Available online at www.elixirpublishers.com (Elixir International Journal)

Applied Mathematics

Elixir Appl. Math. 96 (2016) 41336-41340

ISSN: 2229-712X

Averaged Equations of Flow of Fluid with Pressure-Dependent Viscosity through Porous Media

M.S. Abu Zaytoon[1], F.M. Allan[2], T.L. Alderson[1] and M.H. Hamdan[1]

[1]Department of Mathematics and Statistics, University of New Brunswick, P.O. Box 5050,Saint John, New Brunswick, Canada E2L 4L5.

[2]Department of Mathematical Sciences, United Arab Emirates University, P.O. Box 15551,Al-Ain, UAE.

ARTICLE INFO

Article history:
Received: 4 May 2016;
Received in revised form:
25 June 2016;
Accepted: 30 June 2016;

Keywords
Pressure-dependent Viscosity,
Intrinsic volume averaging.

ABSTRACT

Equations governing the flow of a fluid with variable viscosity through an isotropic porous structure are derived using the method of intrinsic volume averaging. Viscosity of the fluid is assumed to be a variable function of pressure, and the effects of the porous microstructure are modelled in terms of Darcy resistance, Brinkman shear term, and Forchheimer effects.

1.Introduction

Dependence of fluid viscosity on pressure is not a recent knowledge or phenomenon; rather, it has been known, investigated and modelled since the days of Stokes and Barus, (*cf.* [1], [2] and the references therein). However, studies of flow of pressure-dependent fluids through porous structures and its applications is relatively more recent and is motivated by both industrial applications and theoretical fundamentals of the nature of flow. Many applications both in industry and nature involve processes with high pressures that warrant consideration of pressure-dependent viscosity variations of fluids flowing in free space or in porous media, (*cf.* [3-6] and the references therein]. These applications involve chemical and process technologies, such as pharmaceutical tablet production, ground water and crude oil pumping, food processing technologies, lubrication theory, and microfluidics [1-9]. From a theoretical point of view, concern arises in how high pressures affect fluid viscosity, and what parameters in porous media are important in modelling the phenomena.

The pioneering work of Rajagopal and co-workers (*cf.* [5-9] and the references therein), among others, provided detailed analysis, models and many answers to questions that arise when this type of flow is considered. They developed a number of flow models based on mixture theory and reported on various functional forms of dependence of viscosity on pressure, in addition to introducing the limitations of the Darcy drag force in the modelling process. Others have considered modelling the flow through porous media using averaging theorems (which have worked well in other flow problems, [10-13]) and ended with a Darcy-Lapwood-Brinkman model. An advantage of using averaging theorems is the development of models that are valid for variable medium porosity, hence permeability.

The Darcy-Lapwood-Brinkman model developed in [11] remains silent about Forchheimer effects that arise due in part to the porous microstructure. This is the subject matter of the current work where intrinsic volume averaging will be applied to the Navier-Stokes equations with pressure-dependent viscosity. The effects of the porous microstructure will be accounted for using mathematical idealizations of the pore structure, based on the concept of Representative Unit Cell (RUC) that was introduced by Du Plessis and Masliyah, [14,15] and the geometric factors of Du Plessis , [16] and of Du Plessis and Diedericks, [17]. Both granular and consolidated media microstructures are considered.

2. Model Equations

The unsteady, Navier-Stokes flow of an incompressible fluid with pressure-dependent viscosity is governed by the equations of continuity and momentum, expressed as:

$$\nabla \bullet \vec{v} = 0 \qquad \ldots(1)$$

$$\rho\left(\vec{v}_t + \nabla \bullet \vec{v}\vec{v}\right) = -\nabla p + \nabla \bullet \vec{T} + \rho\vec{g} \qquad \ldots(2)$$

where

$$\vec{T} = \mu\left(\nabla\vec{v} + (\nabla\vec{v})^T\right) \qquad \ldots(3)$$

$\vec{v}$ is the velocity vector field, p is the pressure, ρ is the fluid density, $\mu = \mu(p)$ is the pressure-dependent viscosity of the fluid.

