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# Analytical Solution of Temporally Dispersion of Solute through Semi-Infinite Porous Medium

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**ABSTRACT**

An analytical solution is obtained for advection-dispersion equation in one-dimension semi-infinite longitudinal domain. The solute dispersion parameter is considered temporally dependent and flow velocity is uniform. The zero order production term which is inversely proportional to the dispersion coefficient is also considered. Initially the space domain is the linear combination of uniform input and ratio of zero order production and flow velocity with position variable. Laplace transform used to get the analytical solution.



## Introduction

Modeling of solute transport in porous media is a key issue in the area of soil physics, hydrology and environmental science. Because of anthropogenic chemicals frequently enter the soil, aquifers and groundwater either by accident or by human and other responsible activities, and tend to hazards position the environment and groundwater. Dispersion is essentially a microscopic phenomenon caused by a combination of molecular diffusion and hydrodynamic mixing occurring in porous medium. Dispersion phenomenon arise in a variety of application such as heat flow in material, transport of pollutant in aquifers, lakes and miscible displacement in hydrocarbon saturated porous medium. Dispersion is one of the most prominent parameter in miscible displacement.

According to Bear (1972) the porous media is defined as a material medium made of heterogeneous or multiphase matter. At least one of the considered phase is not solid, is usually called the solid matrix. The solid matrix or phase is always continuous and fully connected. A list of previous workers, who obtained analytical solution of advection-dispersion equation in a one-dimensional porous medium under different initial and boundary conditions are as Bastian and Lapidus (1956), Banks and Ali (1964), Al-Niami and Rushton (1977), Kumar (1983). In most of works, porous parameters are taken adsorption, first order decay, zero order production. Such solutions have been compiled by van Genuchten and Alves (1982) and Lindstrom and Boersma (1989).

Yates (1990, 1992) obtained an analytical solution for one-dimensional advection-dispersion equation with linearly or exponentially increasing dispersion coefficient. De Smedt (2006) presented analytical solutions for solute transport in rivers including the effect of transient storage and first order decay. Kumar et al. (2010), and Yadav et al. (2010, 2012a,b), Jaiswal et al. (2009, 2011, 2012, 2013, 2014, and 2015) obtained analytical solutions for temporally and spatially dependent solute dispersion in semi-infinite/finite porous medium. Many kinds of porous material are seen in every day of life and environment. Soil is simple example of porous media.

In the present study, advection-dispersion equation is considered in one-dimension semi-infinite longitudinal domain. The dispersion parameter is temporally dependent and zero order production term is inversely proportional to the dispersion coefficient is considered. Initially the space domain is the linear combination of uniform input and ratio of zero order production and flow velocity with position variable. The input condition is assumed at the origin of the domain.

## Mathematical Formulation

Let us consider a one-dimensional homogeneous porous medium of semi-infinite domain. The governing parabolic partial differential equation describing the concentration distribution with zero order production in a one-dimensional porous medium is,

$$\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left(D(x,t)\frac{\partial c}{\partial x}-u(x,t)c\right)+\gamma(x,t) \qquad (1)$$

where, $D$ is the longitudinal dispersion coefficient, $u$ is the flow velocity, $c$ is the concentration at position $x$ and time $t$. $\gamma$ is zero order production term, it means regular increment of contaminants in the domain. The dimension of $D$, $u$ and $\gamma$ are $L^2T^{-1}$, $LT^{-1}$ and $ML^{-3}T^{-1}$ respectively.

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Let us assume that $D(x,t) = D_0 f(mt)$ and $u(x,t) = u_0$, are the function of position or time and $m$ is a flow resistance coefficient whose dimension is inverse of the time variable $t$. $f(mt)$ is chosen such that $f(mt) = 1$ for $m = 0$ or $t = 0$, Thus $f(mt)$ is an expression of non-dimensional variable $(mt)$. If both the parameters are independent to independent variables $x$ and $t$, then these are called constant dispersion and uniform flow velocity respectively. The zero order production term is considered inversely proportional to the dispersion coefficient i.e. Consider $\gamma(x,t) = \frac{\gamma_0}{f(mt)}$ So under these considerations the partial differential equation (1) may be written as,

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D_0 f(mt)\frac{\partial c}{\partial x} - u_0 c\right) + \frac{\gamma_0}{f(mt)} \quad (2)$$

where, $D_0$ , $u_0$ and $\gamma_0$ are constants.

