Available online at www.elixirpublishers.com (Elixir International Journal)

Discrete Mathematics

Elixir Dis. Math. 96 (2016) 41669-41672

ISSN: 2229-712X

Minimal Total Dominating Color Transversal Set of Generalised Petersen Graph P (n, 1)

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ARTICLE INFO

Article history:

Received: 9 June 2016;

Received in revised form: 13 July 2016;

Accepted: 18 July 2016;

Keywords

Total Dominating Color Transversal Set,

Minimal Total Dominating Color Transversal Set,

χ – Partition of a graph.

ABSTRACT

Total Dominating Color Transversal Set of a graph is a Total Dominating Set which is also Transversal of Some χ - Partition of vertices of G. Here χ is the Chromatic number of the graph G. A Total Dominating Color Transversal Set of a graph G is called Minimal Total Dominating Color Transversal Set of the graph if no proper subset of it is a Total Dominating Color Transversal Set of G. In this paper, we determine a necessary and sufficient condition under which a Total Dominating Color Transversal Set becomes Minimal. We also obtain Minimal Total Dominating Color Transversal set of Generalised Petersen graph P (n, 1).

1.Introduction

We begin with simple, finite, connected and undirected graph without isolated vertices. We know that proper coloring of vertices of graph G partitions the vertex set V of G into equivalence classes (also called the color classes of G). Using minimum number of colors to properly color all the vertices of G yields χ equivalence classes. Transversal of a χ - Partition of G is a collection of vertices of G that meets all the color classes of the χ – Partition. That is, if T is a subset of V(the vertex set of G) and $\{V_1, V_2, \ldots, V_\chi\}$ is a χ - Partition of G then T is called a Transversal of this χ - Partition if $T \cap V_i \neq \emptyset$, $\forall\, i \in \{1, 2, \ldots, \chi\}$.Total Dominating Color Transversal Set of graph G is a Total Dominating Set with the extra property that it is also Transversal of some such χ - Partition of G.

We first mention definitions.

2. Definitions

Definition 2.1[4]: (Total Dominating Set) Let G = (V, E) be a graph. Then a subset S of V (the vertex set of G) is said to be a Total Dominating Set of G if for each $v \in V$, v is adjacent to some vertex in S.

Definition 2.2[4]: (Minimum Total Dominating Set/Total Domination number) Let G = (V, E) be a graph. Then a Total Dominating set S is said to be a Minimum Total Dominating set of G if $|S|$ = minimum $\{|D|$: D is a Total Dominating set of G$\}$. Here S is called γ_t –set and its cardinality, denoted by γ_t(G) or just by γ_t, is called the Total Domination number of G.

Definition 2.3[1]: (χ -partition of a graph) Proper coloring of vertices of a graph G, by using minimum number of colors, yields minimum number of independent subsets of vertex set of G called equivalence classes (also called color classes of G). Such a partition of a vertex set of G is called a χ - Partition of the graph G.

Definition 2.4[1]: (Transversal of a χ - Partition of a graph) Let G = (V, E) be a graph with χ – Partition $\{V_1, V_2, \ldots, V_\chi\}$. Then a set $S \subset V$ is called a Transversal of this χ – Partition if $S \cap V_i \neq \emptyset$, $\forall\, i \in \{1, 2, 3, \ldots, \chi\}$.

Definition 2.5[1]: (Total Dominating Color Transversal Set) Let G = (V, E) be a graph. Then a Total Dominating Set $S \subset V$ is called a Total Dominating Color Transversal Set of G if it is Transversal of at least one χ – partition of G.

Definition 2.6[1]: (Minimal Total Dominating Color Transversal Set) Let G = (V, E) be a graph and $S \subset V$ be a Total Dominating Transversal Set of G. Then S is called Minimal Total Dominating Color Transversal of G if no proper subset of S is a Total Dominating Color Transversal Set of G.