When the flow domain is a porous structure, the above governing equations are valid locally (microscopically) in the pore space. However, due to complexity of the pore structure and its boundary (that is, porous matrix) it is customary to seek a macroscopic form of the equations, obtained by averaging the governing equations over a control volume, V, referred to as a Representative Elementary Volume (REV), [14,15]. An REV is composed of a fluid-phase contained in the pore space, V_φ, and a (stationary) solid-phase contained in the porous matrix solid of volume V_s.

Tele:
E-mail address: hamdan@unb.ca

The fraction of pore space in the REV is the same as pore space fraction in the whole porous medium, thus having the same porosity as the medium. Porous microstructure interactions with the flowing fluid are accounted for through an idealization of the pore geometry and the concept of a Representative Unit Cell (RUC), introduced in [14,15].

Typical condition on the velocity vector is the no-slip assumption, $\vec{v} = \vec{0}$ on the stationary solid matrix. Porosity, φ, of the REV (hence of the porous medium) is defined as, [18]:

$$\varphi = \frac{1}{V}\int_V \kappa_\varphi dV = \frac{1}{V}\int_{V_\varphi} 1dV = \frac{V_\varphi}{V} \quad \ldots(4)$$

where κ_φ is a fluid-phase function define at position $\vec{x}$ in V by:

$$\kappa_\varphi(\vec{x}) = \begin{cases} 1; & \vec{x} \in V_\varphi \\ 0; & \vec{x} \in V_s \end{cases}. \quad \ldots(5)$$

In order to average the governing equations we first define the volumetric phase average of a fluid quantity *F* per unit volume, as:

$$< F >= \frac{1}{V}\int_V \kappa_\varphi F dV = \frac{1}{V}\int_{V_\varphi} F dV \quad \ldots(6)$$

and the intrinsic phase average (or the volumetric average of *F* over the effective pore space, V_ϕ) as:

$$< F >_\varphi = \frac{1}{V_\varphi}\int_V \kappa_\varphi F dV = \frac{1}{V_\varphi}\int_{V_\varphi} F dV. \quad \ldots(7)$$

Relationship between the volumetric phase average and the intrinsic phase average can be seen from equations (6), and (7), and the definition of porosity, (4), as:

$$< F >= \varphi < F >_\varphi \quad \ldots(8)$$

and the deviation of an averaged quantity from its true (microscopic) value is given by the quantity

$$F^\circ = F - < F >_\varphi. \quad \ldots(9)$$

Averaging theorems (*cf.* **Appendix 1**) are applied to equations (1) and (2) to obtain [11]:

$$< \nabla \bullet \vec{v} >= \nabla \bullet \varphi < \vec{v} >_\varphi + \frac{1}{V}\int_S \vec{v} \bullet \vec{n} dS = 0. \quad \ldots(10)$$

$$\rho\left\{(\varphi < \vec{v} >_\varphi)_t + \nabla \bullet \varphi < \vec{v} >_\varphi < \vec{v} >_\varphi\right\}$$
$$+ \rho \nabla \bullet \varphi < \vec{v}^\circ \vec{v}^\circ >_\varphi + \frac{\rho}{V}\int_S \vec{v}\vec{v} \bullet \vec{n} dS$$
$$= -\varphi \nabla < p >_\varphi - \frac{1}{V}\int_S p^\circ \vec{n} dS$$
$$+ \nabla \bullet \varphi < \vec{T} >_\varphi + \frac{1}{V}\int_S \vec{T} \bullet \vec{n} dS + \rho \varphi < \vec{g} >_\varphi. \quad \ldots(11)$$

Equations (10) and (11) represent the intrinsic volume averaged continuity and momentum equations. The deviation terms and surface integrals appearing in the averaged equations contain information on the interactions between the fluid-phase and solid-phase in the control volume. These are analyzed in what follows.

3. Analysis of the Deviation Terms and Surface Integrals

Using Gauss' divergence theorem, namely $\int_S \vec{F} \bullet \vec{n} dS = \int_V \nabla \bullet \vec{F} dV$, equation (10) takes the following form when equation (1) is invoked:

$$\nabla \bullet \varphi < \vec{v} >_\varphi = 0. \quad \ldots(12)$$

Equation (12) represents the final form of the continuity equation which, for the incompressible flow at hand, translates into vanishing normal component of velocity.

The term $\frac{\rho}{V}\int_S \vec{v}\vec{v} \bullet \vec{n} dS$ contains the velocity vector explicitly, hence vanishes due to the no-slip velocity condition on the porous matrix.

