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On the Numerical Solution of Volterra-Fredholm Integral Equations with Exponential Kernal using Chebyshev and Legendre Collocation Methods

Kirtiwant P Ghadle and Ahmed Abdullah Hamoud

Dr.Babasaheb Ambedkar Marathwada University,Aurangabad,431004,India.

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ABSTRACT

Legendre and Chebyshev collocation methods are presented to solve numerically the Voltterra-Fredholm integral equations with exponential kernel. We transform the Volterra Fredholm integral equations to a system of Fredholm integral equations of the second kind,a system Fredholm integral equation with exponential kernel is obtained and will be solved using Legendre and Chebyshev polynomials. This lead to a system of algebraic equations with Legendre or Chebychev coeffcients. Thus, by solving the matrix equation, Legendre and Chebychev coefficients are obtained.A numerical example is included to certify the validity and applicability of the proposed technique.

Introduction

We consider the Volterra-Fredholm integral equation of the second kind with Exponential Kernel:

$$\psi(x,t)-\int_0^t\int_{-1}^{1}e^{|x-y|}\psi(y,\tau)dyd\tau = f(x,t), \quad (x,t)\in[-1,1]\times[0,T] \tag{1}$$

Where $0\le T$ and f is a given function. The elements $K(x,y)=e^{|x-y|}$ is exponential kernel. For solving Volterra-Fredholm integral equations, many methods with enough accuracy and efficiency have been used before by many researches [1,5,6,7,8,10,14]. The properties of Chebyshev or Legendre polynomials are used to reduce the system of Fredholm integral equations to a system of nonlinear algebraic equations [3,4,9]. Maleknejad and Fadaei Yami [12] solved the system of Volterra-Fredholm integral equations byAdomian decomposition method. In [13], Yalsinbas developed numerical solution of nonlinear Volterra- Fredholm integral equations by using Taylor polynomials. In [11],used continuous time collocation method for Volterra-Fredholm integral equations. Legendre wavelets also were applied for solving Volterra-Fredholm integral equations. In this paper,we use numerical technique based on trapezoidal rule, to reduce the Volterra-Fredholm integral equations to a system of Fredholm integral equations which will be solved using Legendre,Chebyshev collocation method (this technique is presented for the exponential kernel. The paper is organized as follows. In section 2 ,a system of Fredholm integral equations of the second kind is obtained from the Volterra-Fredholm integral equation. In Sections 3 and 4,we present the Legendre and Chebychev collocation method to solve the system obtained with special choice of the nodes. In the remainder of the paper,we give a practical example to certify the validity of the proposed technique.

Tele:

E-mail address: drahmed985@yahoo.com

System of Fredholm Integral Equations

We consider the Volterra-Fredholm integral equation of the second kind with exponential Kernel (1). First, if $t=0$ the Volterra-Fredholm integral equations is reduced to: $\psi(x,0)=f(x,0)$. For $t\neq o$,we apply trapezoidal method to solve the Volterra integral equations according to the variable τ .For a given t ,we divide the interval of integration (0,t) in to m equal subintervals, $\delta\tau=\frac{t_m-0}{m}$, where $t_m=t$.

Let $\tau_0=0; t_0=\tau_0; \tau_m=t_m=t; \tau_j=j\delta\tau; t_j=\tau_j$.

Using the trapezoid rule,

$$\int_0^t\int_{-1}^{1}e^{|x-y|}\psi(y,\tau)dyd\tau \approx \delta\tau\sum_{j=0}^{m''}\int_{-1}^{1}e^{|x-y|}\psi(y,\tau_j)dy \tag{2}$$

where the double prime indicates that the first and last term to be halved, where

$$\delta\tau=\frac{\tau_j-0}{j}=\frac{t-0}{m}, \tau_j\le t, j\ge 1, t=t_m=\tau_m$$