Definition 2.7 [5]: (Generalised Petersen Graph) Let n, k be the positive integers such that $n \geq 3$ and $1 \leq k \leq [\frac{n}{2}]$. The generalised Petersen graph P (n, k) is the graph whose vertex set is $\{a_i, b_i \,/\, 1 \leq i \leq n\}$ and whose edge set is $\{\{a_i, b_i\}, \{a_i, a_{i+1}\}, \{b_i, b_{i+k}\} / 1 \leq i \leq n\}$ where $a_{n+c} = a_c$ and $b_{n+c} = b_c$ for every $c \geq 1$.

3. Main results

First we state the following theorem taken from [1].

Theorem 3.1 [1]: If G is a graph with χ (G) = 2 then γ_{tstd}(G) = γ_t(G).

Remark 3.2: Let G be a graph with χ (G) = 2. Then any Total Dominating Set of G will be a Transversal of every χ - Partition of G. Hence any Total Dominating Set of G will be Total Dominating Color Transversal Set of G.

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We now state necessary and sufficient condition under which Total Dominating Color Transversal Set of a graph is Minimal.

Theorem 3.3: A Total Dominating Color Transversal Set D of a graph G = (V, E) is Minimal iff for every u $\in$ D at least one of the following conditions hold:

I) There exists v $\in$ V $\setminus$ {u} such that N(v) $\cap$ D = {u}.

II) For every χ- Partition $\{V_1, V_2, V_3, \ldots V_\chi\}$ there exists V_i such that $V_i \cap$ D = {u} or ϕ.

Proof: Let D be a Total Dominating Color Transversal Set of a graph G. First assume that D is a Minimal Total Dominating Color Transversal Set of G. Then for every u $\in$ D, D $\setminus$ {u} is not a Total Dominating Color Transversal Set of G. Then D $\setminus$ {u} is not a Total Dominating Set of G or D $\setminus$ {u} is not a transversal for every χ- Partition of G.

Case (i): D $\setminus$ {u} is not a Total Dominating of G.

In such case there exists v $\in$ V such that it is not adjacent to any vertex of D $\setminus$ {u}. Note that v $\neq$ u as if v = u then u is an isolate of D which is not possible as D is a Total Dominating Set of G. Hence we are left with two possibilities viz., (a) v $\in$ V $\setminus$ D or (b) v $\in$ D $\setminus$ {u}.

(a) v $\in$ V $\setminus$ D. v is not adjacent to any of D $\setminus$ {u}. v $\in$ V $\setminus$ D and D is a Total Dominating set together implies that v is adjacent to u only. Hence N(v) $\cap$ D = {u}.

(b) v $\in$ D $\setminus$ {u}. v is not adjacent to any of D $\setminus$ {u} implies that v is an isolate of D $\setminus$ {u}. As D is a Total Dominating set, v is adjacent to u only in D. Hence N(v) $\cap$ D = {u}.

Case (ii): D $\setminus$ {u} is not a transversal of every χ- Partition $\{V_1, V_2, V_3, \ldots V_\chi\}$ of G.

In such case D $\setminus$ {u} $\cap$ V_i = ϕ, for some i $\in$ {1, 2,….,n}. Hence D $\cap$ V_i = ϕ or {u} for some i $\in$ {1, 2,….,n}.

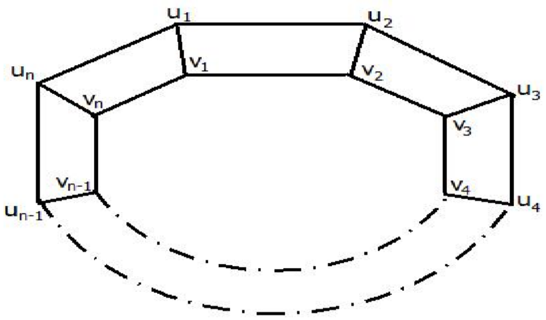
Conversely assume at least one of the two conditions. Suppose D is Total Dominating Color Transversal Set of G but not a Minimal Total Dominating Color Transversal Set of G. Hence D and D $\setminus$ {u} are total Dominating Color Transversal Sets of G for some u $\in$ D. As D and D $\setminus$ {u} are Total Dominating Sets of G there does not exist any v $\in$ V $\setminus$ {u} such that N(v) $\cap$ D = {u}. So by our assumption condition (II) must hold.