The volume filter $\nabla \bullet \varphi < \vec{v}^\circ \vec{v}^\circ >_\varphi$ is related to the hydrodynamic dispersion of the average velocity. Hydrodynamic dispersion through porous media is the sum of mechanical dispersion due to tortuosity of the flow path within the porous microstructure, and molecular diffusion arising from diffusion of fluid vorticity, [11,14,15]. Now, the above deviation term is an inertial term representative of mechanical dispersion. In the absence of high velocity and high porosity gradients, the deviations in the velocity vector are small, and the product of deviations is negligible. Hence, the term $\nabla \bullet \varphi < \vec{v}^\circ \vec{v}^\circ >_\varphi$ is negligible, [14,15].

The term $\frac{1}{V}\int_S p^\circ \vec{n} dS$ represents pressure fluctuations on the fluid-solid interface. Le Bars and Grae Worster [18] argue that this term is small, hence can be neglected. However, in case of high pressure flow, it may be of significance, hence can be combined with $\frac{1}{V}\int_S \vec{T} \bullet \vec{n} dS$ to form a surface filter, in the sense introduced by Whitaker, [19,20]. These latter two surface integrals, combined, are related to the following surface integral that involves the normal derivative of $\vec{v}$:

$$-\frac{1}{V}\int_S [-p^\circ \vec{n} + \mu \frac{\partial \vec{v}}{\partial \vec{n}}] dS \quad \ldots(13)$$

where $\vec{n}$ is a unit normal vector pointing into the solid. Expression (13) was elegantly analyzed by Du Plessis and Masliyah [14,15] and, whether (13) is used, or the analysis of Le Bars and Grae Worster [18] is implemented, the end result will be the same.

As recognized and discussed by Le Bars and Grae Worster [18], the term $\frac{1}{V}\int_S \vec{T} \bullet \vec{n} dS$ is the interfacial viscous stress exchange, which corresponds to the microscopic momentum exchange of the Newtonian fluid with the solid matrix, and it depends on the morphology of the porous matrix, on the viscosity of the fluid (hence depends on pressure for fluids with pressure-dependent viscosities) and, if present, on the relative velocity of the fluid-phase and the solid-phase. As a first approximation, the following expression can be used for this surface integral, [14,15,18,20,20]:

$$\frac{1}{V}\int_S \vec{T}\bullet\vec{n}dS = -\mu f\varphi<\vec{v}>_\varphi = -\mu\frac{\varphi^2<\vec{v}>_\varphi}{k} \quad \ldots(14)$$

wherein $f=\frac{\varphi}{k}$ and k is the hydrodynamic permeability.

With the above analysis of the surface integrals and deviation terms, equation (11) reduces to:

$$\rho\left\{(\varphi<\vec{v}>_\varphi)_t+\nabla\bullet\varphi<\vec{v}>_\varphi<\vec{v}>_\varphi\right\}$$
$$=-\varphi\nabla<p>_\varphi+\nabla\bullet\varphi<\vec{T}>_\varphi-\mu f\varphi<\vec{v}>_\varphi$$
$$+\rho\varphi<\vec{g}>_\varphi. \quad \ldots(15)$$

Equation (14) is the form implemented in the formulation by Alharbi and Hamdan [11]. Clearly, the RHS of (14) is the Darcy viscous damping (drag) term. Following [7-9], the coefficient of velocity in the Darcy term can be replaced by a function of pressure, $\alpha(p)$, as will be seen in the final form of governing equations.

4. Forchheimer Effects

Integral (13) can be identified with the force that gives rise to Darcy resistance and the Forchheimer inertial terms. The first approximation in (14) involves the Darcy resistance. In order to account for the porous microstructure and Forchheimer effects, we follow the analyses provided in Whitaker, [19,20], Ma and Ma and Ruth, [21], Ruth and Ma, [22,23], and Du Plessis and Diedericks, [17] in decomposing the surface integral as follows.

