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Wigner Distribution function for charged particle in external magnetic field

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ABSTRACT
In this article we express the wigner distribution function in convenient differential forms. In addition We find the temporal evolution of the distribution function for charged particles in an external magnetic field. As a special case we report the wigner function for uniform magnetic field.

Introduction

The wigner distribution function (WDF) was introduced in 1932[1] for handling problems in quantum Statistical mechanics as

$$P_W(q,p) = \frac{1}{(\pi\hbar)^n}\int_{-\infty}^{+\infty} \psi^*(\vec{q}+\vec{y})\psi(\vec{q}-\vec{y})e^{\frac{-2i\vec{p}.\vec{y}}{\hbar}}dy \quad (1)$$

or

$$P_W(q,p) = \frac{1}{(\pi\hbar)^n}\int_{-\infty}^{+\infty} \psi^*(\vec{p}+\vec{y})\psi(\vec{p}-\vec{y})e^{\frac{-2i\vec{q}.\vec{y}}{\hbar}}dy \quad (2)$$

Where $\psi(\vec{q})\ or\ \phi(\vec{p})$ is the wave function that is obtained by solving schrodinger equation in coordinate or momentum space .

The quantities $\vec{p}, \vec{q}\ and\ \vec{y}$ are n-dimensional vectors and n is the number of degrees of freedom.

Using this function most of the techniques developed for classical statistical mechanics can be applied to the problems encountered in quantum domain .

In particular, the quantum mechanical averages of the dynamical variables may be calculated by integrations in phase space.

The wave function $\psi(\vec{q})\ or\ \phi(\vec{p})$ must first be transformed to obtain $\psi(\vec{q}\pm\vec{y})$ or $\phi(\vec{p}\pm\vec{y})$ before performing the integral in (1) or (2) .

In most practical cases this transformation leads to complicated expression for the integrand and hence to non-trivial integral .

In this paper we attempt to find expression for P_W in differential forms which in stead of $\psi(\vec{q}\pm\vec{y})$ or $\phi(\vec{p}\pm\vec{y})$ involve derivatives of $\psi(\vec{q})\ or\ \phi(\vec{p})$ and of delta functions.

These new forms turn out to be more convenient than the original integral forms.

In addition we calculate the WDF for a particle in an external uniform magnetic field. We also obtain the equation governing the evolution of the WDF for charged particles in non-uniform external magnetic field. The equation contains the well-known liouville equation in the sense of the correspondence principle i.e. $\hbar \to 0$.

It also contains the derivatives of the vector potential of orders two and higher.

2. The WDF in differential forms

In order to obtain the differential forms of WDF we consider one dimensional case, extension to higer dimensions is straightforward. We expand $\psi(\vec{q}\pm\vec{y})$ as

$$\psi(\vec{q}\pm\vec{y}) = \sum_{n=0}^{\infty}\frac{(\pm 1)^n}{n!}y^n\frac{\partial^n\psi(q)}{\partial q^n} \quad (3)$$

Introducing (3) into (1) and integrating only, we obtain the desired expression.

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$$P_W(q,p) = \sum_{n,m=o}^{\infty} \frac{(-1)^m}{n!m!} \left(\frac{\hbar}{2i}\right)^{n+m} \frac{\partial^n \psi^*(q)}{\partial q^n} \frac{\partial^m \psi(q)}{\partial q^m} \frac{\partial^{n+m}\delta(p)}{\partial p^{n+m}} \tag{4}$$

Using the alternative expression (2) for P_w and similar manipulations lead to another expression for P_w

$$P_W(q,p) = \sum_{n,m=o}^{\infty} \frac{(-1)^m}{n!m!} \left(\frac{\hbar}{2i}\right)^{n+m} \frac{\partial^n \psi^*(p)}{\partial p^n} \frac{\partial^m \psi(p)}{\partial p^m} \frac{\partial^{n+m}\delta(q)}{\partial q^{n+m}} \tag{5}$$

It should be emphasized that practical quantum mechanical calculations of dynamical variables involve the integrals of P_(W) (q,p) on phase space, therefore, the appearance of δ(p) or δ(q) in (4) or (5) leads to the straightforward differentiations of classical dynamical variables.

Furthermore the WDF'S , as expressed in powers of $\hbar$, are appropriate for approximate calculations.

