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# Frattini Subgroups and its properties

Saurabh Singla
IGN college Ladwa(kkr).

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**ABSTRACT**

This manuscript deals with the Frattini Subgroups and some results based on frattini subgroups. In this paper we define the Frattini subgroup and then discuss some of its properties. We denote it by $Fr(G)$. In this we show that Fr(G) consist of non genrator elements of G. If G is finite group then Fr(G) is nilpotent and Frattini subgroup of a normal subgroup N is always contained in Frattini subgroup of a group.



**Introduction**

**Frattini subgroup**

The Frattini subgroup of a group is the intersection of all maximal subgroups of $G$ i.e.

$Fr(G)=\bigcap_{\lambda\in\Lambda} H_\lambda$ where $\{H_\lambda \mid \lambda\in\Lambda\}$ is the class of all maximal subgroups of *G*. If there is no maximal subgroup of *G*. Then we defined $Fr(G)=G$, for example $Fr(Q)=Q$. $Fr(G)$ is characteristic subgroup of *G* and hence normal in *G*. ($\because$ image of a maximal subgroup under an automorphism is again a maximal subgroup).

**Theorem**

Let $G$ be a group, then $Fr(G)$ consists of all non generator of *G*.

**Proof**

Let $x\in G$ be a non generator element, we prove that $x\in Fr(G)$.

Suppose $x\notin Fr(G)=\bigcap_{\lambda\in\Lambda} M_\lambda$ where $\{M_\lambda \mid \lambda\in\Lambda\}$ is the class of all maximal subgroup of *G*. Therefore there exists a maximal subgroup $M$ of $G$ such that $x\in M$. Now $<M,\ x> \underset{\neq}{\supseteq} M$ and $M$ is a maximal subgroup of *G*. Therefore $G=<M,\ x>$. Since $x$ is a non generator element of *G*, therefore $G=<M>=M$ a contradiction. Thus $x\in Fr(G)$. Hence $Fr(G)$ consist of all non generator elements of *G*. Conversely let $y\in Fr(G)$ and $G=<X,\ y>$ where $X$ is a subset of *G*. We prove that $G=<X>$. Suppose $S=<X>$ and $S\underset{\neq}{\subset} G$. Consider the set $f$ of all subgroups of $G$ defined as follows $f=\{H \mid H\subseteq G,\ S\subseteq H\ \ y\notin H\}$ Since $S\neq G$ and $G=<X,\ y>$, therefore $y\notin S$. Hence $S\in f$, therefore $f\neq\phi$. $f$ is a partially ordered set under the set theoretic inclusion. In *f* every chain has an upper bound in *f*. Thus by Zorn's lemma *f* has a maximal element say $H_0$. We prove that $H_0$ is a maximal subgroup of *G*. Suppose there exists $H_1\subseteq G$ such that $H_0 \underset{\neq}{\subseteq} H_1\subseteq G$. If $y\notin H_1$ then $H_1\in f$. But this contradicts the maximality of $H_0$. Hence $y\in H_1$ implies that

$H_1\supseteq <X,\ y>=G$ implies that

$H_1=G$

Hence $H_0$ is a maximal subgroup of *G* and $y\notin H_0$ again a contradiction. Since $y\in Fr(G)$ therefore $G=<X>$. Therefore *y* is a non generator element of *G*.

**Theorem**

If *G* is a finite group then $Fr(G)$ is nilpotent.

**Proof**

Since $Fr(G)$ is a characteristic subgroup of *G* and hence a normal in *G*. Now $K=Fr(G)$ is a finite group. Let $K_p$ be a Sylow *p*-subgroup of $K$ then $G=KN(K_p)$. Since $K=Fr(G)$ consists of non generator element of *G*. Therefore $G=N(K_p)$. Hence $K_p\ \Delta\ G$. Thus every Sylow *p*-subgroup of *K* is normal in *G*. Therefore $K_p\ \Delta\ K$. Hence $K=Fr(G)$ is nilpotent.