Consider χ- Partition $\{V_1, V_2, V_3, \ldots V_\chi\}$ of G such that D $\setminus$ {u} and D are transversals of it. Then D $\setminus$ {u} $\cap$ V_i $\neq$ ϕ, and D $\cap$ V_i $\neq$ ϕ for every i $\in$ {1, 2,….,n}. This means that D $\cap$ V_i $\neq$ ϕ, {u}. Therefore condition (II) also fails to hold. Hence both the conditions fails to hold, which is a contradiction.

Hence D is a Minimal Total Dominating Color Transversal Set of G.

Hence the theorem. ■

Now we obtain the Minimal Total Dominating Color Transversal Set of very Interesting and Special Graph called Generalised Petersen Graph P (n, 1).



Generalised Petersen Graph P (n, 1)

Figure 1

Proposition 3.4 - χ (P (n, 1)) = 2 if n is even

= 3 if n is odd.

Proof: Consider the above drawn Generalised Petersen Graph P (n, 1). In our proof color pair (i, j) assigned to vertex pair (u, v) will mean colors i and j are assigned to u and v respectively.

Case 1**:** n is even

Assign color pair (1, 2) to each pairs (u_1, v_1), (u_3, v_3), ……., (u_{n-1}, v_{n-1}) and color pair (2, 1) to (u_2, v_2), (u_4, v_4), ……. ,(u_n, v_n).

Therefore in this case χ (P (n, 1)) = 2.

Case 2**:** n is odd

In this case P (n, 1) contains odd cycle. So at least three colors are required to color it. We Prove that exactly three colors are required to properly color P (n, 1).

Divide then vertices pairs (u_i, v_i) into groups of three pairs are $\{(u_1, v_1), (u_2, v_2), (u_3, v_3)\}$, $\{(u_4, v_4), (u_5, v_5), (u_6, v_6)\}$,……..where the last group may contain one, two or three pairs.

Sub Case 1: Last group has just one pair.

In such case, n $\geq$ 4. Then P (n, 1) can be properly colored by using three colors 1, 2 and 3 as {(1, 2), (2, 3), (3, 1)}, {(2,, 3), (3, 1), (1,2)},……. ,{(2,3)}.

Sub Case 2. Last group has two pairs.

In such case, n $\geq$ 5. hen P (n, 1) can be properly colored by using three colors 1, 2 and 3 as {(1, 2), (2, 3), (3, 1)}, {(2,, 3), (3, 1), (1,2)},……. ,{(2,3), (3,1)}.

Sub Case 3. Last group has three pairs.

In such case n $\geq$ 3. Then P (n, 1) can be properly colored by using three colors 1, 2 and 3 as {(1, 2), (2, 3), (3, 1)}, {(1, 2), (2, 3), (3, 1)}, ……………, {(1, 2), (2, 3), (3, 1)}.

Therefore in this case χ (P (n, 1)) = 3. ■

Result 3.5

1) If n is even and n $\equiv$ 0 (mod 3) then 3k = n for some even k.

2) If n is odd and n $\equiv$ 0 (mod 3) then 3k = n for some odd k.

Theorem 3.6: A Minimal Total Dominating Color Transversal Set of the generalised Petersen graph P (n, 1) is as follows:

⇨ When n is even

$$D = \begin{cases} \{u_1, u_2, v_4, v_5, u_7, u_8, v_{10}, v_{11}, \ldots\ldots\ldots, u_{n-5}, u_{n-4}, v_{n-2}, v_{n-1}\} \text{ with } |D| = \frac{2n}{3}, n \equiv 0 \pmod 3 \\ \{u_1, u_2, v_4, v_5, u_7, u_8, v_{10}, v_{11}, \ldots\ldots\ldots, u_{n-3}, u_{n-2}, v_n, v_1\} \text{ with } |D| = \frac{2n+4}{3}, n \equiv 1 \pmod 3 \\ \{u_1, u_2, v_4, v_5, u_7, u_8, v_{10}, v_{11}, \ldots\ldots\ldots, v_{n-4}, v_{n-3}, u_{n-1}, u_n\} \text{ with } |D| = \frac{2n+2}{3}, n \equiv 2 \pmod 3 \end{cases}$$