Surface integral (13) can be decomposed into two parts: one is a shear force integral (which accounts for the viscous drag effects that predominate in the Darcy regime, that is, for small Reynolds number flow), and the other is an inertial force integral (which accounts for inertial drag effects that predominate in the Forchheimer regime, that is, for high Reynolds number flow). This type of integral has received extensive analysis, and its quantification gives closure to the problem of flow through a porous structure, [16,17]. To accomplish this, we let f_1 be the *velocity-independent* viscous shear geometric factor that depends on the geometry of the porous medium and gives rise to the Darcy resistance, and f_2 the *velocity-dependent* inertial geometric factor that gives rise to the Forchheimer inertial term. Following Du Plessis and Diedericks, [17], the Churchill-Usagi total frictional effects, *f*, of the porous matrix on the fluid may be expressed as:

$$f^r = f_1^{\,r} + f_2^{\,r} \quad \ldots(16)$$

where *r* is a shifting factor that Du Plessis and Diedericks, [17], have shown to produce reasonable correlation when its value is unity. Furthermore, in terms of the factor f_1, hydrodynamic permeability, η, is given by, [17]:

$$\eta=\frac{\varphi}{f_1} \quad \ldots(17)$$

and expressions (13) or (14) can be expressed as:

$$-\frac{1}{V}\int_S[-p^\circ\vec{n}+\mu\frac{\partial\vec{v}}{\partial n}]dS=-\mu f\varphi<\vec{v}>_\varphi=-\mu(f_1+f_2)\varphi<\vec{v}>_\varphi$$
. $\ldots(18)$

Expressions for f_1 and f_2 require a mathematical description of the porous matrix and its microstructure. Du Plessis and Diedericks, [17], carried out extensive analysis on evaluating these geometric factors for isotropic porous media, based on Du Plessis and Masliyah's concept of a Representative Unit Cell (RUC), [14,15], which they defined as the minimal REV in which the average properties of the porous medium are embedded. For granular and consolidated isotropic porous media, the following expressions, summarized in **Table 1**, as given in Du Plessis and Diedericks, [17], are adopted in this work for f_1 and f_2, and for the hydrodynamic permeability:

Table 1. Friction Factors for Granular and Consolidated Media.

Granular	**Consolidated** (Sponges and Metallic Foams)
$f_1=\frac{36(1-\varphi)^{2/3}}{d^2[1-(1-\varphi)^{1/3}][1-(1-\varphi)^{2/3}]}$	$f_1=\frac{36\tau(\tau-1)}{d^2\varphi}$
$f_2=\frac{\rho d\lvert\varphi<\vec{v}>_\varphi\rvert C_d(1-\varphi)^{2/3}}{d^2\mu[1-(1-\varphi)^{2/3}]^2}$	$f_2=\frac{\rho d\lvert\varphi<\vec{v}>_\varphi\rvert C_d\tau(\tau-1)}{d^2\mu\varphi(3-\tau)}$
$\eta=\frac{\varphi}{f_1}=\frac{d^2\varphi[1-(1-\varphi)^{1/3}][1-(1-\varphi)^{2/3}]}{36(1-\varphi)^{2/3}}$	$\eta=\frac{\varphi}{f_1}=\frac{d^2\varphi}{36\tau(\tau-1)}$

where *d* is a microscopic length (such as the mean pore diameter) and C_d is the Forchheimer drag coefficient, and

$$\frac{1}{\tau}=\frac{3}{4\varphi}+\frac{\sqrt{9-8\varphi}}{2\varphi}\cos\left\{\frac{4\pi}{3}+\frac{1}{3}\cos^{-1}\left(\frac{8\varphi^2-36\varphi+27}{(9-8\varphi)^{3/2}}\right)\right\} \quad \ldots(19)$$

Now, upon using (18) in (15), and substituting the friction factors of **Table 1**, we obtain:

For Granular Media

$$\rho\left\{(\varphi<\vec{v}>_\varphi)_t+\nabla\bullet\varphi<\vec{v}>_\varphi<\vec{v}>_\varphi\right\}$$
$$=-\varphi\nabla<p>_\varphi+\nabla\bullet\varphi<\vec{T}>_\varphi+\rho\varphi<\vec{g}>_\varphi$$
$$-\frac{36(1-\varphi)^{2/3}}{d^2[1-(1-\varphi)^{1/3}][1-(1-\varphi)^{2/3}]}\mu\varphi<\vec{v}>_\varphi-\frac{\rho C_d(1-\varphi)^{2/3}}{d[1-(1-\varphi)^{2/3}]^2}\varphi<\vec{v}>_\varphi\lvert\varphi<\vec{v}>_\varphi\rvert \quad \ldots(20)$$

