



Homotopy Perturbation Method for Solving Proposal Fuzzy Integral Classes with Analytic Approach

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ABSTRACT

In this paper, the application of homotopy perturbation method (HPM) is applied to solve nonlinear fuzzy volterra integral classes equation. Comparison are made between the exact solution and solution of homotopy perturbation method, also we proved the convergence of the solution for the nonlinear fuzzy volterra integral classes equation with high computational and complexity to find the solution by using analytical method, so we describable this solution by using Homotopy perturbation method, in Banach fixed point theory for existence and uniqueness. Numerical examples are given to represent the efficiency and the accuracy of the method and all of the results reveal that the homotopy perturbation method is very effective and simple to solve our problem. Finally using the MAPLE program version (18).

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1.Introduction

In the last two decades with the rapid development of nonlinear science, there has appeared ever-increasing interest of physicists and engineers in the analytical techniques for nonlinear problems. It is well known, that perturbation methods provide the most versatile tools available in nonlinear analysis of engineering problems. The perturbation methods, like other nonlinear analytical techniques, have their own limitations. At first, almost all perturbation methods are based on the assumption that a small parameter must exist in the equation. This so-called small parameter assumption greatly restricts applications of perturbation techniques. As is well known, an overwhelming majority of nonlinear problems have no small parameters at all. Secondly, the determination of small parameters seems to be a special art requiring special techniques. An appropriate choice of small parameters leads to the ideal results, but an unsuitable choice may create serious problems. Furthermore, the approximate solutions solved by perturbation methods are valid, in most cases, only for the small values of the parameters. It is obvious that all these limitations come from the small parameter assumption. These facts have motivated to suggest alternate techniques, such as variational iteration [14], decomposition [5,7], expunction [3,9], variation of parameters [10,11] and iterative [12]. In order to overcome these drawbacks, combining the standard homotopy and perturbation method, which is called the homotopy perturbation, modifies the homotopy method.

Many problems in natural and engineering sciences are modeled by partial differential equations (PDEs). These equations arise in a number of scientific models such as the propagation of shallow water waves, long wave and chemical reaction-diffusion models [17]. A substantial amount of work has been invested for solving such models. Several techniques including the method of characteristic, Riemann invariants, combination of waveform relaxation and multi-grid,

periodic multi-grid wave form, variational iteration, homotopy perturbation and Adomian's decomposition [18,21] have been used for the solutions of such problems. Most of these techniques encounter the inbuilt deficiencies and involve huge computational work. He [23,24] developed the homotopy perturbation method for solving linear, nonlinear, initial and boundary value problems by merging two techniques, the standard homotopy and the perturbation technique. The homotopy perturbation method was formulated by taking the full advantage of the standard homotopy and perturbation methods and has been applied to a wide class of functional equations. The basic motivation of the present paper is the implementation of his reliable technique for solving PDEs. In particular the proposed homotopy perturbation method (HPM) is tested on Helmholtz, Fisher's, Boussinesq, singular fourth-order partial differential equations, systems of partial differential equations and higher-dimensional initial boundary value problems. The proposed iterative scheme finds the solution without any discretization, linearization or restrictive assumptions and is free from round off errors. The HPM gives the solution in the form of a convergent series with easily computable components. Unlike the method of separation of variables which requires both initial and boundary conditions, the HPM gives the solution by using the initial conditions only. The fact that the proposed HPM solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method.

2.Basic concepts

Basic definitions of fuzzy number are given in [1,2,10,15,17,20] as follows:

Definition 2.1. Fuzzy number. A fuzzy number is a map $u: R \rightarrow [a, b]$, which satisfying

(1) u is upper semi-continuous function,

- (2) $u(x) = 0$ outside some interval $[a, d]$
 (3) There are real numbers b, c such $a \leq b \leq c \leq d$
 i) $u(x)$ is a monotonic increasing function on $[a, b]$
 ii) $u(x)$ is a monotonic decreasing function on $[c, d]$
 iii) $u(x) = 1$ for all $x \in [b, c]$

The set of all fuzzy numbers (as given by Definition 13) is denoted by E^1 and is a convex cone. An alternative definition for parameter from of a fuzzy number is given by Kaleva [].

Definition 2.2. A fuzzy number \tilde{u} in parametric form is a pair (\underline{u}, \bar{u}) of function $\underline{u}(\alpha), \bar{u}(\alpha)$, $0 \leq \alpha \leq 1$, which satisfies the following requirements:

- i) $\underline{u}(\alpha)$ is a bounded left continuous non-decreasing function over $[0, 1]$
 ii) $\bar{u}(\alpha)$ is a bounded left continuous non-increasing function over $[0, 1]$
 iii) $\underline{u}(\alpha) \leq \bar{u}(\alpha)$, $0 \leq \alpha \leq 1$

Definition 2.3. For arbitrary fuzzy $u = (\underline{u}(\alpha), \bar{u}(\alpha))$, $v = (\underline{v}(\alpha), \bar{v}(\alpha))$, $0 \leq \alpha \leq 1$ and scalar k , we define addition, subtraction, scalar product by k and multiplication are respectively as following:

$$1 - \text{addition} : (\underline{u} + \underline{v})(\alpha) = (\underline{u}(\alpha) + \underline{v}(\alpha)), \quad (\overline{u + v})(\alpha) = (\bar{u}(\alpha) + \bar{v}(\alpha)),$$

$$2 - \text{subtraction} : (\underline{u} - \underline{v})(\alpha) = (\underline{u}(\alpha) - \underline{v}(\alpha)), \quad (\overline{u - v})(\alpha) = (\bar{u}(\alpha) - \bar{v}(\alpha)),$$

3 - scalar product :

$$k\tilde{u} = \begin{cases} (k\underline{u}(\alpha), k\bar{u}(\alpha)), & k \geq 0 \\ (k\underline{u}(\alpha), k\bar{u}(\alpha)), & k < 0 \end{cases} \quad (1)$$

4- multiplication:

$$\tilde{u} \cdot \tilde{v} = \begin{cases} \underline{uv}(\alpha) = \max\{\underline{u}(\alpha)\underline{v}(\alpha), \underline{u}(\alpha)\bar{v}(\alpha), \bar{u}(\alpha)\underline{v}(\alpha), \bar{u}(\alpha)\bar{v}(\alpha)\} \\ \overline{uv}(\alpha) = \min\{\underline{u}(\alpha)\underline{v}(\alpha), \underline{u}(\alpha)\bar{v}(\alpha), \bar{u}(\alpha)\underline{v}(\alpha), \bar{u}(\alpha)\bar{v}(\alpha)\} \end{cases} \quad (2)$$

Defined 2.4. For arbitrary Fuzzy numbers $\tilde{u}, \tilde{v} \in E^1$

$$D(\tilde{u}, \tilde{v}) = \max\left\{\sup_{\alpha \in [0,1]} |\underline{u}(\alpha) - \underline{v}(\alpha)|, \sup_{\alpha \in [0,1]} |\bar{u}(\alpha) - \bar{v}(\alpha)|\right\} \quad (3)$$

In the distance between The \tilde{u} and \tilde{v} , it is prove (E^1, D) is a complete metric space.

Definition 2.5. The integral of a fuzzy function was define in [14] by using the Riemann integral concept. Let $f: [a, b] \rightarrow E^1$.

For Fuzzy function, for each partition $p = \{t_0, \dots, t_n\}$ of $[a, b]$ and for arbitrary $\xi_i \in [t_{i-1}, t_i]$, $1 \leq i \leq n$, suppose

$$R_p = \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}) \quad (4)$$

$$\Delta := \max\{t_i - t_{i-1}, 1 \leq i \leq n\}.$$

The define integral of $f(t)$ over $[a, b]$ is

$$\int_a^b f(t) dt = \lim_{\Delta \rightarrow 0} R_p \quad (5)$$

If the fuzzy function $f(t)$ is continuous in metric D , its definite the integral exists and also

$$\left(\int_a^b \underline{f}(t; \alpha) dt\right) = \int_a^b \underline{f}(t; \alpha) dt \quad \cdot \quad \left(\int_a^b \overline{f}(t; \alpha) dt\right) = \int_a^b \bar{f}(t; \alpha) dt \quad (6)$$

It should be noted that the fuzzy integral can be also defined using the Lebesgue - type approach. However, if $f(t)$ is continuous, both approaches yield the same value. More details about the properties of the fuzzy integral

Proposition 2.1. A function $F, G: I \rightarrow E^n$ be integrable and $\varphi \in R, \in$ then

$$1- \int (F + G) = \int F + \int G$$

$$2- \int \varphi F = \varphi \int F$$

3- $D(F, G)$ IS integrable

$$4 - D(\int F, \int G) \leq \int D(F, G)$$

Proposition 2.2. For any

$p, q, r, s \in E^n$ and $\varphi \in$

R , then the following hold

$i-(E^n, D)$ is a complete metric space

ii- $D(\varphi p, \varphi q) = |\varphi|D(p, q)$

iii- $D(p + r, q + s) = D(p, q)$

iv- $D(p + q, r + s) \leq D(p, r) + D(q, s)$

Definition 2.6. A function $F: I \rightarrow E^n$ is called bounded if there exists a constant

$M > 0$ such that $D(F(x), \tilde{0}) \leq M$ for all $x \in I$

Definition 2.7. A function

$F: I \rightarrow$

E^n is said to be continuous if for arbitrary fixed $x_0 \in$

I and $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x -$

$x_0| < \delta$, than $D(F(x), F(x_0)) < \varepsilon$ for each $x \in I$

3. Novel formula fuzzy nonlinear volterra integral equation

The fuzzy nonlinear integral equation with integral kernel which is discussed in this work is the fuzzy nonlinear Volterra integral equation (FNVE2) as follows:

$$\tilde{u}(x) = \tilde{f}(x) + \lambda \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}(t)))) dt \tag{7}$$

where $\lambda \geq 0$, $\tilde{f}(x)$ a fuzzy function of $x : a \leq x \leq b$, $k(x, t, \tilde{k}_1(\tilde{F}(t, \tilde{u}(t))))$ is analytic functions on $[a, b]$ and $\tilde{F}(t, \tilde{u}(t))$

are nonlinear function on $[a, b]$. For solving in parametric form of Eq. (7), consider $(\underline{f}(x, \alpha), \overline{f}(x, \alpha))$ and $(\underline{u}(x, \alpha), \overline{u}(x, \alpha))$

and, $0 \leq \alpha \leq 1$ and $t, s \in [a, b]$ are parametric form of $\tilde{f}(x)$ and $\tilde{u}(x)$, respectively. then, parametric form of Eq. (7) is as follows

$$\underline{u}(x, \alpha) = \underline{f}(x, \alpha) + \int_a^x k(x, t, k_1(t, F(t, u(t, \alpha)))) dt \tag{8}$$

$$\overline{u}(x, \alpha) = \overline{f}(x, \alpha) + \int_a^x k(x, t, k_1(t, F(t, u(t, \alpha)))) dt$$

For each $0 \leq \alpha \leq 1$ and $a \leq x \leq b$. We can see that Eq. (7) convert to a system of nonlinear Volterra integral equations in crisp case for each $0 \leq \alpha \leq 1$ and $a \leq t \leq b$. Now, we explain analysis perturbation methods as approximating solution of this system of nonlinear integral equations in crisp case. then, we find approximate solutions for $\tilde{u}(x)$, $a \leq x \leq b$

We write the system (7) we obtain

$$\underline{u}(x, \alpha) = \underline{f}(x, \alpha) + \int_a^c k(x, t, k_1(t, F(t, u(t, \alpha)))) + \int_c^x k(x, t, k_1(t, F(t, u(t, \alpha)))) \tag{9}$$

$$\overline{u}(x, \alpha) = \overline{f}(x, \alpha) + \int_a^c k(x, t, k_1(t, F(t, u(t, \alpha)))) + \int_c^x k(x, t, k_1(t, F(t, u(t, \alpha))))$$

where $0 \leq t \leq c, c \leq t \leq x, 0 \leq \alpha \leq 1$

Now we will find the parameter for the Eq(7), as follows

$$\underline{k}(x, t, k_1(t, F(t, u(t, \alpha)))) = \begin{cases} \underline{k}(x, t, k_1(t, F(t, u(t, \alpha)))) , & \underline{k}(x, t, k_1(t, F(t, u(t, \alpha)))) \geq 0 \\ \overline{k}(x, t, k_1(t, F(t, u(t, \alpha)))) , & \underline{k}(x, t, k_1(t, F(t, u(t, \alpha)))) < 0 \end{cases}$$

$$\underline{k}_1(t, F(t, u(t, \alpha))) = \begin{cases} \underline{k}_1(t, \underline{F}(t, \underline{u}(t, \alpha)), \underline{k}_1(t, F(t, u(t, \alpha)))) \geq 0 \\ \overline{k}_1(t, \overline{F}(t, \overline{u}(t, \alpha)), \underline{k}_1(t, F(t, u(t, \alpha)))) < 0 \end{cases}$$

$$\overline{k}_1(t, F(t, u(t, \alpha))) = \begin{cases} \overline{k}_1(\overline{F}(t, \overline{u}(t, \alpha)), \overline{k}_1(t, F(t, u(t, \alpha)))) \geq 0 \\ \underline{k}_1(\underline{F}(t, \underline{u}(t, \alpha)), \overline{k}_1(t, F(t, u(t, \alpha)))) < 0 \end{cases}$$

$$\overline{k}(x, t, k_1(t, F(t, u(t, \alpha)))) = \begin{cases} \overline{k}(x, t, k_1(t, F(t, u(t, \alpha)))) , \overline{k}(x, t, k_1(t, F(t, u(t, \alpha)))) \geq 0 \\ \underline{k}(x, t, k_1(t, F(t, u(t, \alpha)))) , \overline{k}(x, t, k_1(t, F(t, u(t, \alpha)))) < 0 \end{cases}$$

$$\overline{k_1(t, F(t, u(t, \alpha)))} = \begin{cases} \overline{k_1(\overline{F}(t, \overline{u}(t, \alpha)), k_1(t, F(t, u(t, \alpha)))} \geq 0 \\ \overline{k_1(t, \underline{F}(t, \underline{u}(t, \alpha)), k_1(t, F(t, u(t, \alpha)))} < 0 \end{cases}$$

$$k_1(t, F(t, u(t, \alpha))) = \begin{cases} \underline{k_1(t, \underline{F}(t, \underline{u}(t, \alpha)), k_1(t, F(t, u(t, \alpha)))} \geq 0 \\ \underline{k_1(t, \overline{F}(t, \overline{u}(t, \alpha)), k_1(t, F(t, u(t, \alpha)))} < 0 \end{cases} \tag{10}$$

than

$$\underline{u}(x, \alpha) = \underline{f}(x, \alpha) + \int_a^d k(x, t, \underline{k_1}(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt + \int_a^c k(x, t, \overline{k_1}(\overline{F}(t, \overline{u}(t, \alpha)))) dt + \int_c^e k(x, t, \underline{k_1}(t, \overline{F}(t, \overline{u}(t, \alpha)))) dt + \int_e^x k(x, t, \overline{k_1}(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt$$

$$\overline{u}(x, \alpha) = \overline{f}(x, \alpha) + \int_a^d k(x, t, \overline{k_1}(\overline{F}(t, \overline{u}(t, \alpha)))) dt + \int_a^c k(x, t, k(x, t, \underline{k_1}(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt + \int_c^e k(x, t, \overline{k_1}(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt + \int_e^x k(x, t, \underline{k_1}(t, \overline{F}(t, \overline{u}(t, \alpha)))) dt \tag{11}$$

where $0 \leq t \leq d$, $d \leq t \leq c$, $c \leq t \leq e$, $e \leq t \leq x$, $0 \leq \alpha \leq 1$