**Theorem**

A finite group *G* is nilpotent iff $Fr(G)\supseteq\delta(G)$.

**Proof**

Suppose $G$ is a nilpotent group. We are to prove that $Fr(G)\supseteq\delta(G)$. Since $G$ is finite, therefore maximal subgroups of $G$ exists. Let *M* be a maximal subgroup of *G*. Since maximal subgroups of a nilpotent group are normal,

Tele: 9812068268
E-mail address: Saurabh.singla88@gmail.com

therefore $M \Delta G$ and $G | M$ has no proper subgroup. This implies that $G | M$ is of prime order. Therefore $G | M$ is an abelian group.But then $\delta(G) \subseteq M$. From it follows that $\delta(G) \subseteq Fr(G)$ Conversely, suppose $Fr(G) \supseteq \delta(G)$ We prove that *G* is nilpotent. Let $G_p$ be a Sylow *p*-subgroup of *G*, if $N(G_p) \neq G$. Then $N(G_p) \subseteq M$ for some maximal subgroup *M* of *G*. Now $Fr(G) \subseteq M$ implies that $\delta(G) \subseteq M$ and therefore $M \Delta G$. Again $N(G_p) \subseteq M$ implies that $M = N(M) \neq G$. Therefore *M* can not be normal in *G*, a contradiction. Hence $N(G_p) = G$. Therefore every Sylow *p*-subgroup of *G* is normal. Hence *G* is nilpotent.

**Theorem**

Let *U* be a subgroup of *G*, then $U \subseteq Fr(G)$ if for any proper subgroup *H* of *G*, $< H, U > \underset{\neq}{\subset} G$. …(1) Further if *G* is finitely generated then the above condition is necessary as well as sufficient. In particular for any subgroup of *G*, $H\ Fr(G) \underset{\neq}{\subset} G$. Suppose (1) hold the for a maximal subgroup *M* of *G*. $M \subseteq < U, M > \underset{\neq}{\subset} G$. Since *M* is a maximal subgroup of *G*. Therefore $M = < U, M >$. Hence $U \subseteq M$. Thus for every maximal subgroup *M* of $G$, $U \subseteq M$. But then $U \subseteq Fr(G)$. Next assume that *G* is finitely generated group and $U \subseteq Fr(G)$. We are to prove that $< U, H > \underset{\neq}{\subset} G$ for every proper subgroup of *G*. Let *H* be a proper subgroup of *G*. But since *G* is finitely generated, so there exists a maximal subgroup *M* of *G* such that $H \subseteq M$. Now $U \subseteq Fr(G) \subseteq M$. Hence $< U, H > \subseteq M \underset{\neq}{\subset} G$. Since $H \underset{\neq}{\subset} G$ and $Fr(G)$ consists of non generator elements of *G*, therefore $H\ Fr(G) \underset{\neq}{\subset} G$. This completes the proof.

**Theorem**

Let *G* be a group and *N* be finitely generated normal subgroup of *G*. Then

$Fr(N) \subseteq Fr(G)$

**Proof**

Since characteristic subgroup of a normal subgroups normal, therefore $Fr(N)$ is a normal subgroup of *G*. Let *M* be a maximal subgroup of *G* such that $Fr(N) \not\subseteq M$ .Since M is a maximal subgroup of *G*. therefore $Fr(N)\,M = G$. Since $Fr(N) \subseteq N$. Thus by using modular-law. $N = Fr(N)\,M \cap N = Fr(N)(M \cap N)$. Thus $M \cap N$ together with $Fr(N)$ generates *N*. Since $Fr(N)$ consists of non generator elements of *N*, therefore $M \cap N = N$. Hence $N \subseteq M$. But $Fr(N) \subseteq N \subseteq M$, a contradiction. Hence $Fr(N) \subseteq M$ for every maximal subgroup *M* of *G*. Therefore $Fr(N) \subseteq Fr(G)$ and this completes the proof.

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