For Consolidated Media

$$\rho\left\{(\varphi<\vec{v}>_\varphi)_t+\nabla\bullet\varphi<\vec{v}>_\varphi<\vec{v}>_\varphi\right\}$$
$$=-\varphi\nabla<p>_\varphi++\rho\varphi<\vec{g}>_\varphi+\nabla\bullet\varphi<\vec{T}>_\varphi$$
$$-\mu\frac{36\tau(\tau-1)}{d^2\varphi}\varphi<\vec{v}>_\varphi-\frac{\rho\lvert\varphi<\vec{v}>_\varphi\rvert C_d\tau(\tau-1)}{d\varphi(3-\tau)}\varphi<\vec{v}>_\varphi \quad \ldots(21)$$

Letting $\vec{q}=\varphi<\vec{v}>_\varphi$, $\vec{G}=\varphi<\vec{g}>_\varphi$,

$p^*=<p>_\varphi$, $\vec{I}=\varphi<\vec{T}>_\varphi$, $C_g=\frac{C_d(1-\varphi)^{2/3}}{d[1-(1-\varphi)^{2/3}]^2}$

and $C_c=\frac{C_d\tau(\tau-1)}{d\varphi(3-\tau)}$, and using the definitions of hydrodynamic permeability for each medium, given in Table 1, we can write (12), (20), and (21) in the following forms.

Continuity equation:

$$\nabla \bullet \vec{q} = 0. \quad \ldots(22)$$

Momentum equations:

For Granular Media:

$$\rho\left\{\vec{q}_t + \nabla \bullet \frac{\vec{q}\vec{q}}{\varphi}\right\} = -\varphi\nabla p^* + \nabla \bullet \vec{I} + \rho\vec{G} - \frac{\mu}{\eta}\vec{q} - \rho C_g \vec{q}|\vec{q}| \quad \ldots(23)$$

For Consolidated Media:

$$\rho\left\{\vec{q}_t + \nabla \bullet \frac{\vec{q}\vec{q}}{\varphi}\right\} = -\varphi\nabla p^* + \nabla \bullet \vec{I} + \rho\vec{G} - \frac{\mu}{\eta}\vec{q} - \rho C_c \vec{q}|\vec{q}| \quad \ldots(24)$$

The following observations can be made regarding equations (23) and (24):

(1) The term $\nabla \bullet \vec{I}$ in equations (23) and (24) can be written in the expanded form:

$$\nabla \bullet \vec{I} = \nabla \bullet \varphi < \vec{T} >_\varphi = \nabla \bullet \varphi < \mu >_\varphi \left\{\nabla < \vec{v} >_\varphi + \left(\nabla < \vec{v} >_\varphi\right)^T\right\}. \quad \ldots(25)$$

Defining $\mu^* = \varphi < \mu >_\varphi$ and using $< \vec{v} >_\varphi = \frac{\vec{q}}{\varphi}$ in (25), gives:

$$\nabla \bullet \vec{I} = \nabla \bullet \mu^* \left\{\nabla\left(\frac{\vec{q}}{\varphi}\right) + \left(\nabla\left(\frac{\vec{q}}{\varphi}\right)\right)^T\right\}. \quad \ldots(26)$$

In taking the viscosity a function of pressure, we take $\mu^* = \mu^*(p^*)$ and identify $\alpha(p)$ as follows:

For granular media:

$$\alpha(p^*) = \frac{36(1-\varphi)^{2/3}}{d^2[1-(1-\varphi)^{1/3}][1-(1-\varphi)^{2/3}]}\mu \quad \ldots(27)$$

For consolidated media:

$$\alpha(p^*) = \frac{36\tau(\tau-1)}{d^2\varphi}\mu. \quad \ldots(28)$$

Equations (27) and (28) clearly indicate that the Darcy drag coefficient is indeed a function of pressure. Various forms of $\alpha(p)$ have been discussed and implemented in the work of Rajagopal and co-workers (*cf.* [5-9] and the references therein).

(2) It is customary to express the Forchheimer term in a form that involves the square root of permeability, namely and $\frac{\rho C_d}{\sqrt{\eta}}\vec{q}|\vec{q}|$. This can be accomplished here by mathematically adjusting the definitions of C_c and C_g in terms of the porosity, such that $\rho C_g = \frac{\rho C^*_g}{\sqrt{\eta}}$ and $\rho C_c = \frac{\rho C^*_c}{\sqrt{\eta}}$ for granular and consolidated media, respectively.