3.1 Homotopy Perturbation Method

Consider the fuzzy nonlinear volterra integral equation of the second kind

$$\tilde{u}(x) = \tilde{f}(x) + \lambda \int_a^x k(x, t, \tilde{k_1}(t, \tilde{F}(t, \tilde{u}(t)))) dt$$

Let

$$L(\underline{u}) = \underline{u}(x, \alpha) - \underline{f}(x, \alpha) - \int_a^d k(x, t, \underline{k_1}(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt - \int_a^c k(x, t, \overline{k_1}(\overline{F}(t, \overline{u}(t, \alpha)))) dt - \int_c^e k(x, t, \underline{k_1}(t, \overline{F}(t, \overline{u}(t, \alpha)))) dt - \int_e^x k(x, t, \overline{k_1}(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt = 0$$

$$L(\overline{u}) = \overline{u}(x, \alpha) - \overline{f}(x, \alpha) - \int_a^d k(x, t, \overline{k_1}(\overline{F}(t, \overline{u}(t, \alpha)))) dt - \int_a^c k(x, t, k(x, t, \underline{k_1}(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt - \int_c^e k(x, t, \overline{k_1}(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt - \int_e^x k(x, t, \underline{k_1}(t, \overline{F}(t, \overline{u}(t, \alpha)))) dt = 0 \tag{12}$$

then we defined the homotopy

$$H(\underline{u}, p), H(\overline{u}, p) \text{ by}$$

$$\begin{cases} H(\underline{u}, 0) = F(\underline{u}), & H(\underline{u}, 1) = L(\underline{u}) \\ H(\overline{u}, 0) = F(\overline{u}), & H(\overline{u}, 1) = L(\overline{u}) \end{cases}$$

where $F(\underline{u}), F(\overline{u})$ are functional operators with solution say $\underline{u}_0, \overline{u}_0$ which can be obtained easily. We choose a convex homotopy

$$\begin{aligned} H(\underline{u}, p) &= (1 - p) F(\underline{u}) + pL(\underline{u}) = 0 \\ H(\overline{u}, p) &= (1 - p) F(\overline{u}) + pL(\overline{u}) = 0 \end{aligned} \tag{13}$$

and continuously trace an implicitly defined curve from a starting points $H(\underline{u}_0, 0), H(\overline{u}_0, 0)$ to a solution $H(\underline{u}, 1), H(\overline{u}, 1)$. The embedding parameter p monotonically increases from 0 to 1 as the trivial problem $F(\underline{u}) = 0, F(\overline{u}) = 0$ continuously deformed to the original problem $L(\underline{u}) = 0$,

$L(\overline{u}) = 0$. The parameter can be considered as an expanding parameter. In fact HPM uses the homotopy parameter p as an expanding parameter to obtain

$$\begin{aligned} \underline{u} &= \underline{u}_0 + p\underline{u}_1 + p^2\underline{u}_2 + \dots \dots \\ \overline{u} &= \overline{u}_0 + p\overline{u}_1 + p^2\overline{u}_2 + \dots \dots \end{aligned} \tag{14}$$

when $p \rightarrow 1$, (7) corresponding to (6) and gives an approximate as follows

$$\begin{aligned} \underline{u} &= \lim_{p \rightarrow 1} \underline{u}_2 + \dots \dots \\ \overline{u} &= \lim_{p \rightarrow 1} \overline{u}_2 + \dots \dots \end{aligned} \tag{15}$$

The series (8) converges in most cases, and the rate of convergence depends on $L(\underline{u}), L(\overline{u})$

Now we applied the HPM for solving system for fuzzt volterra nonlinear integral equation

$$F(\underline{u}) = \underline{u}(x, \alpha) - \underline{f}(x, \alpha)$$

$$F(\overline{u}) = \overline{u}(x, \alpha) - \overline{f}(x, \alpha)$$

$$H(\underline{u}, p) =$$

$$\underline{u}(x, \alpha) - \underline{f}(x, \alpha) - \int_a^d k(x, t, \underline{k_1}(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt - \int_a^c k(x, t, \overline{k_1}(\overline{F}(t, \overline{u}(t, \alpha)))) dt - \int_c^e k(x, t, \underline{k_1}(t, \overline{F}(t, \overline{u}(t, \alpha)))) dt - \int_e^x k(x, t, \overline{k_1}(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt = 0$$

$$H(\overline{u}, p) = \overline{u}(x, \alpha) - \overline{f}(x, \alpha) - \int_a^d k(x, t, \overline{k_1}(\overline{F}(t, \overline{u}(t, \alpha)))) dt - \int_a^c k(x, t, k(x, t, \underline{k_1}(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt - \int_c^e k(x, t, \overline{k_1}(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt - \int_e^x k(x, t, \underline{k_1}(t, \overline{F}(t, \overline{u}(t, \alpha)))) dt = 0$$

and equation the term with identical power of p , we obtain

$$p^0: \underline{u}_0(x, \alpha) - \underline{f}(x, \alpha) = 0 \Rightarrow \underline{u}_0(x, \alpha) = \underline{f}(x, \alpha)$$

$$p^0: \bar{u}_0(x, \alpha) - \bar{f}(x, \alpha) = 0 \Rightarrow \bar{u}_0(x, \alpha) = \bar{f}(x, \alpha)$$

$$p^1: \underline{u}_1(x, \alpha)$$

$$- \int_a^d k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}_0(t, \alpha))) dt - \int_a^c k(x, t, \bar{k}_1(\bar{F}(t, \bar{u}_0(t, \alpha)))) dt - \int_c^e k(x, t, \underline{k}_1(t, \bar{F}(t, \bar{u}_0(t, \alpha)))) dt - \int_e^x k(x, t, \bar{k}_1(t, \underline{F}(t, \underline{u}_0(t, \alpha)))) dt = 0$$

\Rightarrow

$$\Rightarrow \underline{u}_1(x, \alpha) = \int_a^d k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}_0(t, \alpha))) dt + \int_a^c k(x, t, \bar{k}_1(\bar{F}(t, \bar{u}_0(t, \alpha)))) dt + \int_c^e k(x, t, \underline{k}_1(t, \bar{F}(t, \bar{u}_0(t, \alpha)))) dt + \int_e^x k(x, t, \bar{k}_1(t, \underline{F}(t, \underline{u}_0(t, \alpha)))) dt$$

$$p^1: \bar{u}_1(x, \alpha)$$

$$- \int_a^d k(x, t, \bar{k}_1(\bar{F}(t, \bar{u}_0(t, \alpha)))) dt - \int_a^c k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}_0(t, \alpha))) dt - \int_c^e k(x, t, \bar{k}_1(t, \underline{F}(t, \underline{u}_0(t, \alpha)))) dt - \int_e^x k(x, t, \underline{k}_1(t, \bar{F}(t, \bar{u}_0(t, \alpha)))) dt = 0 \Rightarrow \bar{u}_1(x, \alpha) =$$

$$\int_a^d k(x, t, \bar{k}_1(\bar{F}(t, \bar{u}_0(t, \alpha)))) dt + \int_a^c k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}_0(t, \alpha))) dt + \int_c^e k(x, t, \bar{k}_1(t, \underline{F}(t, \underline{u}_0(t, \alpha)))) dt + \int_e^x k(x, t, \underline{k}_1(t, \bar{F}(t, \bar{u}_0(t, \alpha)))) dt$$

Now we will write the general formula for HAM to solve our system

$$\underline{u}_0(x, \alpha) = \underline{f}(x, \alpha)$$

$$\underline{u}_{n+1}(x, \alpha) = \int_a^d k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}_n(t, \alpha))) dt + \int_a^c k(x, t, \bar{k}_1(\bar{F}(t, \bar{u}_n(t, \alpha)))) dt + \int_c^e k(x, t, \underline{k}_1(t, \bar{F}(t, \bar{u}_n(t, \alpha)))) dt + \int_e^x k(x, t, \bar{k}_1(t, \underline{F}(t, \underline{u}_n(t, \alpha)))) dt$$

$n=1, 2, \dots$

$$\bar{u}_0(x, \alpha) = \bar{f}(x, \alpha)$$

$$\bar{u}_{n+1}(x, \alpha) = \int_a^d k(x, t, \bar{k}_1(\bar{F}(t, \bar{u}_n(t, \alpha)))) dt + \int_a^c k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}_n(t, \alpha))) dt + \int_c^e k(x, t, \bar{k}_1(t, \underline{F}(t, \underline{u}_n(t, \alpha)))) dt + \int_e^x k(x, t, \underline{k}_1(t, \bar{F}(t, \bar{u}_n(t, \alpha)))) dt$$

$n = 1, 2, \dots$

Then if $k(x, t)$ is non negative and non positive we get

$$\underline{u}_0(x, \alpha) = \underline{f}(x, \alpha)$$

$$\underline{u}_{n+1}(x, \alpha) = \int_a^d k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}_n(t, \alpha))) dt + \int_a^c k(x, t, \bar{k}_1(\bar{F}(t, \bar{u}_n(t, \alpha)))) dt + \int_c^e k(x, t, \underline{k}_1(t, \bar{F}(t, \bar{u}_n(t, \alpha)))) dt + \int_e^x k(x, t, \bar{k}_1(t, \underline{F}(t, \underline{u}_n(t, \alpha)))) dt$$

$$\bar{u}_0(x, \alpha) = \bar{f}(x, \alpha)$$

$$\bar{u}_{n+1}(x, \alpha) = \int_a^d k(x, t, \bar{k}_1(\bar{F}(t, \bar{u}_n(t, \alpha)))) dt + \int_a^c k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}_n(t, \alpha))) dt + \int_c^e k(x, t, \bar{k}_1(t, \underline{F}(t, \underline{u}_n(t, \alpha)))) dt + \int_e^x k(x, t, \underline{k}_1(t, \bar{F}(t, \bar{u}_n(t, \alpha)))) dt$$

$n = 1, 2, \dots$

Theorem 1: (. **Existence and uniqueness**)

The following conditions are satisfy

1- $f: [a, b] \rightarrow E^n$ is a continuous and bounded

2- $K: [a, b] \times [a, b] \times E^n \rightarrow E^n$ is a continuous function

3- if $u, u^*: [a, b] \rightarrow E^n$ are continuous, then the lipschitz condition

$$D(k(x_1, t_1, k_1(t_1, H_1(t))), k(x_2, t_2, k_1(t_2, H_2(t))) \leq L(D(x_1, x_2) + D(t_1, t_2) + D(k_1(t_1, H_1(t)), k_1(t_2, H_2(t))))$$

$$4- D(F(t_1, \tilde{u}(t)), F(t_2, \tilde{u}^*(t))) \leq L_1(D(t_1, t_2) + D(\tilde{u}(x), \tilde{u}^*(x)))$$

$$5- D(k_1(t_1, H_1(t)), k_1(t_2, H_2(t))) \leq L_2(D(t_1, t_2) + L_1(D(t_1, t_2) + D(\tilde{u}(x), \tilde{u}^*(x))))$$

We take an initial guess $\bar{u}_0(x) = \bar{f}(x)$

Then there exist a unique fuzzy solution of $u(x)$ in Eq(19) and the successive iteration

$$\omega_0(x) = f(x)$$

$$\omega_{n+1}(x) = f(x) + \sum_{i=1}^{n+1} \int_a^x k((x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt, n \geq 0 \tag{14}$$

and $u(x)$ is convergent uniformly on $[a, b]$ where

$$u_0(x) = f(x)$$

$$u_n(x) = \int_a^x k((x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt, n \geq 1 \tag{15}$$

Lemma 1: Consider $f(x)$ is a bounded on $[a, b]$ then we will prove the $u_n(x)$ is a bounded and $u_n(x)$ is continuous, where $n \geq 0$

Proof: we will proved the part 1 from lemma 1 $u_0(x) = f(x)$ is a bounded. Now we proved $u_{n-1}(x)$ is a bounded. and proposition (2.1), we have

$$D(\tilde{u}_n(x), \tilde{0}) = D(\int_a^x k((x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt, \tilde{0})$$

$$\leq \int_a^x D(k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt, \tilde{0})$$

$$\leq (b - a)L(D(x, \tilde{0}) + D(t, \tilde{0}) + D(t, F(t, \tilde{u}_{n-1}(t))), \tilde{0})$$

$$\leq (b - a)L(D(x, \tilde{0}) + D(t, \tilde{0}) + L_2[D(t, \tilde{0}) + L_1(D(t, \tilde{0}) + D(\tilde{u}_{n-1}(t), \tilde{0}))])$$

$$\leq (b - a)[LsupD(x, \tilde{0}) + L supD(t, \tilde{0}) + L_2 supD(t, \tilde{0}) + L_1 supD(t, \tilde{0}) + L_1 supD((\tilde{u}_{n-1}(t)), \tilde{0})]$$

$$\leq (b - a)[LsupD(x, \tilde{0}) + (L + L_1 + L_2) supD(t, \tilde{0}) + L_1 supD((\tilde{u}_{n-1}(t)), \tilde{0})]$$