As an application , we show that quantum mechanical averages of the dynamical variable are obtained by integration on phase space . consider a function of coordinate , F(q) then we have

$$< F >= \int P_W(q,p)\, F(q)\, dq\, dp \tag{6}$$

Substituting P_W from (4) into (6) we obtain

$$< F >= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m}{n!\,m!} \left(\frac{\hbar}{2i}\right)^{n+m} \int_{-\infty}^{+\infty} \frac{\partial^n \psi^*(p)}{\partial p^n} \frac{\partial^m \psi(p)}{\partial p^m} F(q)\,dq \int_{-\infty}^{+\infty} \frac{\partial^{n+m}\delta(q)}{\partial q^{n+m}}\, dp \tag{7}$$

Substituting P_W from (4) into (6) we obtain

$$< F >= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m}{n!\,m!} \left(\frac{\hbar}{2i}\right)^{n+m} \int_{-\infty}^{+\infty} \frac{\partial^n \psi^*(p)}{\partial p^n} \frac{\partial^m \psi(p)}{\partial p^m} F(q)\,dq \int_{-\infty}^{+\infty} \frac{\partial^{n+m}\delta(q)}{\partial q^{n+m}}\, dp$$

The integral on p vanish except for n=m=0 , hence

$$\int_{-\infty}^{+\infty} P_W\, F(q)\,dp\,dq = \int_{-\infty}^{+\infty} \psi^*(p) F(q) \psi(q)\, dq \tag{8}$$

For function of momentum F(p) , using the alternative expression (5) for P_W and similar manipulation , we obtain

$$\int_{-\infty}^{+\infty} P_W(q,p)\, F(q)\,dp\,dq = \int_{-\infty}^{+\infty} \phi^*(p) F(q) \phi(q)\, dq \tag{9}$$

Equation (8) and (9) are usually proved using weyl correspondence rule[2]

3. Evolution of WDF for charged particle in magnetic field.

In this section we obtain the equation governing the temporal evolution of WDF for electron in an external magnetic field described by a vector potential , $\overrightarrow{A(q)}$. the Hamiltonian for this system is

$$H(\vec{q} \pm \vec{y}) = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial {q_j}^2} + \frac{e^2}{2\mu c^2} A_j(\vec{q} \pm \vec{y}) A_j(\vec{q} \pm \vec{y}) + \frac{e\hbar}{i\mu c} A_j(\vec{q} \pm \vec{y}) \frac{\partial}{\partial q_j} \tag{10}$$

Where sum over the repeated indices is implied. Expanding the functions $A_j(\vec{q} \pm \vec{y})$ we obtain

$$A_j(\vec{q} \pm \vec{y}) = \sum_{r=o}^{\infty} (\pm 1)^r \frac{1}{r!} (\vec{y}.\vec{\nabla})^{r} A_j(\vec{q}) \tag{11}$$

Substituting (11) for $A_j(\vec{q} \pm \vec{y})$ in to (10) we find

$$H(\vec{q} \pm \vec{y}) = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial {q_j}^2} + \frac{e^2}{2\mu c^2} \sum_{m=n=0} \sum_{m_1,m_2,m_3,} \sum_{n_1,n_2,n_3,} \frac{(\pm 1)^{m+n}}{m_{1!},m_{2!},m_{3!},n_{1!},n_{2!},n_{3!},}$$
$$\times\, {y^{m_1+n_1}}_1 {y^{m_2+n_2}}_2 {y^{m_3+n_3}}_3 \left[\frac{\partial^m A_{J(q)}}{\partial {q_1}^{m_1} \partial {q_2}^{m_2} \partial {q_3}^{m_3}}\right]\left[\frac{\partial^n A_{J(q)}}{\partial {q_1}^{n_1} \partial {q_2}^{n_2} \partial {q_3}^{n_3}}\right] +$$
$$\frac{e\hbar}{i\mu c} \sum_{m=0} \sum_{m_1,m_2,m_3,} \frac{(\pm 1)^m}{m_{1!},m_{2!},m_{3!},} {y^{m_1}}_1 {y^{m_2}}_2 {y^{m_3}}_3 \frac{\partial^m A_{J(q)}}{\partial {q_1}^{m_1} \partial {q_2}^{m_2} \partial {q_3}^{m_3}} \frac{\partial}{\partial q_j} \tag{12}$$

In order to obtain the evolution of P_W , we differential (1) with respect to t , multiply by $i\hbar$, use the schrodinger equation , and obtain

$$i\hbar\frac{\partial P_W}{\partial t} = \int\{[-H^*(\vec{q}-\vec{y})\psi^*(\vec{q}+\vec{y})]\psi(\vec{q}-\vec{y}) + \psi^*(\vec{q}+\vec{y})[-H(\vec{q}-\vec{y})\psi(\vec{q}+\vec{y})]\, e^{\frac{-2ip.y}{\hbar}} dy \tag{13}$$