In all our approximation, the error assumed negligible, this help us to get a system of Fredholm Integral equations. Now, for $0\le r\le m$,the Volterra-Fredholm integral equations become a system of Fredholm integral equations

$$\psi(x,t_r)-\delta\tau\sum_{j=0}^{r''}\int_{-1}^{1}e^{|x-y|}\psi(y,\tau_j)dy = f(x,t_r) \tag{3}$$

and $\psi(x,0)=f(x,0)$,we get the system:

$$\psi(x,0)=f(x,0)$$

$$\psi(x,t_1)-\frac{\delta\tau}{2}\int_{-1}^{1}e^{|x-y|}\psi(y,t_1)dy =$$

$$f(x,t_1)+\frac{\delta\tau}{2}\int_{-1}^{1}e^{|x-y|}\psi(y,0)dy$$

$$\psi(x,t_2)-\frac{\delta\tau}{2}\int_{-1}^{1}e^{|x-y|}\psi(y,t_2)dy=$$

$$f(x,t_2)+\frac{\delta\tau}{2}\int_{-1}^{1}e^{|x-y|}\psi(y,0)dy+\delta\tau\int_{-1}^{1}e^{|x-y|}\psi(y,t_1)dy$$

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$$\psi(x,t_m)-\frac{\delta\tau}{2}\int_{-1}^{1}e^{|x-y|}\psi(y,t_m)dy=$$

$$f(x,t_m)+\delta\tau\sum_{j=0}^{m-1}{}'\int_{-1}^{1}e^{|x-y|}\psi(y,\tau_j)dy$$

where the prime indicates that the first term to be halved.

Denote: $f(x,t_n)=f^n(x),\psi(y,\tau_n)=\psi^n(y),n=0,...,m$

Putting

$$F^m(x)=f^m(x)+\sum_{j=0}^{m-1}{}'\int_{-1}^{1}e^{|x-y|}\psi^j(y)dy$$

An astitute computation gives

$$F^m(x)=f^m(x)+2\sum_{j=0}^{m-1}(-1)^{j+m}\left(f^j(x)-\psi^j(x)\right)$$

$$+(-1)^{m+1}\frac{\delta\tau}{2}\int_{-1}^{1}e^{|x-y|}\psi^0(y)dy$$

Now, our problem become:

$$\psi^n(x)-\frac{\delta\tau}{2}\int_{-1}^{1}e^{|x-y|}\psi^n(y)dy=F^n(x),n=1,...,m \quad (4)$$

$$\psi(x,0)=f(x,0)$$

Equations (4) represents a system of Fredholm integral equations of the second kind. In the next,we will present the well known techniques of Legendre and Chebychev collocation methods to solve the system of Fredholm integral equation with exponential kernel.

Legendre Collocation Method

Orthogonal polynomials are widely used in applications in mathematics, mathematical physics, engineering and computer science. One of the most common set of orthogonal polynomials is the Legendre polynomials. The Legendre polynomials P_n satisfy the recurrence formula:

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x),n\in N$$

$$P_0(x)=1$$

$$P_1(x)=x$$

We choice $x_k,k\in[0,n]$ the zeros of the Legendre polynomial of degree equal $n+1$. Here,$[a,b]$ used to indicate the interval of all integers between a and b. We determine a suitable interpolating elements $\phi_j(x),j=0,1,...,n$, such that

$$\psi_n(x)=\sum_{j=0}^{n}\phi_j(x)\psi(x_j) \quad (5)$$

is the unique interpolating polynomial of degree n, which interpolates at the points $x_i,i=0,1,...,n$ The elements $\phi_j(x),j=0,1,...,n$ are called the basic functions associated with the Legendre interpolation polynomial and they satisfy $\phi_j(x)=\delta_{ij}$ Then we get an approximation of the exactly integral,let say:

$$I_n(\psi)=\int_{-1}^{1}K(x,y)\psi_n(y)dy \quad (6)$$

This type of approximation must be chosen so that the integral (6) can be evaluated (either explicitly or by an efficient numerical technique). The functions $P_o(x),P_1(x),...,P_n(x)$ will be called interpolating elements. In this dissertation, the interpolating function ψ_n will be assumed to be the interpolating polynomial

$$\psi_n(x)=\sum_{j=0}^{n}\beta_jP_j(x) \quad (7)$$

where P_j are Legendre polynomials of degree j,n is the number of Legendre polynomials, and β_j are unknown parameters, to be determined. The coefficients β_j are obtained by multiplying both sides of Eq. (7) by $P_m,m\le n$ (as weight functions), and integrating the resulting equation with respect to x over the interval [-1,1] to obtain

$$\int_{-1}^{1}p_m(x)\psi_n(x)dx=\sum_{j=0}^{n}\beta_j\int_{-1}^{1}p_m(x)P_j(x)dx=\beta_m\frac{2}{2m+1}$$