$\tilde{u}_n(x)$ is a bounded , $\tilde{0}$ is the zero function

Now we will proof part 2 from lemma 2.1 , $\tilde{u}_n(x)$ is a continuous by using propostion (2.1)and (2.2), we have ,

$$a \leq x \leq x_1 \leq b$$

$$D(\tilde{u}_n(x), \tilde{u}_n(x_1)) = D(\int_a^x k((x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt, \int_a^{x_1} k((x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt)$$

$$= D(\int_a^x k((x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt, \int_a^x k((x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt + \int_x^{x_1} k((x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt)$$

$$\leq D(\int_a^x k((x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt, \int_a^x k((x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt) + D(\int_x^{x_1} k((x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt, \tilde{0})$$

$$\leq \int_a^x D(k((x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt, \int_a^x k((x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt) + \int_x^{x_1} D(k((x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t))), \tilde{0}) dt$$

$$\leq (b - a) supD(D(k((x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t))), k((x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t))))))$$

$$+ (x_1 - x)[LsupD(x, \tilde{0}) + (L + L_1 + L_2) supD(t, \tilde{0}) + L_1 supD((\tilde{u}_{n-1}(t)), \tilde{0})]$$

K is continuous , we have

$$D(\tilde{u}_n(x), \tilde{u}_n(x_1)) \rightarrow 0 \text{ as } x \rightarrow x_1$$

$u_n(x)$ is continuous on $[a, b]$

proof

we will prove that $\omega_n(x)$, $n \geq 0$ are bounded on $[a, b]$. Then $\omega_0(x) = f(x)$

is a bounded by the assumption that $\omega_{n-1}(x)$ is a bounded, from Eq(20) and propostion (2.1) we will get .

$$D(\omega_n(x), \tilde{0}) = D(f(x) + \sum_{i=1}^n \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt, \tilde{0})$$

$$= D(f(x) + \sum_{i=1}^{n-1} \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt + \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt, \tilde{0})$$

$$= D(\omega_{n-1}(x) + \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt, \tilde{0})$$

$$\leq D(\omega_{n-1}(x), \tilde{0}) + D(\int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t)))) dt, \tilde{0})$$

$$= D(\omega_{n-1}(x), \tilde{0}) + D(\omega_n(x), \tilde{0})$$

From lemma (1) we have that $\omega_n(x)$ is a bounded . and $\{\omega_n(x)\}_{n=0}^\infty$ is a sequence of bounded functions on $[a, b]$

We will prove $\omega_n(x)$ are continuous on $[a, b]$, by using Lemma (2.1) and proposition (2.1) and(2.2) for $a \leq x \leq x_1 \leq b$, we get

$$D(\omega_n(x), \omega_n(x_1)) = D((f(x) + \sum_{i=1}^n \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt$$

$$, f(x_1) + \sum_{i=1}^n \int_a^{x_1} k(x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt)$$

$$\leq D(f(x), f(x_1)) + D(\sum_{i=1}^n \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt, \sum_{i=1}^n \int_a^{x_1} k(x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt)$$

$$= D(f(x), f(x_1)) + D(\sum_{i=1}^n \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt, \sum_{i=1}^n \int_a^x k(x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt)$$

$$+ \sum_{i=1}^n \int_x^{x_1} k(x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt)$$

$$\leq D(f(x), f(x_1)) + D(\sum_{i=1}^n \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt, \sum_{i=1}^n \int_a^x k(x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt)$$

$$+ D(\sum_{i=1}^n \int_x^{x_1} k(x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt, \tilde{0})$$

$$\leq D(f(x), f(x_1)) + \int_a^x D(\sum_{i=1}^n k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt, \int_a^x D(\sum_{i=1}^n k(x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t)))) dt,$$

$$\begin{aligned}
& + \int_x^{x_1} D\left(\sum_{i=1}^n k(x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t))), \tilde{\theta}) dt\right) \\
& \leq D(f(x), f(x_1)) + (b-a) \sup D\left[\left(\sum_{i=1}^n k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t))) dt, \sum_{i=1}^n k(x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t))) dt, \right.\right. \\
& \left. \left. + (x_1 - x) \sup D\left(\sum_{i=1}^n k(x_1, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t))), \tilde{\theta}\right)\right)
\end{aligned}$$

$$D(\omega_n(x), \omega_n(x_1)) \rightarrow 0, \quad x \rightarrow x_1$$

and the sequence $\{\omega_n(x)\}_{n=0}^{\infty}$ is continuous on $[a, b]$

in this part we will prove the sequence $\{\omega_n(x)\}_{n=0}^{\infty}$ is convergent uniformly for $n \geq 0$ we get

$$\begin{aligned}
D(\omega_{n+1}(x), \omega_n(x)) & = D\left(f(x) + \sum_{i=1}^{n+1} \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t))) dt, \omega_n(x)\right) \\
& = D\left(f(x) + \sum_{i=1}^n \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{i-1}(t))) dt + \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_n(t))) dt, \omega_n(x)\right) \\
& = D\left(\omega_n(x) + \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_n(t))) dt, \omega_n(x)\right) \\
& = D\left(\int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_n(t))) dt, \tilde{\theta}\right) \\
& \leq \int_a^x D(k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_n(t))), \tilde{\theta}) dt \\
& \leq (b-a)[L \sup D(x, \tilde{\theta}) + (L + L_1 + L_2) \sup D(t, \tilde{\theta}) + L_1 \sup D((\tilde{u}_n(t)), \tilde{\theta})]
\end{aligned}$$

Now we obtain

$$\sup D(\omega_{n+1}(x), \omega_n(x)) \leq (b-a)[L \sup D(x, \tilde{\theta}) + (L + L_1 + L_2) \sup D(t, \tilde{\theta}) + L_1 \sup D((\tilde{u}_n(t)), \tilde{\theta})]$$

$$D(u_n(x), \tilde{\theta}) = D\left(\int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t))) dt, \tilde{\theta}\right)$$

$$\leq \int_a^x D(k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}_{n-1}(t))), \tilde{\theta}) dt$$

$$\leq (b-a)[L \sup D(x, \tilde{\theta}) + (L + L_1 + L_2) \sup D(t, \tilde{\theta}) + L_1 \sup D((\tilde{u}_{n-1}(t)), \tilde{\theta})]$$

$$D(u_{n-1}(x), \tilde{\theta}) \leq (b-a)[L \sup D(x, \tilde{\theta}) + (L + L_1 + L_2) \sup D(t, \tilde{\theta}) + L_1 \sup D((\tilde{u}_{n-2}(t)), \tilde{\theta})]$$

we obtain

$$D(u_n(x), \tilde{\theta}) \leq (b-a)[L \sup D(x, \tilde{\theta}) + (L + L_1 + L_2) \sup D(t, \tilde{\theta}) + L_1 \sup D((\tilde{u}_{n-1}(t)), \tilde{\theta})]$$

$$\leq (b-a)[L \sup D(x, \tilde{\theta}) + (L + L_1 + L_2) \sup D(t, \tilde{\theta}) + L_1 \sup D((\tilde{u}_{n-2}(t)), \tilde{\theta})]$$

$$\leq [(b-a)[L \sup D(x, \tilde{\theta}) + (L + L_1 + L_2) \sup D(t, \tilde{\theta})]^n \sup D(f(x), \tilde{\theta})$$

$$D(\omega_{n+1}(x), \omega_n(x)) \leq \sup D(\omega_{n+1}(x), \omega_n(x)) \leq$$

$$[(b-a)[L \sup D(x, \tilde{\theta}) + (L + L_1 + L_2) \sup D(t, \tilde{\theta})]^{n+1} \sup D(f(x), \tilde{\theta})$$

$$\begin{aligned}
& \text{the series } [(b-a)[L \sup D(x, \tilde{\theta}) + (L + L_1 + L_2) \sup D(t, \tilde{\theta})] \sup D(f(x), \tilde{\theta}) \sum_{n=0}^{\infty} [(b-a)[L \sup D(x, \tilde{\theta}) \\
& \quad + (L + L_1 + L_2) \sup D(t, \tilde{\theta})]^n
\end{aligned}$$

the series $\sum_{n=0}^{\infty} D(\omega_{n+1}(x), \omega_n(x))$ is uniformly convergent on $[a, b]$, and the uniformly convergent of the sequence $\{\omega_n(x)\}_{n=0}^{\infty}$. if denoted $u(x) = \lim_{n \rightarrow \infty} \omega_n(x)$, then $u(x)$ is satisfy all the conditions for theorem (1)

We prove the uniqueness of solution > Let $u(x)$ and $u^*(x)$ are two continuous solution of Eq(7) on $[a, b]$

$$D(\tilde{u}(x), \tilde{u}^*(x)) \geq 0$$

and

$$D(\tilde{u}(x), \tilde{u}^*(x)) = D(\tilde{u}(x) + \omega_n(x), \tilde{u}^*(x) + \omega_n(x))$$

$$\leq D(\tilde{u}(x), \omega_n(x)) + D(\tilde{u}^*(x), \omega_n(x))$$

$$D(\tilde{u}(x), \omega_n(x)) \rightarrow 0$$

$$D(\tilde{u}^*(x), \omega_n(x)) \rightarrow 0$$

when $n \rightarrow \infty$, then $D(\tilde{u}(x), \tilde{u}^*(x)) = 0$, then $\tilde{u}(x) = \tilde{u}^*(x)$

4. Numerical example

We will apply the Homotopy perturbation method to solve our problem and compared with the exact solution

Example 4.1

Consider the fuzzy nonlinear volterra integral equation

$$\tilde{u}(x) = \tilde{f}(x) + \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}(t)))) dt \quad (16)$$

Substituting eq (7) in Eq(16), we get

$$\underline{u}(x, \alpha) = \underline{f}(x, \alpha) + \int_a^c k(x, t, \underline{k}_1(t, \underline{F}(t, u(t, \alpha)))) + \int_c^x k(x, t, \overline{k}_1(t, \overline{F}(t, u(t, \alpha))))$$

$$\overline{u}(x, \alpha) = \overline{f}(x, \alpha) + \int_a^x k(x, t, \overline{k}_1(t, \overline{F}(t, u(t, \alpha)))) + \int_c^x k(x, t, \underline{k}_1(t, \underline{F}(t, u(t, \alpha))))$$

Substituting Eq(7) in (11) and find the upper and lower for Eq(16), we obtain

$$\underline{u}(x, \alpha) = \underline{f}(x, \alpha) + \int_a^d k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt + \int_d^c k(x, t, \overline{k}_1(t, \overline{F}(t, \overline{u}(t, \alpha)))) dt + \int_c^e k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt + \int_e^x k(x, t, \overline{k}_1(t, \overline{F}(t, \overline{u}(t, \alpha)))) dt$$

$$\overline{u}(x, \alpha) = \overline{f}(x, \alpha) + \int_a^d k(x, t, \overline{k}_1(t, \overline{F}(t, \overline{u}(t, \alpha)))) + \int_d^c k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}(t, \alpha)))) + \int_e^e k(x, t, \overline{k}_1(t, \overline{F}(t, \overline{u}(t, \alpha)))) dt + \int_e^x k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt$$

where $0 \leq t \leq d$, $d \leq t \leq c$, $c \leq t \leq e$, $e \leq t \leq x$, $0 \leq \alpha \leq 1$

$$\underline{f}(x, \alpha) = x^{r-0.47541e-1} x^{r^2-0.3312e-1} x^{(2-r)^2-(1/5)x^6 r^2}$$

$$\overline{f}(x, \alpha) = x^{(2-r)+0.29505e-1} x^{(2-r)^2-0.4386e-1} x^{r^2-(1/6)x^7(2-r)^2}$$

Kernal of Eq(16) is

$$k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}(t, \alpha)))) = xt^3(\underline{u}(t, \alpha))^2$$

$$k(x, t, \overline{k}_1(t, \overline{F}(t, \overline{u}(t, \alpha)))) = xt^2(\overline{u}(t, \alpha))^2$$

$$k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}(t, \alpha)))) = xt^3(\underline{u}(t, \alpha))^2$$

$$k(x, t, \overline{k}_1(t, \overline{F}(t, \overline{u}(t, \alpha)))) = xt^2(\overline{u}(t, \alpha))^2$$

$$a = 0, d = \frac{1}{4}, c = \frac{1}{2}, e = \frac{3}{4}, x = 1$$

The exact solution

$$\underline{u}(x, \alpha) = x\alpha, \quad \overline{u}(x, \alpha) = x(2 - \alpha)$$