The momentum equations (23) and (24) can thus be written in the following final forms of momentum equations that are valid for variable porosity media:

For Granular Media

$$\rho\left\{\vec{q}_t + \nabla \bullet \frac{\vec{q}\vec{q}}{\varphi}\right\} = -\varphi\nabla p^* + \nabla \bullet \mu^*\left\{\nabla\left(\frac{\vec{q}}{\varphi}\right) + \left(\nabla\left(\frac{\vec{q}}{\varphi}\right)\right)^T\right\} + \rho\vec{G} - \alpha(p^*)\vec{q} - \frac{\rho C^*_g}{\sqrt{\eta}}\vec{q}|\vec{q}| \quad \ldots(29)$$

For Consolidated Media:

$$\rho\left\{\vec{q}_t + \nabla \bullet \frac{\vec{q}\vec{q}}{\varphi}\right\} = -\varphi\nabla p^* + \nabla \bullet \mu^*\left\{\nabla\left(\frac{\vec{q}}{\varphi}\right) + \left(\nabla\left(\frac{\vec{q}}{\varphi}\right)\right)^T\right\} + \rho\vec{G} - \alpha(p^*)\vec{q} - \frac{\rho C^*_c}{\sqrt{\eta}}\vec{q}|\vec{q}| \quad \ldots(30)$$

6. Conclusion

In this work, intrinsic volume averaging was implemented in deriving equations governing the flow of an incompressible fluid with variable, pressure-dependent viscosity through isotropic porous media. Granular and consolidated media were considered and the effects of the porous microstructure were modelled based on existing models in the literature. Darcian drag and Forchheimer effects were identified in this work, and the dependence of Darcy's drag coefficient on pressure was discussed.

Appendix 1

Averaging Theorems

Letting F and H be volumetrically additive scalar quantities, $\vec{F}$ a vector quantity, and c a constant (whose average is itself), then, [5]:

(i)… $< cF >= c < F >^{=c} \varphi < F >_\varphi$

(*ii*)… $< \nabla F >= \varphi\nabla < F >_\varphi + \frac{1}{V}\int_S F^\circ \vec{n} dS$

where S is the surface area of the solid matrix in the REV that is in contact with the fluid, and $\vec{n}$ is the unit normal vector pointing into the solid, and a surface integral of the form $\iint_S \vec{n} dS$ has been abbreviated as $\int_S \vec{n} dS$.

(*iii*)… $< F \mp H >=< F > \mp < H >$

$= \varphi < F \mp H >_\varphi = \varphi < F >_\varphi \mp \varphi < H >_\varphi$

(iv)…

$< FH >= \varphi < FH >_\varphi$

$= \varphi < F >_\varphi < H >_\varphi + \varphi < F^\circ H^\circ >_\varphi$

(*v*)… $< \nabla \bullet \vec{F} >= \nabla \bullet \varphi < \vec{F} >_\varphi + \frac{1}{V}\int_S \vec{F} \bullet \vec{n} dS.$

(*vi*)… $< F_t >= < F >_t = (\varphi < F >_\varphi)_t.$

Rule *(vi)* is valid under the assumption that the fluid-solid interface in the REV has a zero microscopic velocity.

(*vii*)… $<\nabla\times\vec{F}>=\nabla\times\varphi<\vec{F}>_{\varphi}-\frac{1}{V}\int_{S}\vec{F}\times\vec{n}dS.$

(*vii*)... Due to the no-slip condition, a surface integral is zero if it contains the fluid velocity vector explicitly.

References

[1]. C.J. Barus, "Note on dependence of viscosity on pressure and temperature", Proceedings of the American Academy, vol. 27, pp. 13–19, 1891.

[2]. C.J. Barus, "Isothermals, isopiestics and isometrics relative to viscosity", American Journal of Science, vol. 45, pp. 87–96, 1893.

[3]. L. Fusi, A. Farina and F. Rosso, "Mathematical models for fluids with pressure-dependent viscosity flowing in porous media", International Journal of Engineering Science, vol. 87, pp. 110-118, 2015.