⟹ When n is odd

$$D=\begin{cases}\{u_1,u_2,u_3\} \text{ with } |D|=3, (n=3)\\ \{u_1,u_2,v_4,v_5,u_7,u_8,v_{10},v_{11},\ldots\ldots,u_{n-5},u_{n-4},v_{n-2},v_{n-1}\} \text{ with } |D|=\frac{2n}{3}, n\equiv 0 (\text{mod } 3)(n>3)\\ \{u_1,u_2,v_4,v_5,u_7,u_8,v_{10},v_{11},\ldots\ldots,v_{n-3},v_{n-2},u_n\} \text{ with } |D|=\frac{2n+1}{3}, n\equiv 1 (\text{mod } 3)\\ \{u_1,u_2,v_4,v_5,u_7,u_8,v_{10},v_{11},\ldots\ldots,u_{n-4},u_{n-3},v_{n-1},v_n\}\ |D|=\frac{2n+2}{3}, n\equiv 2 (\text{mod } 3)\end{cases}$$

Proof: Consider the Generalized Petersen Graph drawn in Figure 1.

Divide the vertices into groups of three pairs as $\begin{pmatrix}u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{pmatrix}$, $\begin{pmatrix}u_4 & u_5 & u_6\\ v_4 & v_5 & v_6\end{pmatrix}$, ... where the last group may have one, two or three pairs.

Case 1: n is even

We know by Proposition 3.4 that P (n, 1) is bipartite. So obviously any Total Dominating Set of P (n, 1) is a Total Dominating Color Transversal Set of P (n, 1) .

Sub Case 1(a): n $\equiv$ 0 (mod 3)

By above Result 3.5, the number of groups of three pairs will be even. From each odd group of three pairs select first two u_i's and from each even group of three pairs select first two v_i's. The resultant set D= $\{u_1, u_2, v_4, v_5, u_7, u_8, v_{10}, v_{11}, \ldots\ldots\ldots, u_{n-5}, u_{n-4}, v_{n-2}, v_{n-1}\}$ is a Minimal Total Dominating Color Transversal Set of P (n, 1) with $|D|=\frac{2n}{3}$.

Sub Case 1(b): n $\equiv$ 1 (mod 3)

In this case n $\geq$ 4.

Note that as n-1 $\equiv$ 0 (mod 3) and n-1 is odd, by Result 3.5, the number of groups of three pairs will be odd and last group will have just one pair $\begin{pmatrix}u_n\\ v_n\end{pmatrix}$. From each odd groups of three pairs select first two u_i's and from each even group of three pairs select first two v_i's. Also select v_n and v_1. In this case note that the last group of three pairs of vertices is an even group. The resultant set D = $\{u_1, u_2, v_4, v_5, u_7, u_8, v_{10}, v_{11}, \ldots\ldots\ldots, u_{n-3}, u_{n-2}, v_n, v_1\}$ is a Minimal Total Dominating Color Transversal Set of P (n, 1) with $|D|=\frac{2(n-1)}{3}+2=\frac{2n+4}{3}$.

Sub Case 1(c): n $\equiv$ 2 (mod 3)

In this case n $\geq$ 8.

Note that as n - 2 $\equiv$ 0 (mod 3) and n – 2 is even, by Result 3.5, the number of groups of three pairs will be even and the last group will have $\begin{pmatrix}u_{n-1} & u_n\\ v_{n-1} & v_n\end{pmatrix}$. From each odd groups of three pairs select first two u_i's and from each even group of three pairs select first two v_i's. Also select u_{n-1} and u_n. In this case note that the last group of two pairs of vertices is an even group.