Now we apply the HPM technique to solve the our problem

$$\underline{u}_0(x, \alpha) = \underline{f}(x, \alpha) = x^{r-0.47541e-1} x^{r^2-0.3312e-1} x^{(2-r)^2-(1/5)x^6 r^2}$$

$$\overline{u}_0(x, \alpha) = \overline{f}(x, \alpha) = x^{(2-r)+0.29505e-1} x^{(2-r)^2-0.4386e-1} x^{r^2-(1/6)x^7(2-r)^2}$$

$$\underline{u}_1(x, \alpha) = 5.820766091 * 10^{(-13)} * x^{r^4-8.669766513 * 10^{(-9)} * x^{(r-0.4754100000e-1 * r^2-0.3312000000e-1 * (2-1. * r)^2) * r^2+0.4069010417e-4 * x^{(r-0.4754100000e-1 * r^2-0.3312000000e-1 * (2-1. * r)^2)^2+0.8706749286e-5 * x^{(2-1. * r)^4-0.8879061600e-3 * x^{(2-1. * r+0.2950500000e-1 * (2-1. * r)^2-0.4386000000e-1 * r^2) * (2-1. * r)^2+0.3311360677e-1 * x^{(2-1. * r+0.2950500000e-1 * (2-1. * r)^2-0.4386000000e-1 * r^2)^2+0.2666666667e-2 * x^{r^4} * (x^{15-0.1336346101e-1}-0.4000000000e-1 * x^{(r-0.4754100000e-1 * r^2-0.3312000000e-1 * (2-1. * r)^2) * r^2 * (x^{10-0.5631351471e-1})+2.000000000 * x^{(r-0.4754100000e-1 * r^2-0.3312000000e-1 * (2-1. * r)^2)^2} * (x^5-2.373046875)$$

$$\overline{u}_1(x, \alpha) = 9.511055705 * 10^{(-14)} * x^{(2-1. * r)^4-7.224805427 * 10^{(-9)} * x^{(2-1. * r+0.2950500000e-1 * (2-1. * r)^2-0.4386000000e-1 * r^2) * (2-1. * r)^2+0.1953125000e-3 * x^{(2-1. * r+0.2950500000e-1 * (2-1. * r)^2-0.4386000000e-1 * r^2)^2+0.3559266221e-4 * x^{r^4-0.2231225100e-2 * x^{(r-0.4754100000e-1 * r^2-0.3312000000e-1 * (2-1. * r)^2) * r^2+0.4377441406e-1 * x^{(r-0.4754100000e-1 * r^2-0.3312000000e-1 * (2-1. * r)^2)^2+0.1543209877e-2 * x^{(2-1. * r)^4} * (x^{18-0.5637710114e-2})-0.2777777778e-1 * x^{(2-1. * r+0.2950500000e-1 * (2-1. * r)^2-0.4386000000e-1 * r^2) * (2-1. * r)^2 * (x^{12-0.3167635202e-1})+1.666666667 * x^{(2-1. * r+0.2950500000e-1 * (2-1. * r)^2-0.4386000000e-1 * r^2)^2} * (x^6-1.779785156)$$

$$\underline{u}_2(x, \alpha) = 0.1070811478e-5 * x^3 * (2-1. * r+0.2950500000e-1 * (2-1. * r)^2-0.4386000000e-1 * r^2)^4+0.8706749289e-5 * x^{(2-1. * r+0.2950500000e-1 * (2-1. * r)^2-0.4386000000e-1 * r^2)^4+0.9993489583e-1 * x^{(-0.2231225100e-2 * x^{(r-0.4754100000e-1 * r^2-0.3312000000e-1 * (2-1. * r)^2) * r^2+0.8798986673e-3 * x^{(2-1. * r+0.2950500000e-1 * (2-1. * r)^2-0.4386000000e-1 * r^2) * (2-1. * r)^2)^2-0.1892206101e-3 * x^2 * (2-1. * r+0.2950500000e-1 * (2-1. * r)^2-0.4386000000e-1 * r^2)^2} * (r-$$

$$\begin{aligned}
& 0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2*(x^{14}-0.1781794801e-1)+8.669766513*10^{(-} \\
& 9)*x*(5.820766091*10^{(-13)}*r^4+(0.2252531919e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2))*r^2- \\
& 0.4742024740e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4-(0.8879061600e- \\
& 3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2+0.2666666667e-2*r^4*(x^{15}-0.1336346101e- \\
& 1))*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2-1.038075591*10^{(-10)}*x^2*(2.-1.*r+0.2950500000e-1*(2.- \\
& 1.*r)^2-0.4386000000e-1*r^2)*(2.-1.*r)^6+3.062124243*10^{(-9)}*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e- \\
& 1*r^2)^2*(2.-1.*r)^4+8.748987894*10^{(-9)}*x^3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(2.- \\
& 1.*r)^4+1.290452760*10^{(-7)}*x*(-0.8700169836e-5*(2.-1.*r)^4-7.224805427*10^{(-9)}*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2- \\
& 0.4386000000e-1*r^2))*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e- \\
& 1*r^2)^2+0.3559266221e-4*r^4+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2*(2.- \\
& 1.*r)^4+0.6400000000e-4*x*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2*r^4*(x^{25}-0.7525434582e-3)- \\
& 0.8000000000e-3*x*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^3*r^2*(x^{20}-0.3171211939e-2)+0.1471715856e- \\
& 2*x^2*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)^2*(x^9-0.7508468628e-1)-0.3486072243e-4*x^2*(-0.2231225100e-2*x*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)*r^2+0.8798986673e-3*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)*(2.-1.*r)^2*(2.- \\
& 1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)*(2.-1.*r)^2+0.4000000000e-1*x*(5.820766091*10^{(-} \\
& 13)*r^4+(0.2252531919e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2))*r^2-0.4742024740e-1*(r-0.4754100000e- \\
& 1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4-(0.8879061600e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2- \\
& 0.4386000000e-1*r^2))*(2.-1.*r)^2+0.2666666667e-2*r^4*(x^{15}-0.1336346101e-1))*(r-0.4754100000e-1*r^2-0.3312000000e- \\
& 1*(2.-1.*r)^2)^2*(x^{10}-0.5631351471e-1)+0.1297576500e-2*x*(-0.2231225100e-2*x*(r-0.4754100000e-1*r^2-0.3312000000e- \\
& 1*(2.-1.*r)^2)*r^2+0.8798986673e-3*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)*(2.-1.*r)^2*(2.- \\
& 1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2+0.3655036511e-3*x^3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2- \\
& 0.4386000000e-1*r^2)^4*(x^3-0.4218750000)+3.207552540*10^{(-13)}*x*(2.-1.*r)^8+0.6666666667e-1*x*(-(0.8000000000e- \\
& 1*(5.820766091*10^{(-13)}*r^4+(0.2252531919e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2))*r^2-0.4742024740e- \\
& 1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4-(0.8879061600e-3*(2.- \\
& 1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2+0.2666666667e-2*r^4*(x^{15}-0.1336346101e-1)))*(r- \\
& 0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2+0.4000000000e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)^4*(x^{15}-0.1336346101e-1)-1.644770641*10^{(-13)}*x^2*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e- \\
& 1*r^2)^2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2+1.263183852*10^{(-9)}*x^2*(2.-1.*r+0.2950500000e- \\
& 1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2-1.732370861*10^{(-16)}*x*(r- \\
& 0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^3*r^2+1.366428338*10^{(-20)}*x*(r-0.4754100000e-1*r^2-0.3312000000e- \\
& 1*(2.-1.*r)^2)^2*r^4+1.455191523*10^{(-11)}*x*(-(0.8000000000e-1*(5.820766091*10^{(-13)}*r^4+(0.2252531919e-2*(r- \\
& 0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2))*r^2-0.4742024740e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4-(0.8879061600e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))* \\
& (2.-1.*r)^2+0.2666666667e-2*r^4*(x^{15}-0.1336346101e-1)))*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)*r^2+0.4000000000e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^4)+0.3311360677e-1*x*(- \\
& 0.8700169836e-5*(2.-1.*r)^4-7.224805427*10^{(-9)}*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)*(2.-1.*r)^2- \\
& 0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4+0.4377441406e-1*(r- \\
& 0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2)^2+1.117187500*x*(-0.2231225100e-2*x*(r-0.4754100000e-1*r^2- \\
& 0.3312000000e-1*(2.-1.*r)^2)*r^2+0.8798986673e-3*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)*(2.- \\
& 1.*r)^2*(-0.8700169836e-5*(2.-1.*r)^4-7.224805427*10^{(-9)}*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e- \\
& 1*r^2)*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2+0.3559266221e- \\
& 4*r^4+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2)+0.1293500264e-4*x^2*(2.- \\
& 1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(5.820766091*10^{(-13)}*r^4+(0.2252531919e-2*(r-0.4754100000e- \\
& 1*r^2-0.3312000000e-1*(2.-1.*r)^2))*r^2-0.4742024740e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4-(0.8879061600e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))* \\
& (2.-1.*r)^2+0.2666666667e-2*r^4*(x^{15}-0.1336346101e-1))+0.4069010417e-4*x*(5.820766091*10^{(-13)}*r^4+(0.2252531919e- \\
& 2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2))*r^2-0.4742024740e-1*(r-0.4754100000e-1*r^2-0.3312000000e- \\
& 1*(2.-1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4-(0.8879061600e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e- \\
& 1*r^2))* \\
& (2.-1.*r)^2+0.2666666667e-2*r^4*(x^{15}-0.1336346101e-1))^2+0.5817326449e-4*x*((0.3086419754e-2*(- \\
& 0.2231225100e-2*x*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2+0.8798986673e-3*x*(2.-1.*r+0.2950500000e- \\
& 1*(2.-1.*r)^2-0.4386000000e-1*r^2)*(2.-1.*r)^2)*(2.-1.*r)^4-0.9259259262e-2*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2- \\
& 0.4386000000e-1*r^2)^3*x*(2.-1.*r)^2)-0.2459323412e-4*x^2*(-0.8700169836e-5*(2.-1.*r)^4-7.224805427*10^{(-9)}*(2.- \\
& 1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2- \\
& 0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)^2*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)*(2.-1.*r)^2+0.1655680338e-1*x^2*(2.- \\
& 1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(5.820766091*10^{(-13)}*r^4+(0.2252531919e-2*(r-0.4754100000e- \\
& 1*r^2-0.3312000000e-1*(2.-1.*r)^2))*r^2-0.4742024740e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4-(0.8879061600e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))* \\
& (2.-1.*r)^2+0.2666666667e-2*r^4*(x^{15}-0.1336346101e-1))*(x^4-0.3164062500)+0.8879061602e-3*x*(-0.8700169836e-5*(2.- \\
& 1.*r)^4-7.224805427*10^{(-9)}*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)*(2.-1.*r)^2-0.2946777344e-1*(2.- \\
& 1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4+0.4377441406e-1*(r-0.4754100000e-1*r^2- \\
& 0.3312000000e-1*(2.-1.*r)^2)^2*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e- \\
& 1*r^2)^2+2.000000000*x*(5.820766091*10^{(-13)}*r^4+(0.2252531919e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-
\end{aligned}$$

$$1.*r^2)*r^2-0.4742024740e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4-(0.8879061600e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2+0.2666666667e-2*r^4*(x^15-0.1336346101e-1))^2*(x^5-.2373046875)$$

$$\overline{u_2}(x, \alpha) = 0.8898165553e-3*x*(-(0.8000000000e-1*(5.820766091*10^{(-13)}*r^4+(0.2252531919e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2-0.4742024740e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4-(0.8879061600e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2+0.2666666667e-2*r^4*(x^15-0.1336346101e-1)))*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2+0.4000000000e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^4)+0.4377441406e-1*x*(5.820766091*10^{(-13)}*r^4+(0.2252531919e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2-0.4742024740e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4-(0.8879061600e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2+0.2666666667e-2*r^4*(x^15-0.1336346101e-1))^2+0.1245710686e-3*x^3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^4+1.201175561*10^{(-32)}*x*(2.-1.*r)^8+6.178632485*10^{(-16)}*x*((0.3086419754e-2*(-0.8700169836e-5*(2.-1.*r)^4+(0.8798914425e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2)-0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2)*(2.-1.*r)^4-0.9259259262e-2*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^3*(2.-1.*r)^2+3.423980054*10^{(-12)}*x*(-(0.5555555556e-1*(-0.8700169836e-5*(2.-1.*r)^4+(0.8798914425e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2)*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(2.-1.*r)^2+0.277777779e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^4)+0.1953125000e-3*x*(-(0.8700169836e-5*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2)^2+5.670230296*10^{(-8)}*x*(2.-1.*r)^8*(x^42-0.5656829153e-5)+0.4166666667e-1*x*((0.3086419754e-2*(-0.8700169836e-5*(2.-1.*r)^4+(0.8798914425e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2)*(2.-1.*r)^4-0.9259259262e-2*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^3*(2.-1.*r)^2*(x^24-0.1003391278e-2)+0.5555555556e-1*x*(-(0.5555555556e-1*(-0.8700169836e-5*(2.-1.*r)^4+(0.8798914425e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2)-0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2)*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(2.-1.*r)^2+0.277777779e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^4*(x^18-0.5637710114e-2)+.1666666667*x*(-(0.8700169836e-5*(2.-1.*r)^4+(0.8798914425e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2)-0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2)^2*(x^6-.1779785156)+0.2777777778e-1*x*(-(0.8700169836e-5*(2.-1.*r)^4+(0.8798914425e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2)-0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2)*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(x^12-0.3167635202e-1)-0.2381496724e-5*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(2.-1.*r)^6*(x^36-0.3178377533e-4)+0.4286694103e-4*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(2.-1.*r)^4*(x^30-0.1785820902e-3)+4.816179096*10^{(-8)}*x*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2*(2.-1.*r)^4-0.2536569916e-5*x*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2*(2.-1.*r)^2+0.2231225100e-2*x*(5.820766091*10^{(-13)}*r^4+(0.2252531919e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2-0.4742024740e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4-(0.8879061600e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2+0.2666666667e-2*r^4*(x^15-0.1336346101e-1))*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+0.1089211154e-3*x^2*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+0.4604860941e-2*x^2*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(5.820766091*10^{(-13)}*r^4+(0.2252531919e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2-0.4742024740e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4-(0.8879061600e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2+0.2666666667e-2*r^4*(x^15-0.1336346101e-1))+7.224805429*10^{(-9)}*x*(-(0.8700169836e-5*(2.-1.*r)^4+(0.8798914425e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2)-0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2)*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2-0.074840652*10^{(-27)}*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2)-0.4386000000e-1*r^2)^2*(2.-1.*r)^6+1.538530116*10^{(-22)}*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(2.-1.*r)^4-0.3365363305e-5*x^2*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2$$