Now let us introduce a new independent variable $X$ by a transformation (Jaiswal et al. 2009),

$$\frac{\partial X}{\partial x} = \frac{1}{f(mt)} \text{ or } X = \int \frac{dx}{f(mt)} \quad (3)$$

Equation (2) becomes,

$$f(mt)\frac{\partial c}{\partial t} = \frac{\partial}{\partial X}\left(D_0 \frac{\partial c}{\partial X} - u_0 c\right) + \gamma_0 \quad (4)$$

The dimension of $X$ is the same as the dimension of $x$. Further let us introduce a new time variable $T$, by the following transformation (Crank; 1975),

$$T = \int_0^t \frac{dt}{f(mt)} \quad \text{or} \quad \frac{\partial T}{\partial t} = \frac{1}{f(mt)} \quad (5)$$

So (6)

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial T}\frac{\partial T}{\partial t} = \frac{1}{f(mt)}\frac{\partial c}{\partial T}$$

The partial differential equation (4) reduces into constant coefficient

$$\frac{\partial c}{\partial T} = D_0 \frac{\partial^2 c}{\partial X^2} - u_0 \frac{\partial c}{\partial X} + \gamma_0 \quad (7)$$

**Methodology**

Laplace transformation technique is used to get the analytical solutions in the present paper. The Laplace transformation can be defined as;

If $f(x,t)$ is an any function defined in $a \le x \le b$ and $t \ge 0$, then its Laplace transform with respect to $t$ is denoted by and ~~is~~ defined by;

$$L\{f(x,t)\} = F(x,s) = \int_0^\infty e^{-st} f(x,t)dt, \; s > 0,$$

where, $s$ is called the transform variable..

The inverse Laplace transform is denoted by $L^{-1}\{F(x,s)\} = f(x,t)$ and defined by

$$L^{-1}\{F(x,s)\} = f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st} F(x,s)ds, \; c > 0 .$$

**Solution for Uniform Input Source**

The concentration in porous domain at time $t = 0$ is not solute free. The concentration at $x = 0$ is $C_0$ for time $t > 0$ and concentration gradient at $x \to \infty$ is considered zero for all time $t \ge 0$. Thus initial and boundary conditions for equation (1) in mathematical form in a semi-infinite domain may be written as:

$$c(x,t) = c_i + \frac{\gamma x}{u}, \; x \ge 0 \; t = 0 \quad (8)$$

$$c(x,t) = C_0, \; x = 0, \; t > 0 \quad (9)$$

$$\frac{\partial c(x,t)}{\partial x} = 0, \; x \to \infty, \; t \ge 0 \quad (10)$$

These conditions in terms of new space and time variable may be written as,

$$c(X,T) = c_i + \frac{\gamma_0 X}{u_0}, \; X \ge 0, T = 0 \quad (11)$$

$$c(X,T) = C_0, \; X = 0, \; T > 0 \quad (12)$$

$$\frac{\partial c(X,T)}{\partial X} = 0, \; X \to \infty, \; T \ge 0 \quad (13)$$

Now introducing a new dependent variable $K(X,T)$ by following transformation,

$$c(X,T) = K(X,T) exp\left[\frac{u_0}{2D_0}X - \frac{u_0^2 T}{4D_0}\right] + \gamma_0 T \quad (14)$$

Then,

$$\frac{\partial c}{\partial T} = \left[\frac{\partial K}{\partial T} - \frac{u_0^2 K}{4D_0}\right] exp\left[\frac{u_0}{2D_0}X - \left\{\frac{u_0^2}{4D_0}\right\}T\right] + \gamma_0 \quad (15)$$

$$\frac{\partial c}{\partial X} = \left\{\frac{\partial K}{\partial X} + \frac{u_0}{2D_0}K\right\} exp\left[\frac{u_0}{2D_0}X - \frac{u_0^2}{4D_0}T\right] \quad (16)$$

$$\frac{\partial^2 c}{\partial X^2} = \left\{\frac{\partial^2 K}{\partial X^2} + \frac{u_0}{D_0}\frac{\partial K}{\partial X} + \frac{u_0^2}{4D_0^2}K\right\} exp\left[\frac{u_0}{2D_0}X - \frac{u_0^2}{4D_0}T\right] \quad (17)$$

