564136 | 853178 |
| 3 | 2.150041784 | 2.920857715 | 1.999883480 | 3.000852426 |
|  | 777900 | 581494 | 688993 | 126139 |
| 4 | 2.017538608 | 2.983715102 | 1.999999972 | 3.000000122 |
|  | 199260 | 632965 | 746525 | 584301 |
| 5 | 2.000271959 | 2.999799009 | 2.000000000 | 3.000000000 |
|  | 387804 | 514695 | 000000 | 000002 |
| 6 | 2.000000066 | 2.999999947 | 2.000000000 | 3.000000000 |
|  | 547217 | 567330 | 000000 | 000000 |
| 7 | 2.000000000 | 2.999999999 |  |  |
| 8 | 000004 | 999997 |  |  |
|  | 2.000000000 | 3.000000000 |  |  |

As it can easily be seen from the table1, the Newton's method took 7 iterations for $F(x)$ to converge while $G(x)$ took only
5 iterations to converge to the exact solution. This clearly shows the impact of convexity on a solution of system of nonlinear equations of functions of two variables.
Example2.
Consider the following transcendental equation
$F(x, y)=0$
,where $\left\{\begin{array}{c}F_{1}(x, y)=x^{2}-4 y^{3}=0 \\ F_{2}(x, y)=\sin x+3 \cos (3 y)=0\end{array}\right.$
$F^{\prime}(x, y)=\left[\begin{array}{cc}2 x & -12 y^{2} \\ \cos x & -9 \sin (3 y)\end{array}\right]$
$x_{0}=0.5$ and $y_{0}=0.5$
Table2. Comparison of Newton's method and accelerated Newton-Raphson's method.

| $n$ | Newton's Method |  | Accelerated Newton's <br> Method |  |
| :--- | :--- | :--- | :--- | :--- |
|  | $x_{n}$ | $y_{n}$ | $x_{n}$ | $y_{n}$ |
| 0 | 0.500000000 | 0.500000000 | 0.500000000 | 0.500000000 |
|  | 000000 | 000000 | 000000 | 000000 |
| 2 | 1.180768240 | 0.643589413 | 0.985347061 | 0.617810357 |
|  | 168025 | 389342 | 575724 | 191540 |
| 3 | 0.990939171 | 0.619367958 | 0.967635891 | 0.616294048 |
|  | 194292 | 182828 | 847071 | 453753 |
| 4 | 0.967975939 | 0.616337724 | 0.967636004 | 0.616294108 |
|  | 488200 | 192481 | 449559 | 232118 |
| 5 | 0.967636079 | 0.616294117 | 0.967636004 | 0.616294108 |
|  | 475339 | 831845 | 449559 | 232118 |
| 6 | 0.967636004 | 0.616294108 |  |  |
|  | 449563 | 232118 |  |  |
| 7 | 0.967636004 | 0.616294108 |  |  |
| 8 | 449559 | 232118 |  |  |
| 8967636004 | 0.616294108 |  |  |  |
|  | 449559 | 232118 |  |  |

As revealed in table2, the Newton's method took 7 iterations to converge while the accelerated NewtonRaphson's method took only 3 iterations to converge to the exact solution. This clearly shows that the proposed method accelerates Newton-Raphson's method.

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