$$\begin{aligned}
& 1.*r^2+0.2666666667e-2*r^4*(x^{15}-0.1336346101e-1))*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2*(x^{10}- \\
& 0.5631351471e-1)-0.3486072243e-4*x^2*(-0.2231225100e-2*x*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)*r^2+0.8798986673e-3*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)*(2.-1.*r)^2*(2.- \\
& 1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)*(2.-1.*r)^2+0.1471715856e-2*x^2*(2.-1.*r+0.2950500000e-1*(2.- \\
& 1.*r)^2-0.4386000000e-1*r^2)^2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2*(x^9-0.7508468628e-1)- \\
& 0.8000000000e-3*x*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^3*r^2*(x^{20}-0.3171211939e-2)+0.6400000000e- \\
& 4*x*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2*r^4*(x^{25}-0.7525434582e-3)-0.1892206101e-3*x^2*(2.- \\
& 1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2*(x^{14}- \\
& 0.1781794801e-1)+0.3311360677e-1*x*(-0.8700169836e-5*(2.-1.*r)^4-7.224805427*10^{-9})*(2.-1.*r+0.2950500000e-1*(2.- \\
& 1.*r)^2-0.4386000000e-1*r^2)*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e- \\
& 1*r^2)^2+0.3559266221e-4*r^4+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)^2+0.3311360677e-1*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2+0.4069010417e-4*x*(r- \\
& 0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+0.8706749286e-5*x*(2.-1.*r)^4-(1/5)*x^6*r^2+x*r-0.47541e-1*x*r^2- \\
& 0.3312e-1*x*(2-r)^2 \\
\bar{u}(x, \alpha) = & 1.666666667*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(x^6-1.779785156)+x*(2- \\
& r)+0.29505e-1*x*(2-r)^2-0.4386e-1*x*r^2-(1/6)*x^7*(2-r)^2+0.1953125000e-3*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2- \\
& 0.4386000000e-1*r^2)^2+0.3559266221e-4*x*r^4+0.4377441406e-1*x*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)^2+4.816179096*10^{-8}*x*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2*r^4+9.511055705*10^{-14} \\
& *x*(2.-1.*r)^4+1.201175561*10^{-32}*x*(2.-1.*r)^8+0.5555555556e-1*x*(-(0.5555555556e-1*(-0.8700169836e-5*(2.- \\
& 1.*r)^4+(0.8798914425e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2-0.2946777344e-1*(2.- \\
& 1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r-0.4754100000e-1*r^2- \\
& 0.3312000000e-1*(2.-1.*r)^2))*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2)*(2.- \\
& 1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(2.-1.*r)^2+0.2777777779e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2- \\
& 0.4386000000e-1*r^2)^4*(x^{18}-0.5637710114e-2)+5.670230296*10^{-8}*x*(2.-1.*r)^8*(x^4-0.5656829153e- \\
& 5)+0.8898165553e-3*x*(-(0.8000000000e-1*(5.820766091*10^{-13})*r^4+(0.2252531919e-2*(r-0.4754100000e-1*r^2- \\
& 0.3312000000e-1*(2.-1.*r)^2))*r^2-0.4742024740e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4-(0.8879061600e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)))*(2.- \\
& 1.*r)^2+0.2666666667e-2*r^4*(x^{15}-0.1336346101e-1))*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)*r^2+0.4000000000e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^4-0.3365363305e-5*x^2*(2.- \\
& 1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)*r^2+0.4377441406e-1*x*(5.820766091*10^{-13})*r^4+(0.2252531919e-2*(r-0.4754100000e-1*r^2-0.3312000000e- \\
& 1*(2.-1.*r)^2))*r^2-0.4742024740e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4- \\
& (0.8879061600e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2+0.2666666667e-2*r^4*(x^{15}- \\
& 0.1336346101e-1)^2+0.1953125000e-3*x*(-(0.8700169836e-5*(2.-1.*r)^4+(0.8798914425e-3*(2.-1.*r+0.2950500000e-1*(2.- \\
& 1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e- \\
& 1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2))*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+1.666666667*x*(- \\
& 0.8700169836e-5*(2.-1.*r)^4+(0.8798914425e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2- \\
& 0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r- \\
& 0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2))*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)^2)^2*(x^6-1.779785156)+0.1245710686e-3*x^3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e- \\
& 1*r^2)^4+0.4166666667e-1*x*((0.3086419754e-2*(-0.8700169836e-5*(2.-1.*r)^4+(0.8798914425e-3*(2.-1.*r+0.2950500000e- \\
& 1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e- \\
& 1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2))*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2)*(2.-1.*r)^4-0.9259259262e-2*(2.- \\
& 1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^3*(2.-1.*r)^2*(x^{24}-0.1003391278e-2)-0.2231225100e-2*x*(r- \\
& 0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)*r^2+0.1543209877e-2*x*(2.-1.*r)^4*(x^{18}-0.5637710114e- \\
& 2)+3.423980054*10^{-12}*x*(-(0.5555555556e-1*(-0.8700169836e-5*(2.-1.*r)^4+(0.8798914425e-3*(2.-1.*r+0.2950500000e- \\
& 1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e- \\
& 1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2))*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2)*(2.-1.*r+0.2950500000e-1*(2.- \\
& 1.*r)^2-0.4386000000e-1*r^2)*(2.-1.*r)^2+0.2777777779e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^4)- \\
& 0.2536569916e-5*x*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^3*r^2+6.178632485*10^{-16}*x*((0.3086419754e- \\
& 2*(-0.8700169836e-5*(2.-1.*r)^4+(0.8798914425e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2- \\
& 0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r- \\
& 0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2))*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)^2)*(2.-1.*r)^4-0.9259259262e-2*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^3*(2.- \\
& 1.*r)^2+0.2231225100e-2*x*(5.820766091*10^{-13})*r^4+(0.2252531919e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2))*r^2-0.4742024740e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4- \\
& (0.8879061600e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2+0.2666666667e-2*r^4*(x^{15}- \\
& 0.1336346101e-1))*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2+0.460486094e-2*x^2*(2.-1.*r+0.2950500000e- \\
& 1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(5.820766091*10^{-13})*r^4+(0.2252531919e-2*(r-0.4754100000e-1*r^2- \\
& 0.3312000000e-1*(2.-1.*r)^2))*r^2-0.4742024740e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
& 1.*r)^2)^2+0.8706749286e-5*(2.-1.*r)^4-(0.8879061600e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-
\end{aligned}$$

$$\begin{aligned}
 & 1.*r^2+0.2666666667e-2*r^4*(x^15-0.1336346101e-1)-0.2777777778e-1*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2- \\
 & 0.4386000000e-1*r^2)*(2.-1.*r)^2*(x^12-0.3167635202e-1)-0.2381496724e-5*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2- \\
 & 0.4386000000e-1*r^2)*(2.-1.*r)^6*(x^36-0.3178377533e-4)+0.4286694103e-4*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2- \\
 & 0.4386000000e-1*r^2)^2*(2.-1.*r)^4*(x^30-0.1785820902e-3)-7.224805427*10^(-9)*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2- \\
 & 0.4386000000e-1*r^2)*(2.-1.*r)^2-2.074840652*10^(-27)*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)*(2.- \\
 & 1.*r)^6+1.538530116*10^(-22)*x*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2)^2*(2.- \\
 & 1.*r)^4+7.224805429*10^(-9)*x*(-0.8700169836e-5*(2.-1.*r)^4+(0.8798914425e-3*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2- \\
 & 0.4386000000e-1*r^2))*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e- \\
 & 1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
 & 1.*r)^2))*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2*(2.-1.*r+0.2950500000e-1*(2.- \\
 & 1.*r)^2-0.4386000000e-1*r^2)^2+0.2777777778e-1*x*(-0.8700169836e-5*(2.-1.*r)^4+(0.8798914425e-3*(2.- \\
 & 1.*r+0.2950500000e-1*(2.-1.*r)^2-0.4386000000e-1*r^2))*(2.-1.*r)^2-0.2946777344e-1*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2- \\
 & 0.4386000000e-1*r^2)^2+0.3559266221e-4*r^4-(0.2231225100e-2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.- \\
 & 1.*r)^2))*r^2+0.4377441406e-1*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2*(2.-1.*r+0.2950500000e-1*(2.- \\
 & 1.*r)^2-0.4386000000e-1*r^2)^2*(x^12-0.3167635202e-1)+0.1089211154e-3*x^2*(2.-1.*r+0.2950500000e-1*(2.-1.*r)^2- \\
 & 0.4386000000e-1*r^2)^2*(r-0.4754100000e-1*r^2-0.3312000000e-1*(2.-1.*r)^2)^2
 \end{aligned}$$

Table1. HPM solve the equation 1 and comperd with exact solution with different value for α for lower side (A)

x	Exact $\underline{u}(x, \alpha)$ $\alpha = 0.1$	HPM $\underline{u}(x, \alpha)$ $\alpha = 0.1$	Exact $\underline{u}(x, \alpha)$ $\alpha = 0.4$	HPM $\underline{u}(x, \alpha)$ $\alpha = 0.4$	Exact $\underline{u}(x, \alpha)$ $\alpha = 0.7$	HPM $\underline{u}(x, \alpha)$ $\alpha = 0.7$	Exact $\underline{u}(x, \alpha)$ $\alpha = 1$	HPM $\underline{u}(x, \alpha)$ $\alpha = 1$
0	0.000000	0.0000000	0.000000	0.00000000	0.000000	0.00000000	0.000000	0.00000000
0.2	0.020000	0.0213973	0.080000	0.07834603	0.140000	0.13920386	0.200000	0.18251966
0.4	0.040000	0.0426968	0.160000	0.15659395	0.280000	0.26374526	0.400000	0.36491658
0.6	0.060000	0.0639038	0.240000	0.23543220	0.420000	0.39471045	0.600000	0.54598664
0.8	0.080000	0.0853819	0.320000	0.31992652	0.560000	0.52122930	0.800000	0.78191312
1.0	0.100000	0.1082735	0.400000	0.37849267	0.700000	0.68367956	1.000000	0.98842474

HPM solve the equation 1 and comperd with exact solution with different value for α for upper side (B)

x	Exact $\bar{u}(x, \alpha)$ $\alpha = 0.1$	HPM $\bar{u}(x, \alpha)$ $\alpha = 0.1$	Exact $\bar{u}(x, \alpha)$ $\alpha = 0.4$	HPM $\bar{u}(x, \alpha)$ $\alpha = 0.4$	Exact $\bar{u}(x, \alpha)$ $\alpha = 0.7$	HPM $\bar{u}(x, \alpha)$ $\alpha = 0.7$	Exact $\bar{u}(x, \alpha)$ $\alpha = 1$	HPM $\bar{u}(x, \alpha)$ $\alpha = 1$
0	0.0000000	0.00000000	0.000000	0.00000000	0.000000	0.00000000	0.000000	0.00000000
0.2	0.3800000	0.37874172	0.320000	0.318838731	0.260000	0.2589034517	0.200000	0.1985695810
0.4	0.7600000	0.75768423	0.640000	0.637777673	0.520000	0.5178385084	0.400000	0.3971294639
0.6	1.1400000	1.13829735	0.960000	0.957590047	0.780000	0.7770337618	0.600000	0.5955423963
0.8	1.5200000	1.51969408	1.280000	1.277275239	1.040000	1.0350154250	0.800000	0.7918457018
1.0	1.9000000	1.79120078	1.600000	1.528025145	1.300000	1.2515325978	1.000000	0.9632121320

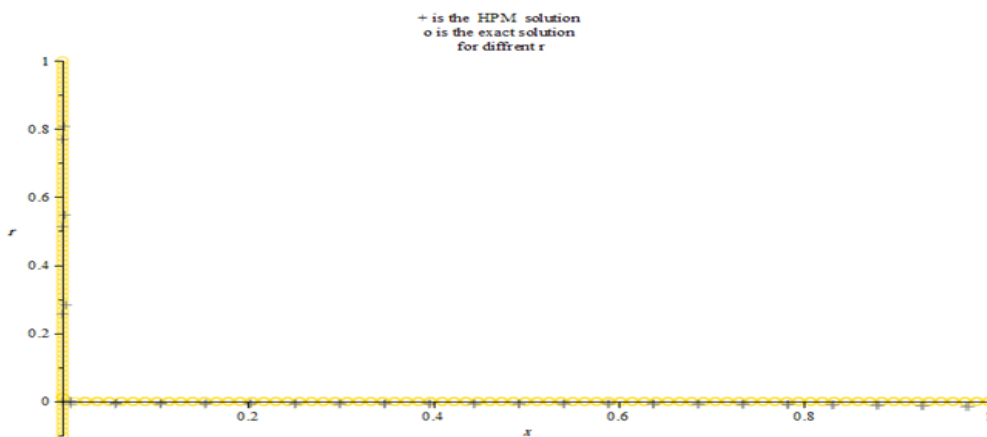


Fig 1. compares the HPM with the exact solution for lower side for different r .

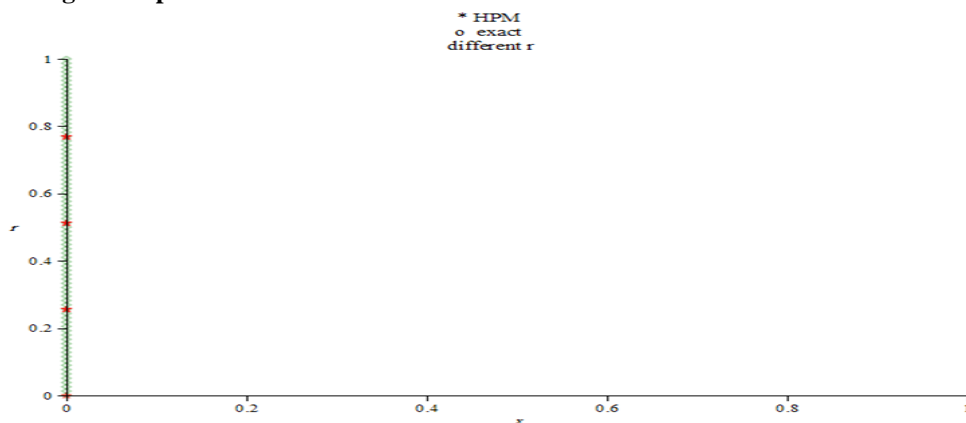


Fig 2. Compares the HPM with the exact solution for lower side for different r .