Therefore,

$$\beta_m=\frac{2m+1}{2}\int_{-1}^{1}p_m(x)\psi_n(x)dx \quad (8)$$

Here the integrand $P_m\psi_n$ is a polynomial of degree $n+m\le 2n$ then its integration in (8) can exactly be obtained from just $n+1$ point Gauss-Legendre method, by using the following formula

$$\beta_m=\frac{2m+1}{2}\sum_{j=0}^{n}\omega_j p_m(x_j)\psi(x_j) \quad (9)$$

Where $\omega_j,j=0,...,n$ are the $(n+1)$ point Gauss-Legendre weights. The $n+1$ grid points (x_i) of Gauss Legendre integration in formula (9) giving us the exact integral of an integrand polynomial of degree $n+m\le 2n$ can be obtained as the zeros of the $n+1^{th}$ degree Legendre polynomial. Then, given the $n+1$ grid point x_i, we can get the corresponding weight ω_i of the i point Gauss Legendre integration formula by solving the system of linear equations. Now, the interpolating polynomial ψ_n can be written as:

$$\psi_n(x)=\sum_{m=0}^{n}\left(\frac{2m+1}{2}\sum_{j=0}^{n}\omega_j p_m(x_j)\psi(x_j)\right)p_m(x)$$

$$=\sum_{j=0}^{n}\left[\omega_j\sum_{m=0}^{n}\frac{2m+1}{2}p_m(x_j)p_m(x)\right]\psi(x_j) \quad (10)$$

Using (5) and (10) we get

$$\phi_j(x) = \omega_j \sum_{m=0}^{n} (\frac{2m+1}{2} p_m(x_j) p_m(x)),\ j = 0,...,n \tag{11}$$

Substituting ψ_n into Eq. (1) and collocating at the points x_i, we obtain:

$$\psi(x_i) - \sum_{j=0}^{n} \psi(x_j) \int_{-1}^{1} K(x_i, y)\phi_j(y)dy = f(x_i), i = 0,...,n \tag{12}$$

To simplify the presentation let us define

$$a_{i,j} = \int_{-1}^{1} K(x_i, y)\phi_j(y)dy \tag{13}$$

Then $a(n+1)\times(n+1)$ linear system is obtained:

$$(Id - A)\psi = F \tag{14}$$

Where $A = (a_{i,j})_{(i,j)} \in [0,n]^2$ is square matrix, $\psi = (\psi(x_0),...,\psi(x_n))^t$ and $F = (f(x_0),..., f(x_n))^t$, where t indicate the transpose. Obviously,the system (14) has a unique solution if the determinant of the matrix $Id - A$ is nonzero, which also depends on the choice of collocation point. Substituting (11) into (13) we obtain

$$a_{i,j} = \omega_j \sum_{k=0}^{n} [\frac{2k+1}{2} p_k(x_j) u_k(x_i)] \tag{15}$$

where $u_k(x_i), (i,k) \in [[0,n]]^2$ are defined

$$u_k(x_i) = \int_{-1}^{1} e^{|x_i - y|} p_k(y)dy$$

Chebyshev Collocation Method

Like Legendre methods, here we will use the Chebyshev polynomials T_n of the first kind. The polynomial T_{n+1} has $n+1$ zeros in the interval [-1,1],which are located at the points

$$x_k = \cos(\frac{2k+1}{2n+2}\pi), k \in [[0,n]] \tag{16}$$

The Chebyshev polynomials of the first kind of degree n, T_n, satisfy discrete orthogonality relationships on the grid of the $(n+1)$ zeros of T_{n+1} (which are referred to as the Chebyshev nodes):

$$\sum_{k=0}^{n} T_i(x_k)T_j(x_k) = \begin{cases} 0 & ,i \neq j. \\ n+1 & ,i = j = 0. \\ \frac{n+1}{2} & ,i = j \neq 0. \end{cases} \tag{17}$$