In tro ducing (12) into (13) we find

$$i\hbar\frac{\partial P_W}{\partial t} = J_1 + J_2 + J_3 \tag{14}$$

Where

$$J_1=\frac{\hbar^2}{2\mu}\int\left[\frac{\partial^2\psi^*(\vec{q}+\vec{y})}{\partial q_j{}^2}\psi(\vec{q}-\vec{y})-\psi^*(\vec{q}+\vec{y})\frac{\partial^2\psi(\vec{q}-\vec{y})}{\partial q_j{}^2}\right]e^{\frac{-2i\vec{p}.\vec{y}}{\hbar}}dy$$

$$J_2=-\frac{e^2}{2\mu c^2}\sum_{m=n=0}\sum_{m_1,m_2,m_3,}\sum_{n_1,n_2,n_3,}\frac{1}{m_1!,m_2!,m_3!,n_1!,n_2!,n_3!,}\times\left[\frac{\partial^m A_{J(q)}}{\partial q_1{}^{m_1}\partial q_2{}^{m_2}\partial q_3{}^{m_3}}\right]\left[\frac{\partial^n A_{J(q)}}{\partial q_1{}^{n_1}\partial q_2{}^{n_2}\partial q_3{}^{n_3}}\right](1-(-1)^{m+n})\int y^{m_1+n_1}{}_1 y^{m_2+n_2}{}_2 y^{m_3+n_3}{}_3\psi^*(\vec{q}+\vec{y})\psi(\vec{q}-\vec{y})e^{\frac{-2i\vec{p}.\vec{y}}{\hbar}}dy$$

$$J_3=\frac{e\hbar}{i\mu c}\sum_{m=0}\sum_{m_1,m_2,m_3,}\frac{1}{m_1!,m_2!,m_3!,}\frac{\partial^m A_{J(q)}}{\partial q_1{}^{m_1}\partial q_2{}^{m_2}\partial q_3{}^{m_3}}\int y^{m_1}{}_1 y^{m_2}{}_2 y^{m_3}{}_3\left[\frac{\partial\psi^*(\vec{q}+\vec{y})}{\partial q_j}\psi(\vec{q}-\vec{y})+(-1)^m\psi^*(\vec{q}+\vec{y})\frac{\partial\psi(\vec{q}-\vec{y})}{\partial q_j}\right]e^{\frac{-2i\vec{p}.\vec{y}}{\hbar}}dy$$

Replacing $\frac{\partial}{\partial q_j}$ by $\pm\frac{\partial}{\partial y_j}$, integrating on y and replacing $\pm\frac{\partial}{\partial y_j}$ by $\frac{\partial}{\partial q_j}$ again , we obtain

$$J_1=-\frac{i\hbar}{\mu}P_j\frac{\partial P_W}{\partial q_j} \quad (15)$$

Using the following identity

$$\int y_j^r\,[\psi^*(\vec{q}+\vec{y})\psi(\vec{q}-\vec{y})]e^{\frac{-2i\vec{p}.\vec{y}}{\hbar}}dy=\left(\frac{\hbar}{2i}\right)^r\frac{\partial^r P_W}{\partial p_j{}^r} \quad (16)$$

We obtain

$$J_2=-\frac{e^2}{2\mu c^2}\sum_{m=n=0}\sum_{m_1,m_2,m_3,}\sum_{n_1,n_2,n_3,}\frac{1}{m_1!,m_2!,m_3!,n_1!,n_2!,n_3!,}\times\left[\frac{\partial^m A_{J(q)}}{\partial q_1{}^{m_1}\partial q_2{}^{m_2}\partial q_3{}^{m_3}}\right]\left[\frac{\partial^n A_{J(q)}}{\partial q_1{}^{n_1}\partial q_2{}^{n_2}\partial q_3{}^{n_3}}\right](1-(-1)^{m+n})\left(\frac{\hbar}{2i}\right)^{m+n}\frac{\partial^{m+n}P_W}{\partial p_1{}^{m_1+n_1}\partial p_2{}^{m_2+n_2}\partial p_3{}^{m_3+n_3}} \quad (17)$$