Example 4.2

Consider the fuzzy nonlinear volterra integral equation

$$\tilde{u}(x) = \tilde{f}(x) + \int_a^x k(x, t, \tilde{k}_1(t, \tilde{F}(t, \tilde{u}(t)))) dt \quad (17)$$

Substituting eq (7) in Eq(16), we get

$$\underline{u}(x, \alpha) = \underline{f}(x, \alpha) + \int_a^c k(x, t, \underline{k}_1(t, \underline{F}(t, u(t, \alpha)))) + \int_c^x k(x, t, \overline{k}_1(t, \overline{F}(t, u(t, \alpha))))$$

$$\overline{u}(x, \alpha) = \overline{f}(x, \alpha) + \int_a^x k(x, t, \overline{k}_1(t, \overline{F}(t, u(t, \alpha)))) + \int_c^x k(x, t, \underline{k}_1(t, \underline{F}(t, u(t, \alpha))))$$

Substituting Eq(7) in (11) and find the upper and lower for Eq(16) , we obtain

$$\underline{u}(x, \alpha) = \underline{f}(x, \alpha) + \int_a^d k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt + \int_d^c k(x, t, \overline{k}_1(t, \overline{F}(t, \overline{u}(t, \alpha)))) dt + \int_c^e k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt + \int_e^x k(x, t, \overline{k}_1(t, \overline{F}(t, \overline{u}(t, \alpha)))) dt$$

$$\overline{u}(x, \alpha) = \overline{f}(x, \alpha) + \int_a^d k(x, t, \overline{k}_1(t, \overline{F}(t, \overline{u}(t, \alpha)))) + \int_d^c k(x, t, k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}(t, \alpha)))) + \int_e^e k(x, t, \overline{k}_1(t, \overline{F}(t, \overline{u}(t, \alpha)))) + \int_e^x k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}(t, \alpha)))) dt$$

where $0 \leq t \leq d$, $d \leq t \leq c$, $c \leq t \leq e$, $e \leq t \leq x$, $0 \leq \alpha \leq 1$

$$\underline{f}(x, \alpha) = (5/4 - (1/4)*r)*\exp(x) + .2620498398*r^2*x^2 - 2.225551737*r*x^2 + 5.882874723*x^2 - 0.2799479167e-1*x^2*(.7500000000 + .2500000000*r)^3*(\exp(x))^3 + 0.3375612483e-2*r^3*x^2 + (1/16)*\exp(2*x)*r^2*x^5 - (1/4)*\exp(2*x)*r^2*x^4 - (5/8)*\exp(2*x)*r*x^5 + (3/8)*\exp(2*x)*r^2*x^3 + (5/2)*\exp(2*x)*r*x^4 + (25/16)*\exp(2*x)*x^5 - (7/32)*\exp(2*x)*r^2*x^2 - (15/4)*\exp(2*x)*r*x^3 - (25/4)*\exp(2*x)*x^4 + (35/16)*\exp(2*x)*r*x^2 + (75/8)*\exp(2*x)*x^3 - (175/32)*\exp(2*x)*x^2$$

$$\overline{f}(x, \alpha) = (3/4 + (1/4)*r)*\exp(x) - 0.7161458333e-2*x^2*(1.2500000000 - .2500000000*r)^2*(\exp(x))^2 - (63/64)*\exp(3*x)*x^4 + (39/32)*\exp(3*x)*x^3 - (35/64)*\exp(3*x)*x^2 + (39/32)*\exp(3*x)*r*x^3 - (35/64)*\exp(3*x)*r*x^2 + (9/32)*\exp(3*x)*x^5 + (13/32)*\exp(3*x)*r^2*x^3 - (63/64)*\exp(3*x)*r*x^4 - (35/192)*\exp(3*x)*r^2*x^2 + (13/288)*\exp(3*x)*r^3*x^3 - (21/64)*\exp(3*x)*r^2*x^4 + (9/32)*\exp(3*x)*r*x^5 - (35/1728)*\exp(3*x)*r^3*x^2 + (1/96)*\exp(3*x)*r^3*x^5 - (7/192)*\exp(3*x)*r^3*x^4 + (3/32)*\exp(3*x)*r^2*x^5 + 0.2063574710e-1*r^3*x^2 + .1925350109*r^2*x^2 + .4890323008*r*x^2 + .7274973486*x^2$$

Kernal of Eq(17) is

$$k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}(t, \alpha)))) = x^2 t (1 - 2t) (\underline{u}(t, \alpha))^2$$

$$k(x, t, \overline{k}_1(t, \overline{F}(t, \overline{u}(t, \alpha)))) = x^2 (1 - 2t) (1 - t)^2 (\overline{u}(t, \alpha))^3$$

$$k(x, t, \underline{k}_1(t, \underline{F}(t, \underline{u}(t, \alpha)))) = x^2 t (1 - 2t) (\underline{u}(t, \alpha))^3$$

$$k(x, t, \overline{k}_1(t, \overline{F}(t, \overline{u}(t, \alpha)))) = x^2 (1 - 2t) (1 - t)^2 (\overline{u}(t, \alpha))^2$$

$$\alpha = 0, d = \frac{1}{4}, c = \frac{1}{2}, e = \frac{3}{4}, x = 1$$

The exact solution

$$\underline{u}(x, \alpha) = \left(\frac{5-\alpha}{4}\right)\exp(x), \quad \overline{u}(x, \alpha) = \left(\frac{3+\alpha}{4}\right)\exp(x)$$

Now we apply the HPM technique to solve the our problem

$$\underline{u}_0(x, \alpha) = \underline{f}(x, \alpha)$$

$$\overline{u}_0(x, \alpha) = \overline{f}(x, \alpha)$$

$$\underline{u}_1(x, \alpha) =$$

$$\begin{aligned} &.2929687500*\exp(4.*x)*r^2*x^15 + 5.882120848*r^3*\exp(3.*x)*x^4 + 86.05989162*r^2*\exp(3.*x)*x^5 + 0.2316476000e- \\ &2*r^4*\exp(3.*x)*x^3 - 0.1007874868e-2*r^5*\exp(3.*x)*x^2 - 0.1148010294e-5*\exp(6.*x)*r^5*x^9 + 0.1859029134e- \\ &3*\exp(5.*x)*r^4*x^11 - 0.1859029134e-3*\exp(5.*x)*r^5*x^11 - 0.2187093099e-4*\exp(5.*x)*r^4*x^12 - \\ &.5371093750*\exp(4.*x)*r^3*x^14 - 0.1953125000e-2*\exp(4.*x)*r^4*x^15 + 0.2187093099e- \\ &4*\exp(5.*x)*r^5*x^12 + 0.2685546875e-1*\exp(4.*x)*r^4*x^14 + 0.3906250000e- \\ &1*\exp(4.*x)*r^3*x^15 + 2.864543278*\exp(3.*x)*r^2*x^9 + 1.562500000*\exp(3.*x)*r*x^10 - 164.5470897*\exp(2.*x)*r*x^11 - \\ &14.26532454*\exp(3.*x)*r*x^9 - 0.8310953777e- \\ &3*\exp(5.*x)*r^3*x^12 + 4.028320312*\exp(4.*x)*r^2*x^14 + .9765625000*\exp(4.*x)*r*x^15 + 0.4209371078e- \\ &5*\exp(6.*x)*r^5*x^8 + .2949623108*r*\exp(5.*x)*x^10 + 33.05508825*r^2*\exp(3.*x)*x^7 + 61.26677696*r*\exp(3.*x)*x^8 - \\ &.1782226562*r^4*\exp(4.*x)*x^13 + 31.81683167*r^4*\exp(4.*x)*x^4 + 848.4322955*r^3*\exp(4.*x)*x^5 + 6363.253698*r^2*\exp(4.*x)*x^6 + \\ &16963.92287*r*\exp(4.*x)*x^7 + 0.1460659610e-3*r^5*\exp(5.*x)*x^2 + 0.7303298048e-3*r^4*\exp(5.*x)*x^3 - \\ &0.6938133145e-1*r^3*\exp(5.*x)*x^4 + 0.5477473536e- \\ &1*r^2*\exp(5.*x)*x^5 + 1.540539432*r*\exp(5.*x)*x^6 + 8.732234598*r^4*\exp(2.*x)*x^9 + 41.51572158*r^3*\exp(2.*x)*x^10 + 0.38 \\ &03801067e-2*r^5*\exp(5.*x)*x^6 + 0.3765526937e-2*r^4*\exp(5.*x)*x^7 - .1123158915*r^3*\exp(5.*x)*x^8 + 0.3160437012e- \\ &1*r^2*\exp(5.*x)*x^9 - 0.3926544559e-4*r^6*\exp(3.*x)*x^6 + 0.1989780816e-2*r^5*\exp(3.*x)*x^7 - \\ &.4615095192*r^5*\exp(2.*x)*x^8 - 352.2803883*r^4*\exp(2.*x)*x^2 - \\ &14695.44943*r^3*\exp(2.*x)*x^3 + 3.977103959*r^4*\exp(4.*x)*x^2 + 318.1620015*r^3*\exp(4.*x)*x^3 + 4772.436337*r^2*\exp(4.*x)*x^4 - \\ &0.4219515604e-3*\exp(2.*x)*r^5*x^12 + 0.3157028308e-4*\exp(6.*x)*r^4*x^8 - 0.3444030882e- \end{aligned}$$