For an arbitrary interval $[a,b]$, we can find a mapping that transform $[a,b]$ into [-1,+1]:

$$y_k = \frac{b-a}{2} x_k + \frac{b+a}{2} = \frac{b-a}{2} \cos(\frac{2k+1}{2n+2}\pi) + \frac{b+a}{2},$$

$$k \in [[0,n]]$$

and the Chebyshev nodes defined by equation (16) are actually zeros of this Chebyshev polynomial. Based on the discrete orthogonality relationships of the Chebyshev polynomials, various methods of solving linear and nonlinear ordinary differential equations (The solution of linear ordinary differential systems ,with polynomial coefficients, can be approximated by a finite polynomial or a finite Chebyshev series. The computation can be performed so that the solution satisfies exactly a perturbed differential system,the perturbations being computed multiples of one or more Chebyshev polynomials) and integral differential equations,see [2] were devised at about the same time and were found to have considerable advantage over finite-differences methods.Since then,these methods have become standard[15]. They rely on expanding out the unknown function in a large series of Chebyshev polynomials, truncating this series, substituting the approximation in the actual equation,and determining equations for the coefficients. In our approach we follow closely the procedures like Legendre method. Let us say,that similar procedures can be applied for a second grid given by the extrema's of T_n as nodes. It is important to stress that our goal is not to approximate a function f on the interval [-1,1],but rather to approximate the values of the function f corresponding to a given discrete set of points like those given in equation (16). Here, let $(T_0, T_1, T_2,..., T_n)$ the interpolating elements. The equation (7) becomes

$$\psi_n(x) = \sum_{j=0}^{n'} \beta_j T_j(x) \tag{18}$$

Where the prime indicates that the first term is to be halved (which is convenient for obtaining a simple formula for all the coefficients β_j).The function ψ_n interpolates ψ at the $n+1$ Chebyshev nodes, we have at these nodes $\psi(x_k) = \psi_n(x_k)$.Hence, using the discrete orthogonality relation (17) we get

$$\beta_j = \frac{2}{n+1} \sum_{k=0}^{n} \psi(x_k)T_j(x_k),\ j = 0,1,...,n \tag{19}$$

$$\begin{aligned} \psi_n(x) &= \sum_{j=0}^{n'} \beta_j T_j(x) \\ &= \sum_{j=0}^{n'} \frac{2}{n+1} \sum_{k=0}^{n} \psi(x_k)T_j(x_k)T_j(x) \\ &= \sum_{k=0}^{n} \frac{2}{n+1} (\sum_{j=0}^{n'} T_j(x_k)T_j(x))\psi(x_k) \end{aligned} \tag{20}$$

Using (5) and (20) we get:

$$\phi_k(x) = \frac{2}{n+1} \sum_{j=0}^{n'} T_j(x_k)T_j(x) \tag{21}$$

Now,the same system like (14) is obtained with

$$a_{i,j} = \frac{2}{n+1} \sum_{j=0}^{n'} v_k(x_i)T_k(x_j)$$

Where $v_k(x_i), (i,k) \in [[0,n]]^2$ are defined

$$v_k(x_i) = \int_{-1}^{1} e^{|x_i - y|} T_k(y)dy$$

The constants $v_k(x_i), (i,k) \in [0,n]^2$,can be evaluated from the recurrence relation:

$$(1 + \frac{1}{m+1})v_{m+1}(x_i) - 2x_i v_m(x_i) + (1 - \frac{1}{m-1})v_{m-1}(x_i)$$

=

$$\frac{2}{1-m^2}((1-x_i)e^{|1-x_i|}-(-1)^m(1+x_i)e^{|1+x_i|})-$$

$$6\frac{(1-(-1)^m)}{(m^2-1)(m^2-4)}$$

Numerical Experimentation

We confirm our theoretical discussion with numerical example in order to achieve the validity, the accuracy. The computations, associated with the following example,is performed by MATLAB 7.

Example: Here, we will apply the technique presented in previous section to a linear integral equation,in order to show that the method presented can be applied.We consider the equation (12) with :

$$f(x)=x^3-(6-2e)e^x, K(x,y)=e^{|x+y|}, \delta\tau=1$$

The suggested method will be considered with $n=4$,and the approximate solution $\phi(x)$ can be written in the following way

$$\phi_4(x)=\sum_{j=1}^{4}\beta_j P_j(x) \tag{22}$$

Using the same technique presented in previous section and using Equation (12) we obtain

$$\sum_{j=0}^{4}\beta_j P_j(x_i)-(x_i^3-(6-2e)e^{x_i}-\frac{h}{2}-(F(y_0)+ \tag{23}$$