[4]. F.J. Martinez-Boza, M. J., Martin-Alfonso, C. Callegos and M. Fernandez, "High-pressure behavior of intermediate fuel oils", Energy Fuels, vol. 25, pp. 5138–5144, 2011.

[5]. J. Hron, J., Malek, and K.R. Rajagopal, "Simple flows of fluids with pressure-dependent viscosities", Proceedings of the Royal Society, vol. 457, pp. 1603–1622, 2001.

[6]. K.R. Rajagopal, G. Saccomandi and L. Vergori, "Flow of fluids with pressure- and shear-dependent viscosity down an inclined plane", Journal of Fluid Mechanics, vol. 706, pp. 173–189, 2012.

[7]. V.L. Savatorova and K.R. Rajagopal, "Homogenization of a generalization of Brinkman's equation for the flow of a fluid with pressure dependent viscosity through a rigid porous solid", ZAMM, 91(8), pp. 630–648, 2011.

[8]. S. Srinivasan, A. Bonito and K.R. Rajagopal, "Flow of a fluid through a porous solid due to high pressure gradient", Journal of Porous Media, vol. 16, pp. 193–203, 2013.

[9]. S. Srinivasan and K.R. Rajagopal, "A thermodynamic basis for the derivation of the Darcy, Forchheimer and Brinkman models for flows through porous media and their generalizations", International Journal of Non-Linear Mechanics, vol. 58, pp. 162–166, 2014.

[10]. F. M. Allan and M. H. Hamdan, "Fluid-particle model of flow through porous media: the case of uniform particle distribution and parallel velocity fields," Applied Mathematics and Computation, vol. 183(2), pp. 1208-1213, 2006.

[11]. S. O. Alharbi, T. L. Alderson and M. H. Hamdan, "Flow of a fluid with pressure-dependent viscosity through porous media", Advances in Theoretical and Applied Mechanics, vol. 9(1), pp. 1-9, 2016.

[12]. M. H. Hamdan, "A note on polar fluid flow through porous media with Forchheimer effects", Advances in Theoretical and Applied Mechanics, vol. 4(3), pp. 123-134, 2011.

[13]. M.H. Hamdan and M.T. Kamel, "Polar fluid flow through variable-porosity, isotropic porous media", Special Topics & Reviews in Porous Media, vol. 2(2), pp. 145–155, 2011.

[14]. J. P. Du Plessis and J. H. Masliyah, "Mathematical modeling of flow through consolidated isotropic porous media," Transport in Porous Media, vol. 3, pp. 145-161, 1988.

[15]. J. P. Du Plessis and J. H. Masliyah, "Flow through isotropic granular porous media," Transport in Porous Media, vol. 6, pp. 207-221, 1991.

[16]. J. P. Du Plessis, 1994, "Analytical quantification of coefficients in the Ergun equation for fluid friction in a packed bed," Transport in Porous Media, vol. 16, pp. 189-207, 1994.

[17]. J. P. Du Plessis and G. P. J. Diedericks, "Pore-Scale Modeling of Interstitial Phenomena," In: Fluid Transport in Porous Media, J.P. Du Plessis, ed., Computational Mechanics Publications, Southampton, 1997, pp. 61-104.

[18]. M. Le Bars and M. Grae Worster, "Interfacial conditions between a pure fluid and a porous medium: implications for binary alloy solidification", Journal of Fluid Mechanics, vol. 550, pp.149-173, 2006.

[19]. S. Whitaker, "Volume Averaging of Transport Equations," In: Fluid Transport in Porous Media, J.P. Du Plessis, ed., Computational Mechanics Publications, Southampton, 1997, pp. 1-60.

[20]. S. Whitaker, "The method of volume averaging," Kluwer Academic Publishers, Dordrecht, 1999.

[21]. H. Ma and D. W. Ruth, "The microscopic analysis of high Forchheimer Number flow in porous media," Transport in Porous Media, vol. 13, pp.139-160, 1993.

[22]. D.W. Ruth and H. Ma, "On the derivation of the Forchheimer equation by means of the average theorem," Transport in Porous Media, vol. 7(3), pp. 255-264, 1992.

[23]. D.W. Ruth and H. Ma, "Numerical Analysis of Viscous, Incompressible Flow in a Diverging-Converging RUC," Transport in Porous Media, vol. 13, pp. 161-177, 1993.