$$\begin{aligned}
& 4 * \exp(6. * x) * r^3 * x^9 + 0.3346252442e-2 * \exp(5. * x) * r^2 * x^{11} - 0.8610077204e-5 * \exp(6. * x) * r^4 * x^9 + 0.7064310710e- \\
& 2 * \exp(5. * x) * r^3 * x^{11} - 0.3936767579e-3 * \exp(5. * x) * r^2 * x^{12} - 13.42773438 * \exp(4. * x) * r * x^{14} + 0.6672033692e- \\
& 1 * r^3 * \exp(5. * x) * x^9 - 0.1310943604e-1 * r^2 * \exp(5. * x) * x^{10} - 0.3996776579e-5 * r^5 * \exp(6. * x) * x^5 + 0.4496373651e- \\
& 4 * r^4 * \exp(6. * x) * x^6 - 0.1951617500e-3 * r^3 * \exp(6. * x) * x^7 + 0.5995164868e-5 * r^5 * \exp(6. * x) * x^6 - 0.4879043749e- \\
& 4 * r^4 * \exp(6. * x) * x^7 - 0.3765526937e-2 * r^5 * \exp(5. * x) * x^7 - 0.2955681356e- \\
& 2 * r^4 * \exp(5. * x) * x^8 + 7954.104768 * r * \exp(4. * x) * x^3 + 1.307137966 * r^3 * \exp(3. * x) * x^2 + 57.73784441 * r^2 * \exp(3. * x) * x^3 + 431.5 \\
& 076351 * r * \exp(3. * x) * x^4 - 0.1755798340e-2 * r^5 * \exp(5. * x) * x^9 - 0.7283020020e-3 * r^4 * \exp(5. * x) * x^{10} + 0.5915671419e- \\
& 1 * r * \exp(5. * x) * x^2 + 596.5545421 * r^2 * \exp(4. * x) * x^2 + 0.7283020020e- \\
& 3 * r^5 * \exp(5. * x) * x^{10} + 7664794922 * r^4 * \exp(4. * x) * x^{12} + 3.564453125 * r^3 * \exp(4. * x) * x^{13} + 704.5607766 * r^4 * \exp(2. * x) * x^3 \\
& + 14695.44943 * r^3 * \exp(2. * x) * x^4 + 71372.78381 * r^2 * \exp(2. * x) * x^5 - 5.208912897 * r^5 * \exp(2. * x) * x^2 - \\
& 121.9467163 * r^3 * \exp(4. * x) * x^{10} - 364.0411377 * r^2 * \exp(4. * x) * x^{11} - 383.2397461 * r * \exp(4. * x) * x^{12} - 0.2943112134e- \\
& 1 * r^5 * \exp(2. * x) * x^{10} - 12.30883418 * r^2 * \exp(3. * x) * x^8 - \\
& 3.472608598 * r^5 * \exp(2. * x) * x^6 + 93.92146117 * r^4 * \exp(2. * x) * x^7 + 651.0081789 * r^3 * \exp(2. * x) * x^8 + 0.4535436908e- \\
& 2 * r^5 * \exp(3. * x) * x^5 - 0.2606035500e-2 * r^4 * \exp(3. * x) * x^6 - \\
& 2.251201753 * r^3 * \exp(3. * x) * x^7 + 21210.93818 * r * \exp(4. * x) * x^5 + 0.3490261830e-4 * r^6 * \exp(3. * x) * x^3 - 0.5235392746e- \\
& 4 * r^6 * \exp(3. * x) * x^4 - 7.110983277 * r * \exp(5. * x) * x^9 - 33.92793147 * r^4 * \exp(4. * x) * x^7 - 451.1860657 * r^3 * \exp(4. * x) * x^8 - \\
& 1909.107971 * r^2 * \exp(4. * x) * x^9 - 3048.667908 * r * \exp(4. * x) * x^{10} - 1958.976612 * r^3 * \exp(2. * x) * x^7 - \\
& 4742.723957 * r^2 * \exp(2. * x) * x^8 - 4378.383430 * r * \exp(2. * x) * x^9 - \\
& 12.72738647 * r^4 * \exp(4. * x) * x^9 + 1213.470459 * r * \exp(4. * x) * x^{11} + 0.2296864505e-4 * r^6 * \exp(3. * x) * x^7 - 0.3043040853e- \\
& 2 * r^5 * \exp(5. * x) * x^5 - 0.3803801067e-2 * r^4 * \exp(5. * x) * x^6 + 1.430900236 * r^3 * \exp(5. * x) * x^7 - 0.5320226441e- \\
& 1 * r^2 * \exp(5. * x) * x^8 - 0.2997582434e-4 * r^4 * \exp(6. * x) * x^5 + 0.1798549460e-3 * r^3 * \exp(6. * x) * x^6 - 0.4391139374e- \\
& 3 * r^2 * \exp(6. * x) * x^7 + 1.291171775 * r^5 * \exp(2. * x) * x^9 - 1.990435827 * r^4 * \exp(2. * x) * x^{10} + 0.1998388289e- \\
& 5 * r^5 * \exp(6. * x) * x^4 + 0.4856083543e-3 * r * \exp(6. * x) * x^6 - 0.7303298048e-3 * r^5 * \exp(5. * x) * x^3 - 0.1825824512e- \\
& 2 * r^4 * \exp(5. * x) * x^4 + 1.156355524 * r^3 * \exp(5. * x) * x^5 - 0.1163420610e-4 * r^6 * \exp(3. * x) * x^2 - 0.4995970724e- \\
& 5 * r^4 * \exp(6. * x) * x^3 + 0.5995164868e-4 * r^3 * \exp(6. * x) * x^4 - 0.2697824191e- \\
& 3 * r^2 * \exp(6. * x) * x^5 + 95.89058558 * r * \exp(3. * x) * x^2 - 53529.41599 * r^2 * \exp(2. * x) * x^2 + 7347.724713 * r^3 * \exp(2. * x) * x^2 - \\
& 0.5550506516e-2 * r^3 * \exp(5. * x) * x^2 + 0.1314593649e- \\
& 1 * r^2 * \exp(5. * x) * x^3 + 7394589273 * r * \exp(5. * x) * x^4 + 14271.53392 * r^2 * \exp(2. * x) * x^7 + 15649.85907 * r * \exp(2. * x) * x^8 + 0.74939 \\
& 56085e-5 * r^2 * \exp(6. * x) * x^2 - 0.5395648381e-4 * r * \exp(6. * x) * x^3 - 164.6058697 * r * \exp(3. * x) * x^7 - \\
& 10.41782579 * r^5 * \exp(2. * x) * x^4 + 469.7071844 * r^4 * \exp(2. * x) * x^5 + 4898.483142 * r^3 * \exp(2. * x) * x^6 + 0.3023624605e- \\
& 2 * r^5 * \exp(3. * x) * x^3 - 0.3474714000e-2 * r^4 * \exp(3. * x) * x^4 - 5.845662515 * r^3 * \exp(3. * x) * x^5 - \\
& 62.20116872 * r^2 * \exp(3. * x) * x^6 + 6.097335815 * r^4 * \exp(4. * x) * x^{10} + 48.53881836 * r^3 * \exp(4. * x) * x^{11} + 114.9719238 * r^2 * \exp(\\
& 4. * x) * x^{12} + 89.11132812 * r * \exp(4. * x) * x^{13} - 0.6505391666e-5 * r^5 * \exp(6. * x) * x^7 + 0.2955681356e- \\
& 2 * r^5 * \exp(5. * x) * x^8 + 0.1755798340e-2 * r^4 * \exp(5. * x) * x^9 - 0.2767547608e-1 * r^3 * \exp(5. * x) * x^{10} - \\
& 15.32958984 * r^3 * \exp(4. * x) * x^{12} - 26.73339844 * r^2 * \exp(4. * x) * x^{13} - 0.1998388289e-4 * r^3 * \exp(6. * x) * x^3 + 0.1348912095e- \\
& 3 * r^2 * \exp(6. * x) * x^4 - 0.3237389029e-3 * r * \exp(6. * x) * x^5 - \\
& 2.426940918 * r^4 * \exp(4. * x) * x^{11} + 1326.878657 * r^2 * \exp(2. * x) * x^9 + 998.0138701 * r * \exp(2. * x) * x^{10} - 0.9298883381e- \\
& 4 * \exp(6. * x) * r * x^9 - 0.8243377667e-3 * \exp(3. * x) * r^5 * x^8 + 0.1306647326e-3 * \exp(3. * x) * r^4 * x^9 + 0.2083333333e- \\
& 1 * \exp(3. * x) * r^3 * x^{10} + 0.1705526414e- \\
& 3 * \exp(3. * x) * r^5 * x^9 + 3.281722153 * \exp(2. * x) * r^4 * x^{11} + 0.5952074779 * \exp(2. * x) * r^3 * x^{12} + 0.2841325477e- \\
& 3 * \exp(6. * x) * r^2 * x^8 + 0.1262811323e-3 * \exp(6. * x) * r^3 * x^8 - 0.7749069484e-4 * \exp(6. * x) * r^2 * x^9 - 0.7529067994e- \\
& 1 * \exp(5. * x) * r * x^{11} - 0.9515581522e-5 * \exp(3. * x) * r^6 * x^8 + 0.8857727052e-2 * \exp(5. * x) * r * x^{12} + 0.1968741005e- \\
& 5 * \exp(3. * x) * r^6 * x^9 + 0.4852442944e-2 * \exp(2. * x) * r^5 * x^{11} - 0.2853671437e-1 * \exp(2. * x) * r^4 * x^{12} - \\
& 5089.171349 * r^2 * \exp(4. * x) * x^7 - 11279.65164 * r * \exp(4. * x) * x^8 + 6.945217196 * r^5 * \exp(2. * x) * x^5 - \\
& 234.8535922 * r^4 * \exp(2. * x) * x^6 - 15908.20954 * r * \exp(4. * x) * x^4 - 0.7721586668e-3 * r^4 * \exp(3. * x) * x^2 - \\
& 3.921413899 * r^3 * \exp(3. * x) * x^3 - 86.60676662 * r^2 * \exp(3. * x) * x^4 - 35686.42316 * r^2 * \exp(2. * x) * x^6 - \\
& 47092.66165 * r * \exp(2. * x) * x^7 + 0.8992747302e-5 * r * \exp(6. * x) * x^2 + 1.388748073 * r^5 * \exp(2. * x) * x^7 - \\
& 31.21203134 * r^4 * \exp(2. * x) * x^8 - 182.1334882 * r^3 * \exp(2. * x) * x^9 - 302.4502821 * r^2 * \exp(2. * x) * x^{10} - 0.3401577681e- \\
& 2 * r^5 * \exp(3. * x) * x^6 + 0.1524421880e- \\
& 2 * r^4 * \exp(3. * x) * x^7 + 8402796152 * r^3 * \exp(3. * x) * x^8 + 42.42215061 * r^4 * \exp(4. * x) * x^6 + 678.5560048 * r^3 * \exp(4. * x) * x^7 + 33 \\
& 83.895493 * r^2 * \exp(4. * x) * x^8 + 2.338539488 * 10^{(-7)} * \exp(6. * x) * r^6 * x^8 - 6.377834966 * 10^{(-} \\
& 8) * \exp(6. * x) * r^6 * x^9 + 6.167865091 * 10^{(-9)} * r^6 * \exp(6. * x) * x^2 - 2.220431433 * 10^{(-7)} * r^6 * \exp(6. * x) * x^5 - \\
& 1.070590195 * 10^5 * r^2 * \exp(2. * x) * x^4 - 2.355127628 * 10^5 * r * \exp(2. * x) * x^5 + 1.110215716 * 10^{(-7)} * r^6 * \exp(6. * x) * x^4 - \\
& 3.614106481 * 10^{(-7)} * r^6 * \exp(6. * x) * x^7 + 1.177566939 * 10^5 * r * \exp(2. * x) * x^6 - 3.700719054 * 10^{(-} \\
& 8) * r^6 * \exp(6. * x) * x^3 + 1.070588945 * 10^5 * r^2 * \exp(2. * x) * x^3 + 3.532675817 * 10^5 * r * \exp(2. * x) * x^4 + 3.330647149 * 10^{(-} \\
& 7) * r^6 * \exp(6. * x) * x^6 + 8.326617873 * 10^{(-7)} * r^4 * \exp(6. * x) * x^2 + 1.110215716 * 10^{(-7)} * r^5 * \exp(6. * x) * x^2
\end{aligned}$$

$$\begin{aligned}
\bar{u}_1(x, \alpha) &= 0.1255343419e-5 * r^9 * x^9 + 0.3513769932e-4 * r^8 * x^9 + 0.4170892249e-3 * r^7 * x^9 + 1.887509464 * \exp(x) * r^7 * x^6 - \\
& .8162983138 * \exp(x) * r^6 * x^7 - 0.3768631417e- \\
& 1 * \exp(x) * r^7 * x^7 + 6.829036534 * r * \exp(2. * x) * x^5 + 9196115941 * r^4 * \exp(2. * x) * x^2 - 3.233418317 * r^3 * \exp(2. * x) * x^3 - \\
& 497.1284789 * r * \exp(6. * x) * x^5 - \\
& 25.88861406 * r^2 * \exp(4. * x) * x^2 + 113.6460204 * r * \exp(4. * x) * x^3 + 18.86424117 * r * \exp(7. * x) * x^4 + 6.099958909 * r^4 * \exp(6. * x) * x \\
& ^2 - 74.96161284 * r^3 * \exp(6. * x) * x^3 + 304.5923103 * r^2 * \exp(6. * x) * x^4 - 0.1971276049e-2 * r^2 * \exp(8. * x) * x^2 + 0.2782977952e- \\
& 1 * r * \exp(8. * x) * x^3 + 4.277633021 * r^3 * \exp(7. * x) * x^2 - 5.389844648 * r^2 * \exp(7. * x) * x^3 - 2307948285 * r * \exp(9. * x) * x^2 -
\end{aligned}$$

.1384119831*r^3*exp(2.*x)*x^8+.2576557430*r^2*exp(2.*x)*x^9-0.1115979950e-
1*r^9*exp(6.*x)*x^9+.2280156256*r^8*exp(6.*x)*x^10-1.787840594*r^7*exp(6.*x)*x^11+7.038256228*r^6*exp(6.*x)*x^12-
14.94713373*r^5*exp(6.*x)*x^13-.2158319326*r^6*exp(7.*x)*x^11+.8574634822*r^5*exp(7.*x)*x^12-
1.315851038*r^4*exp(7.*x)*x^13+0.2395014339e-1*r^4*exp(8.*x)*x^12+0.4723728277e-1*r^3*exp(8.*x)*x^13-
.1197989029*r^2*exp(8.*x)*x^14+0.6407227367e-1*r^2*exp(8.*x)*x^15+0.1732312765e-
1*r^7*exp(7.*x)*x^10+0.3964903019e-3*r^7*exp(8.*x)*x^9+0.9795787291e-3*r^6*exp(8.*x)*x^10-0.1498147132e-
1*r^5*exp(8.*x)*x^11-0.4343498587e-4*r^8*exp(8.*x)*x^8+.9002582131*r^6*exp(4.*x)*x^7-
5.213070276*r^5*exp(4.*x)*x^8+13.31898751*r^4*exp(4.*x)*x^9-
15.39458391*r^3*exp(4.*x)*x^10+.3592303586*r^3*exp(5.*x)*x^9+.3973136585*r^2*exp(5.*x)*x^10-
.6110118297*r^2*exp(5.*x)*x^11-0.5291370847e-1*r^7*exp(4.*x)*x^6-0.2593650064e-2*r^6*exp(5.*x)*x^6+0.7957676066e-
1*r^5*exp(5.*x)*x^7-.3653555672*r^4*exp(5.*x)*x^8-
939.5636125*r^2*exp(6.*x)*x^9+573.0871675*r^2*exp(6.*x)*x^10+0.5268108378e-4*r^8*exp(5.*x)*x^4-0.7314038979e-
3*r^7*exp(5.*x)*x^5-16.36257291*r^6*exp(6.*x)*x^5+109.4794146*r^5*exp(6.*x)*x^6-
395.2774488*r^4*exp(6.*x)*x^7+809.5876411*r^3*exp(6.*x)*x^8+1.839223188*r^4*exp(2.*x)*x^4-
2.041239456*r^3*exp(2.*x)*x^5-1.738320610*r^2*exp(2.*x)*x^6+3.666954441*r^2*exp(2.*x)*x^7-
8628.486805*r^2*exp(3.*x)*x^6+0.1842863975e-1*r^6*exp(2.*x)*x^2-.5673385090*r^5*exp(2.*x)*x^3-.4138237367*r^2*x^10-
.3882322726*r^3*x^10-0.4393701968e-5*r^9*x^10-0.1229819476e-3*r^8*x^10-0.1459812287e-2*r^7*x^10-0.9862228465e-
2*r^6*x^9+0.4326634935e-1*r^5*x^10-.1315447868*r^4*x^10-.2803533192*r^3*x^10+0.2817779562e-
2*r^6*x^9+0.1236181410e-1*r^5*x^9+0.3758422481e-1*r^4*x^9+0.8010094835e-
1*r^3*x^9+.1182353533*r^2*x^9+.1109235065*r*x^9-
4023.662933*exp(x)*r^2*x^5+1180.760646*exp(x)*r*x^6+4149.831183*exp(x)*r^4*x^2-
12492.20079*exp(x)*r^3*x^3+12070.98880*exp(x)*r^2*x^4-4723.042584*exp(x)*r*x^5-
74.14719453*r^6*exp(3.*x)*x^8+140.5186870*r^5*exp(3.*x)*x^9-
158.6186204*r^4*exp(3.*x)*x^10+152.5983335*r^2*exp(3.*x)*x^11+0.1809742766e-3*r^8*exp(2.*x)*x^7-0.5224328193e-
3*r^7*exp(2.*x)*x^8-0.4658107994e-3*r^6*exp(2.*x)*x^9-65.43033387*r^3*exp(9.*x)*x^6+151.4244870*r^2*exp(9.*x)*x^7-
170.3525478*r^2*exp(9.*x)*x^8-327.9629648*r^5*exp(3.*x)*x^8+424.6006600*r^4*exp(3.*x)*x^9-
332.0391579*r^3*exp(3.*x)*x^10-1.192977517*10^(-7)*r^8*exp(8.*x)*x^2+.1314378641*exp(x)*r^6*x^8-
.1216371704*exp(x)*r^5*x^9+1.155553119*exp(x)*r^5*x^8-.5851425808*exp(x)*r^4*x^9+5.558854518*exp(x)*r^4*x^8-
1.761449633*exp(x)*r^3*x^9+16.73377151*exp(x)*r^3*x^8-3.404114157*exp(x)*r^2*x^9+32.33908449*exp(x)*r^2*x^8-
3.995805908*exp(x)*r*x^9+37.96015613*exp(x)*r*x^8-.1321311429*exp(8.*x)*x^14-
70.34197405*exp(9.*x)*x^12+.3065881334*exp(8.*x)*x^13-3.937117609*exp(6.*x)*x^15+80.94375884*exp(9.*x)*x^11-
.5388782263*exp(8.*x)*x^12+120.5276894*exp(4.*x)*x^7-1104.209279*exp(3.*x)*x^8+.5329995405*exp(2.*x)*x^9-
0.2174201525e-2*exp(8.*x)*x^2-2.309775387*exp(7.*x)*x^3+93.09283540*exp(6.*x)*x^4+9.147654733*exp(5.*x)*x^5-
151.1307965*exp(4.*x)*x^6+2208.467401*exp(3.*x)*x^7-
1.874381717*exp(2.*x)*x^8+.3299679124*exp(7.*x)*x^2+52.81171591*exp(7.*x)*x^8-
287.0736043*exp(6.*x)*x^9+.7586009462*exp(5.*x)*x^10-15.40620635*exp(3.*x)*x^12+9.347190553*exp(9.*x)*x^5-
.3710637270*exp(8.*x)*x^6-46.08017334*exp(7.*x)*x^7+335.1001136*exp(6.*x)*x^8-.1080861379*exp(5.*x)*x^9-
15.74782141*exp(4.*x)*x^10-176.7206816*exp(3.*x)*x^10-140.7045353*exp(6.*x)*x^11+13.38390445*exp(2.*x)*x^5-
14.17087252*exp(4.*x)*x^2-19.46203079*exp(2.*x)*x^4+19.46203079*exp(2.*x)*x^3-
3.045033166*exp(7.*x)*x^13+14.34338461*exp(6.*x)*x^14-
81.88951554*exp(9.*x)*x^10+.7584369853*exp(8.*x)*x^11+9.921254546*exp(7.*x)*x^12-
38.08278113*exp(6.*x)*x^13+72.98486976*exp(9.*x)*x^9-.8926411169*exp(8.*x)*x^10-
22.47544770*exp(7.*x)*x^11+79.99310745*exp(6.*x)*x^12-
56.78418261*exp(9.*x)*x^8+.9032544689*exp(8.*x)*x^9+37.88328097*exp(7.*x)*x^10-31.03094513*exp(6.*x)*x^3-
5.673163152*exp(5.*x)*x^4+151.1496793*exp(4.*x)*x^5-.5256958008*exp(9.*x)*x^18+2.346450806*exp(9.*x)*x^17-
7.423046112*exp(9.*x)*x^16+17.79852464*exp(9.*x)*x^15-33.86779352*exp(9.*x)*x^14+0.4004517104e-
1*exp(8.*x)*x^15+53.01752403*exp(9.*x)*x^13+.6923844854*exp(9.*x)*x^3-0.6957444880e-1*exp(8.*x)*x^4-
18.86316566*exp(7.*x)*x^5+279.2785062*exp(6.*x)*x^6+7.097222417*exp(5.*x)*x^7-
78.48697028*exp(4.*x)*x^8+473.0823385*exp(3.*x)*x^9-0.7693160949e-1*exp(9.*x)*x^2+0.1739361220e-
1*exp(8.*x)*x^3+8.084213854*exp(7.*x)*x^4-186.1856708*exp(6.*x)*x^5-9.997556697*exp(5.*x)*x^6-
9.731015394*exp(2.*x)*x^2+57.39080628*exp(3.*x)*x^11-
3.115730184*exp(9.*x)*x^4+.1855318635*exp(8.*x)*x^5+33.01053990*exp(7.*x)*x^6-335.1287000*exp(6.*x)*x^7-
2.706532019*exp(5.*x)*x^8+40.83063827*exp(4.*x)*x^9+4907.440366*exp(3.*x)*x^5-1090.057928*exp(3.*x)*x^2-
4906.737241*exp(3.*x)*x^4+3270.595660*exp(3.*x)*x^3+4.395493376*exp(2.*x)*x^7+5.171824189*exp(6.*x)*x^2+2.269265
261*exp(5.*x)*x^3-113.3669802*exp(4.*x)*x^4-7.919604001*exp(2.*x)*x^6-
.4538530522*exp(5.*x)*x^2+56.68349008*exp(4.*x)*x^3-3680.791212*exp(3.*x)*x^6