$$F(y_m)+2\sum_{k=1}^{m-1}F(y_k))=0, j=0,1,2,3,4$$

Where $F(y)=e^{(y+x_i)}\sum_{j=0}^{4}\beta_j P_j(y)$ and the nodes $y_{(i+1)}=y_i+h, i=0,1,...,n, y_0=0$ and $h=\frac{1}{n}$, equation (23) represents linear system of 5 algebraic equations in the coefficients $\beta_j, j=0,...,4$ which will be solved by the conjugate gradient method and we get the following coefficients:

$$\beta_0=-0.0048, \beta_1=0.5955, \beta_2=-0.0015,$$

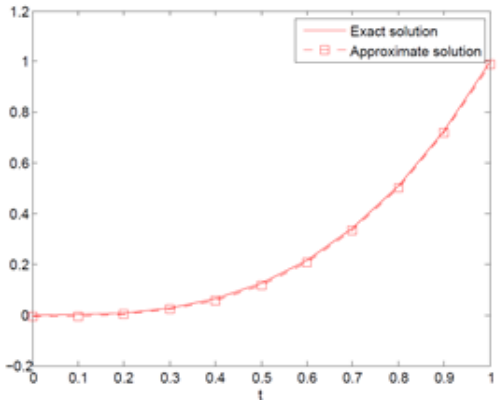
$$\beta_3=0.3998, \beta_4=-0.0001$$

Hence,the approximate solution of equation(23) is as follows:

$$\phi(x)=-0.0048P_0(x)+0.5955P_1(x)-$$

$$0.0015P_2(x)+0.3998P_3(x)-0.0001P_4(x).$$

Corresponding to exact solution $\phi(x)=x^3$.



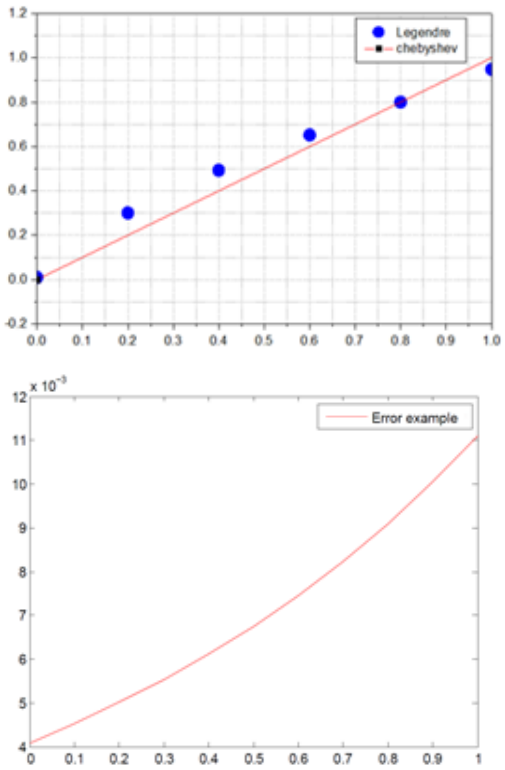


Figure 1. Error between exact solution and the present method with n = 4.

In theis example,we have considered only $n=4$ terms in the expansion of the solution using Legendre polynomials,the Figure 1 gives the behavior of true (exact) and the approximate solution and the behavior of the error between them,we notice that the technique used is much more pertinent and can be considered as a profitable method to solve the linear integral equations.

Conclusion

We solved Volterra-Fredholm integral equations by using Legendre and Chebyshev collocation methods. The properties of Chebyshev or Legendre polynomials are used to reduce the system of Fredholm integral equations to a system of nonlinear algebraic equations.The method presented in this paper based on the Legendre and Chebyshev polynomials is suggested to find the numerical solution which will be compared to the analytic solution.The iterative method conjugate gradient method and Newton's method are used to solve the linear and nonlinear system. Analyzing the numerical solution and the exact solution declare that the technique used is very effective and convenient. The approach used is tested with example to show that the accuracy improves with increasing n. Moreover,using the obtained numerical solution,we can affirm that the proposed method gives the solution in an great accordance with the analytic solution. In addition,one can investigate other type of a nonlinear Fredholm integro differential equation with singular kernel.This method may be applied to solve Volterra Fredholm integral equations with Other type of singular Kernels can be investigate using the same method.

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