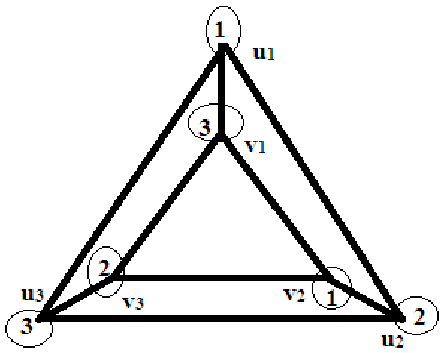
The resultant set D= $\{u_1, u_2, v_4, v_5, u_7, u_8, v_{10}, v_{11}, \ldots\ldots\ldots, v_{n-4}, v_{n-3}, u_{n-1}, u_n\}$ is a Minimal Total Dominating Color Transversal Set of P (n, 1) with $|D|=\frac{2(n-2)}{3}+2=\frac{2n+2}{3}$.

Case 2: n is odd

Sub Case 2(a): n $\equiv$ 0 (mod 3)

In this Case n $\geq$ 3.

⟹ For n = 3



P (3, 1) Figure 2

D = $\{u_1, u_2, u_3\}$ is a Minimal Total Dominating Color Transversal Set of P (3, 1) with proper 3 – coloring is shown in Figure 2.

⟹ For n > 3 (That is in this case n $\geq$ 9)

By Result 3.5, number of groups of three pairs will be odd. Select the vertices as in Sub Case 1(a), we get a Minimal Total Dominating Color Transversal Set of P (n, 1) with $|D|=\frac{2n}{3}$ with proper 3 - coloring of vertices of P (n, 1) as follows: $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$, ...$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$.

Sub Case 2(b): n $\equiv$ 1 (mod 3)

In this Case n $\geq$ 7.

Here as n -1 $\equiv$ 0 (mod 3) and as n - 1 is even by Result3.5, the number of groups of three pairs will be even. From each odd group of three pairs select first two u_i's and from each even group of three pairs select first two v_i's. The last group, which is an odd group, will have $\begin{pmatrix}u_n\\ v_n\end{pmatrix}$. From this group select u_n. The resultant set D = $\{u_1, u_2, v_4, v_5, u_7, u_8, v_{10}, v_{11}, \ldots\ldots\ldots, v_{n-3}, v_{n-2}, u_n\}$ is a Minimal Total Dominating Color Transversal Set of P (n, 1) with $|D|=\frac{2(n-1)}{3}+1=\frac{2n+1}{3}$ with proper 3 - coloring of P (n, 1) is same as follows: $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$, $\begin{pmatrix}2 & 3 & 1\\ 3 & 1 & 2\end{pmatrix}$,......$\begin{pmatrix}2\\ 3\end{pmatrix}$.

Sub Case 2(c): n $\equiv$ 2 (mod 3)

In this case n $\geq$ 5.

Here as n - 2 $\equiv$ 0 (mod 3) and n -2 is odd by Result 3.5, the number of groups of three pairs will be odd. Select first two u_i's and from each even group of three pairs select first two v_i's. From the last group, which is an even group, will have $\begin{pmatrix}u_{n-1} & u_n\\ v_{n-1} & v_n\end{pmatrix}$. Select v_{n-1} and v_n from it. The resultant set D = $\{u_1, u_2, v_4, v_5, u_7, u_8, v_{10}, v_{11}, \ldots\ldots\ldots, u_{n-4}, u_{n-3}, v_{n-1}, v_n\}$ is a Minimal Total Dominating Color Transversal of P(n, 1) with $|D|=\frac{2(n-2)}{3}+2=\frac{2n+2}{3}$ with proper 3 – coloring of P (n, 1) is as follows:

$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$,...... $\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}$. ■

4. Concluding Remarks

Minimal Total Dominating Color Transversal Set of Generalised Petersen graph P (n, 1) is in fact Minimal Total Dominating Set of the graph. This set for Generalised Petersen Graph P (n, k) (k > 2) is still to be obtained and it becomes an open problem.

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