$$\underline{u}(x, \alpha) = \underline{u}_0 + p^1 \underline{u}_1 + \dots$$

$$\bar{u}(x, \alpha) = \bar{u}_0 + p^1 \bar{u}_1 + \dots$$

where $p=1$

Table 2. HPM solve the equation 2 and comperd with exact solution with different value for α for lower side (A)

x	Exact $\underline{u}(x, \alpha)$ $\alpha = 0.1$	HPM $\underline{u}(x, \alpha)$ $\alpha = 0.1$	Exact $\underline{u}(x, \alpha)$ $\alpha = 0.4$	HPM $\underline{u}(x, \alpha)$ $\alpha = 0.4$	Exact $\underline{u}(x, \alpha)$ $\alpha = 0.7$	HPM $\underline{u}(x, \alpha)$ $\alpha = 0.7$	Exact $\underline{u}(x, \alpha)$ $\alpha = 1$	HPM $\underline{u}(x, \alpha)$ $\alpha = 1$
0	1.225000000	1.225000000	1.150000000	1.150000000	1.075000000	1.075000000	1.000000000	1.000000000
0.2	1.496218379	1.496564954	1.404613172	1.405264770	1.313007965	1.314079386	1.2211402758	1.223039258
0.4	1.827415255	1.825885016	1.715598403	1.714274248	1.603711550	1.602953490	1.4918246980	1.491928520
0.6	2.232095530	2.215862000	2.095436620	2.075827800	1.958777771	1.935144458	1.8221188000	1.795144326
0.8	2.726287637	2.656580000	2.559372067	2.470980000	2.392456498	2.277277305	2.2255409280	2.082724609
1.0	3.329895240	3.102925446	3.126024103	2.829797657	2.922152966	2.544335025	2.7182818280	2.255057922

HPM solve the equation 2 and comperd with exact solution with different value for α for upper side (B)

x	Exact $\bar{u}(x, \alpha)$ $\alpha = 0.1$	HPM $\bar{u}(x, \alpha)$ $\alpha = 0.1$	Exact $\bar{u}(x, \alpha)$ $\alpha = 0.4$	HPM $\bar{u}(x, \alpha)$ $\alpha = 0.4$	Exact $\bar{u}(x, \alpha)$ $\alpha = 0.7$	HPM $\bar{u}(x, \alpha)$ $\alpha = 0.7$	Exact $\bar{u}(x, \alpha)$ $\alpha = 1$	HPM $\bar{u}(x, \alpha)$ $\alpha = 1$
0	0.775000000	0.775000000	0.850000000	0.850000000	0.925000000	0.925000000	1.000000000	1.000000000
0.2	0.9465871374	0.946899184	1.038192344	1.038447314	1.129797551	1.129986521	1.221402758	1.221513902
0.4	1.1561641410	1.156232753	1.268050993	1.267959941	1.379937846	1.379596853	1.491824698	1.491120485
0.6	1.4121420700	1.408065360	1.548800980	1.544884944	1.685459890	1.684773740	1.822118800	1.817792636
0.8	1.7247942190	1.706040827	1.891709789	1.875375185	2.058625358	2.044619583	2.225540928	2.213780457
1.0	2.1066689170	2.050912291	2.310539559	2.262256760	2.514410691	2.473557889	2.718281828	2.684895333

Fig 3. comperd the HPM Fig 3a. with the exact solution Fig 3b. for lower side

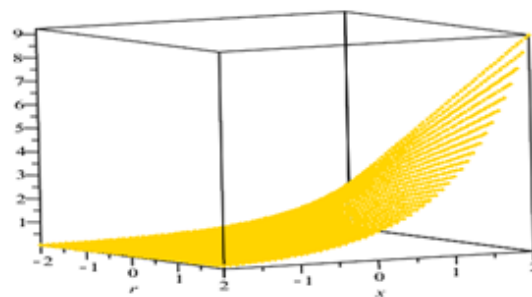
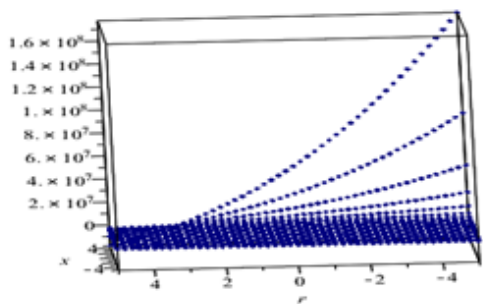


Fig 3a. is present the HPM

Fig 3b. is present the exact solution

Fig 4. comperd the HPM Fig 4(a) with the exact solution Fig 4b. for lower side

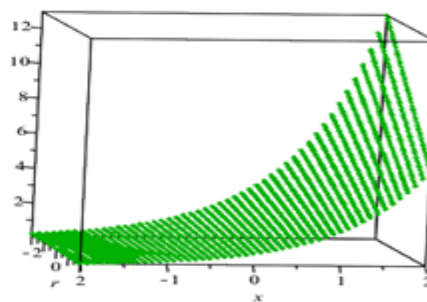
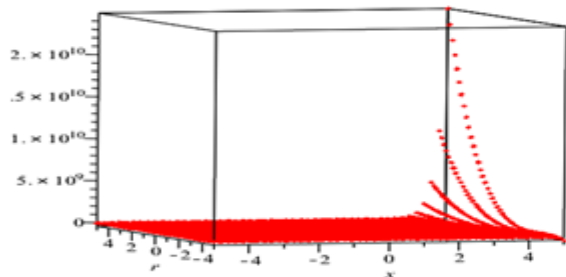


Fig 4a. is present the HPM

Fig 4b. is present the exact solution

5. Conclusion

In this paper, the homotopy perturbation method has been successfully applied to find the solution of nonlinear fuzzy volterra integral classes equation. It is apparently seen that HPM is a very powerful and efficient technique in finding analytical solutions for complex classes of nonlinear problems. It is worth pointing out that this method presents a convergence for the solutions. The computations associated with the examples in this letter were performed using Maple 18

References

[1] C. Chun. Integration using He's homotopy perturbation method. Chaos, Solitons & Fractals, 34(4):1130–1134, 2007.
 [2] L. Cveticanin. Homotopy perturbation method for pure nonlinear differential equation. Chaos, Solitons & Fractals, 30:1221–1230, 2006.

[3] R. Goetschel and W. Voxman. Elementary calculus. Fuzzy Sets and Systems, 18:31–34, 1986.
 [4] J. H. He. Homotopy perturbation technique. Comput. Methods Appl. Mech. Eng., 178(3-4):257–262, 1999.
 [5] J. H. He. A coupling method of homotopy technique and perturbation technique for nonlinear problems. Int. J. Nonlinear Mech., 35(3):37–43, 2000.
 [6] C. Hillermeier. Generalized homotopy approach to multiobjective optimization. Int. J. Optim. Theory Appl., 110(3):557–583, 2001.
 [7] R. Goetschel and Vaxman, Elementary fuzzy clculus, Fuzzy Sets and Systems, 18 (1986), 31-43
 [8] O. Kaleva. Fuzzy differential equations. Fuzzy Sets and Systems, 24:301–317, 1987.

- [9] S. J. Liao. An approximate solution technique not depending on small parameters: a special example. *Int. J. Non-Linear Mech.*, 30(3):371–380,1995.
- [10] S. J. Liao. Boundary element method for general nonlinear differential operators. *Eng. Anal. Boundary Element*, 20(2):91–99, 1997.
- [11] A. H. Nayef. *Problem in perturbation*. John Wiley, Stateplace, New York, 1985.
- [12] H. T. Nguyen. A note on the extension principle for fuzzy sets. *J. Math. Anal. Appl.*, 64:369–380, 1978.
- [13] T. Ozis and A. Yildirim. A note on He's homotopy perturbation method for van der pol oscillator with very strong nonlinearity. *Chaos, Solitons & Fractals*, 34(3):989–991, 2007.
- [14] M. L. Puri and D. Ralescu. Differential for fuzzy function. *J. Math. Anal. Appl.*, 91:552–558, 1983.
- [16] M. L. Puri and D. Ralescu. Fuzzy random variables. *J. Math. Anal. Appl.*, 114:409–422, 1986.
- [15] A. M. Siddiqui, R. Mahmood, and Q. K. Ghori. Homotopy perturbation method for thin film flow of a third grade fluid down an inclined plane. *Chos, Solitons & Fractals*, 35(1):140–147, 2008.
- [16] C. Wu and M. Ma. On the integrals, series and integral equations of fuzzy set-valued functions. *J. Harbin Inst. Technol*, 21:11–19, 1990. [7] Hany. N. Magdy. A, A new technique of using Homotopy Analysis Method for Second Order Non Linear Differential Equation, Department of Basic Science, Faculty of Engineering of Branch Benha University, Egypt (2012)
- [17] N. A. Rajab, A. M. Ahmad, O. M. Alfaour, Reduction Formula for Linear Fuzzy Equation, Applied Science Department, University of Technology Baghdad – Iraq (2013)
- [18] Sushila Rathora, Devendra Kumar, Jagdev Singh, Sumit Gupta, Homotopy Analysis Method for Non Linear Equation, Department of Mathematics, Jagon Meth, University Village – Rampun Tehsil Chaksu, Jaipur – 303901, Ragashtan – India (2012)
- [19] S. Abbasbandy, E. Babolian, M. Alavi, Numerical method for solving linear Fredholm fuzzy integral equations of the second kind, *Chaos Soliton and Fractals* 31 (2007) 138146.
- [20] H.C. Wu, The improper fuzzy Riemann integral and its numerical integration, *Information Science* 111 (1999) 109-137.
- [21] S. Abbasbandy and A. Jafarian, “Steepest descent method for solving fuzzy nonlinear equations,” *Applied Mathematics and Computation*, vol. 174, no. 1, pp. 669–675, 2006.
- [22].H.C.Wu, The proper fuzzy Riemann integral and its numerical integration, *Information Science* 111(1999) 109-137.
- [23] G. J. Klir, U. St. Clair, and B. Yuan, *Fuzzy Set Theory: Foundations and Applications*, Prentice-Hall, Eaglewood Cliffs ,NJ, USA, 1997.
- [24] J.Y. Park, Y.C. Kwan, J.V. Jeong, Existence of solutions of fuzzy integral equations in Banach spaces, *Fuzzy Sets and System* 72 (1995) 373-378
- [25]. Chang, S. S. L., & Zadah, L. A. (1972). On fuzzy mapping and control. *Trans. Systems, Man Cyberetics*, SMC-2(1), 30-34.
- [26].J.Y.Park, Y.C.Kwan,J.V. Jeong, Existence of solution of fuzzy integral equations in Banach spaces, *Fuzzy Sets and System* 72 (1995) 373- 378.
- [27] D. Dubois, H. Prade, Operation on fuzzy numbers, *Int. J. system Science* 9 (1978) 613-626 <http://dx.doi.org/10.1080/00207727808941724>
- [28]. Nanda, S.(1989). On integration of fuzzy mappings. *Fuzzy Sets and Systems*, 32, 95-101.