Similar manipulation leads to following expression for J_3

$$J_3=\frac{e\hbar}{i\mu c}\sum_{m=0}\sum_{m_1,m_2,m_3,}\frac{1}{m_1!,m_2!,m_3!,}\left(\frac{\hbar}{2i}\right)^{2m}\times\frac{\partial^{2m}A_{J(q)}}{\partial q_1{}^{m_1}\partial q_2{}^{m_2}\partial q_3{}^{m_3}}\times\frac{\partial^{2m+1}P_W}{\partial q_j\partial p_1{}^{m_1}\partial p_2{}^{m_2}\partial p_3{}^{m_3}}$$
$$-m_1\frac{\partial^{2m+1}A_{1(q)}}{\partial q_1{}^{m_1}\partial q_2{}^{m_2}\partial q_3{}^{m_3}}\times\frac{\partial^{2m}P_W}{\partial q_j\partial p_1{}^{m_1-1}\partial p_2{}^{m_2}\partial p_3{}^{m_3}}-m_2\frac{\partial^{2m+1}A_{2(q)}}{\partial q_1{}^{m_1}\partial q_2{}^{m_2}\partial q_3{}^{m_3}}\times\frac{\partial^{2m}P_W}{\partial p_1{}^{m_1}\partial p_2{}^{m_2-1}\partial p_3{}^{m_3}}-$$
$$m_3\frac{\partial^{2m+1}A_{1(q)}}{\partial q_1{}^{m_1}\partial q_2{}^{m_2}\partial q_3{}^{m_3}}\times\frac{\partial^{2m}P_W}{\partial p_1{}^{m_1-1}\partial p_2{}^{m_2}\partial p_3{}^{m_3-1}}-p_1\frac{\partial^{2m+1}A_{1(q)}}{\partial q_1{}^{m_1}\partial q_2{}^{m_2}\partial q_3{}^{m_3}}\times\frac{\partial^{2m+1}P_W}{\partial p_1{}^{m_1}\partial p_2{}^{m_2}\partial p_3{}^{m_3}}$$
$$-p_2\frac{\partial^{2m+1}A_{2(q)}}{\partial q_1{}^{m_1}\partial q_2{}^{m_2}\partial q_3{}^{m_3}}\times\frac{\partial^{2m+1}P_W}{\partial p_1{}^{m_1}\partial p_2{}^{m_2}\partial p_3{}^{m_3}}-p_3\frac{\partial^{2m+1}A_{3(q)}}{\partial q_1{}^{m_1}\partial q_2{}^{m_2}\partial q_3{}^{m_3}}\times\frac{\partial^{2m+1}P_W}{\partial p_1{}^{m_1}\partial p_2{}^{m_2}\partial p_3{}^{m_3}} \quad (18)$$

Substituting (16) , (17), (18) in to (14) the final result

$$i\hbar\frac{\partial P_W}{\partial t}=-\frac{i\hbar}{\mu}P_j\frac{\partial P_W}{\partial q_j}$$
$$-\frac{e^2}{2\mu c^2}\sum_{m=n=0}\sum_{m_1,m_2,m_3,}\sum_{n_1,n_2,n_3,}\frac{1}{m_1!,m_2!,m_3!,n_1!,n_2!,n_3!,}\times\left[\frac{\partial^m A_{J(q)}}{\partial q_1{}^{m_1}\partial q_2{}^{m_2}\partial q_3{}^{m_3}}\right]\left[\frac{\partial^n A_{J(q)}}{\partial q_1{}^{n_1}\partial q_2{}^{n_2}\partial q_3{}^{n_3}}\right](1-$$
$$(-1)^{m+n})\left(\frac{\hbar}{2i}\right)^{m+n}\frac{\partial^{m+n}P_W}{\partial p_1{}^{m_1+n_1}\partial p_2{}^{m_2+n_2}\partial p_3{}^{m_3+n_3}}+\frac{e\hbar}{i\mu c}\sum_{m=0}\sum_{m_1,m_2,m_3,}\frac{1}{m_1!,m_2!,m_3!,}\left(\frac{\hbar}{2i}\right)^{2m}\times$$
$$\{\frac{\partial^{2m}A_{J(q)}}{\partial q_1{}^{m_1}\partial q_2{}^{m_2}\partial q_3{}^{m_3}}\times\frac{\partial^{2m+1}P_W}{\partial q_j\partial p_1{}^{m_1}\partial p_2{}^{m_2}\partial p_3{}^{m_3}}$$

-

$$m_1 \frac{\partial^{2m+1} A_{1(q)}}{\partial q_1^{m_1} \partial q_2^{m_2} \partial q_3^{m_3}} \times \frac{\partial^{2m} P_W}{\partial p_1^{m_1-1} \partial p_2^{m_2} \partial p_3^{m_3}} - m_2 \frac{\partial^{2m+1}}{\partial q_1^{m_1} \partial q_2^{m_2} \partial q_3^{m_3}} \times \frac{\partial^{2m} P_W}{\partial p_1^{m_1} \partial p_2^{m_2-1} \partial p_3^{m_3}} -$$

$$m_3 \frac{\partial^{2m+1} A_{1(q)}}{\partial q_1^{m_1} \partial q_2^{m_2} \partial q_3^{m_3}} \times \frac{\partial^{2m} P_W}{\partial p_1^{m_1} \partial p_2^{m_2} \partial p_3^{m_3-1}} - p_1 \frac{\partial^{2m+1} A_{1(q)}}{\partial q_1^{m_1} \partial q_2^{m_2} \partial q_3^{m_3}} \times \frac{\partial^{2m+1} P_W}{\partial p_1^{m_1} \partial p_2^{m_2} \partial p_3^{m_3}}$$

$$- \times \times \frac{\partial^{2m+1} P_W}{\partial p_1^{m_1} \partial p_2^{m_2} \partial p_3^{m_3}} \times \frac{\partial^{2m+1} (\overrightarrow{P.A}_{(q)})}{\partial q_1^{m_1} \partial q_2^{m_2} \partial q_3^{m_3}} \} \quad (19)$$

The first quantum mechanical correction is of order $\hbar^2$ and given by

$$-\frac{e^2 \hbar^2}{4\mu c^2}$$

$$\sum_{m_1, m_2, m_3,} \sum_{n_1, n_2, n_3,} \frac{1}{m_{1!}, m_{2!}, m_{3!}, n_{1!}, n_{2!}, n_{3!},} \frac{\partial^3 P_W}{\partial p_1^{m_1+n_1} \partial p_2^{m_2+n_2} \partial p_3^{m_3+n_3}} \times \{ \frac{\partial^2 A_{J(q)}}{\partial q_1^{m_1} \partial q_2^{m_2} \partial q_3^{m_3}} \frac{\partial A_{J(q)}}{\partial q_1^{n_1} \partial q_2^{n_2} \partial q_3^{n_3}} +$$

$$\frac{\partial A_{J(q)}}{\partial q_1^{m_1} \partial q_2^{m_2} \partial q_3^{m_3}}$$

$$\frac{\partial^2 A_{J(q)}}{\partial q_1^{n_1} \partial q_2^{n_2} \partial q_3^{n_3}} +$$

$$\frac{\partial^3 A_{J(q)}}{\partial q_1^{m_1} \partial q_2^{m_2} \partial q_3^{m_3}} A_{J(q)} + \frac{\partial^3 A_{J(q)}}{\partial q_1^{n_1} \partial q_2^{n_2} \partial q_3^{n_3}} A_{J(q)} \} +$$

$$\frac{e\hbar^2}{4\mu c^2} \sum_{m_1, m_2, m_3,} \frac{1}{m_{1!}, m_{2!}, m_{3!},} \left\{ \frac{\partial^2 A_{J(q)}}{\partial q_1^{m_1} \partial q_2^{m_2} \partial q_3^{m_3}} \frac{\partial^3 P_W}{\partial q_j \partial p_1^{m_1} \partial p_2^{m_2} \partial p_3^{m_3}} - \right.$$

$$m_1 \frac{\partial^3 A_{J(q)}}{\partial q_1^{m_1} \partial q_2^{m_2} \partial q_3^{m_3}} \frac{\partial^2 P_W}{\partial p_1^{m_1-1} \partial p_2^{m_2} \partial p_3^{m_3}} - m_2 \frac{\partial^3 A_{J(q)}}{\partial q_1^{m_1} \partial q_2^{m_2} \partial q_3^{m_3}} \frac{\partial^2 P_W}{\partial p_1^{m_1} \partial p_2^{m_2-1} \partial p_3^{m_3}} -$$

$$\left. m_3 \frac{\partial^3 A_{J(q)}}{\partial q_1^{m_1} \partial q_2^{m_2} \partial q_3^{m_3}} \frac{\partial^2 P_W}{\partial p_1^{m_1} \partial p_2^{m_2} \partial p_3^{m_3-1}} - p_j \frac{\partial^3 P_W}{\partial p_1^{m_1} \partial p_2^{m_2} \partial p_3^{m_3}} \times \frac{\partial^3 A_{j(q)}}{\partial q_1^{m_1} \partial q_2^{m_2} \partial q_3^{m_3}} \right\} \quad (20)$$

Where $m_1 + m_2 + m_3$ or $n_1 + n_2 + n_3$ equal to corresponding order of the derivative of the vector potential $A_j(q)$.

For charged particle in uniform magnetic field all the correction terms vanish and the corresponding WDF satisfies liouville equation . for the special case , afther lengthy.

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