Thus, the set of equations (7), (11), (12) and (13) reduces into,

$$\frac{\partial K}{\partial T}=D_0\frac{\partial^2 K}{\partial X^2} \quad (18)$$

$$K(X,T)=\left(c_i+\frac{\gamma_0 X}{u_0}\right)exp\left(-\frac{u_0 X}{2D_0}\right),\ X\geq 0, T=0 \quad (19)$$

$$K(X,T)=(C_0-\gamma_0 T)exp(\alpha^2 T),\ X=0,\ T>0,\ \alpha^2=\left\{\frac{u_0^2}{4D_0}\right\} \quad (20)$$

$$\frac{\partial K(X,T)}{\partial X}+\frac{u_0}{2D_0}K=0,\ X\to\infty,\ T\geq 0 \quad (21)$$

Applying Laplace transformation on equations (18) to (21), we have,

$$s\bar{K}=D_0\frac{d^2\bar{K}}{dX^2}+\left(c_i+\frac{\gamma_0 X}{u_0}\right)exp\left(-\frac{u_0 X}{2D_0}\right) \quad (22)$$

$$\bar{K}(X,s)=\frac{C_0}{(s-\alpha^2)}-\frac{\gamma_0}{(s-\alpha^2)^2},\ X=0 \quad (23)$$

$$\frac{d\bar{K}}{dX}+\frac{u_0}{2D_0}\bar{K}=0,\ X\to\infty \quad (24)$$

where, $\bar{K}(X,s)=\int_0^\infty K(X,T)\,e^{-sT}\,dT$ and $s$ is the Laplace transformation parameter.

Thus the general solution of equation (22) may be written as,

$$\bar{K}(X,s)=C_1 exp\left(X\sqrt{\frac{s}{D_0}}\right)+C_2 exp\left(-X\sqrt{\frac{s}{D_0}}\right)+\frac{c_i exp\left(-\frac{u_0 X}{2D_0}\right)}{(s-\alpha^2)}+\frac{\gamma_0 exp\left(-\frac{u_0 X}{2D_0}\right)}{u_0(s-\alpha^2)}\left\{X-\frac{u_0}{(s-\alpha^2)}\right\} \quad (25)$$

Using condition (24) in general solution (25), we get,

$$C_1=0 \quad (26)$$

Thus the general solution (25) becomes,

$$\bar{K}(X,s)=C_2 exp\left(-X\sqrt{\frac{s}{D_0}}\right)+\frac{c_i exp\left(-\frac{u_0 X}{2D_0}\right)}{(s-\alpha^2)}+\frac{\gamma_0 exp\left(-\frac{u_0 X}{2D_0}\right)}{u_0(s-\alpha^2)}\left\{X-\frac{u_0}{(s-\alpha^2)}\right\} \quad (27)$$

Now using condition (23) in (27), we obtain,

$$C_2=\frac{C_0-c_i}{(s-\alpha^2)} \quad (28)$$

Thus the particular solution in the Laplacian domain may be written as,

$$\bar{K}(X,s)=\frac{(C_0-c_i)}{(s-\alpha^2)}exp\left(-X\sqrt{\frac{s}{D_0}}\right)+\frac{c_i exp\left(-\frac{u_0 X}{2D_0}\right)}{(s-\alpha^2)}+\frac{\gamma_0 exp\left(-\frac{u_0 X}{2D_0}\right)}{u_0(s-\alpha^2)}\left\{X-\frac{u_0}{(s-\alpha^2)}\right\} \quad (29)$$

Taking inverse Laplace transform of equation (29) , we get,

$$K(X,T)=\frac{(C_0-c_i)}{2}\left[exp\left\{-\frac{u_0}{2D_0}X+\frac{u_0^2 T}{4D_0}\right\}erfc\left\{\frac{X}{2\sqrt{D_0 T}}-\alpha\sqrt{T}\right\}\right]+\frac{(C_0-c_i)}{2}\left[exp\left\{\frac{u_0}{2D_0}X+\frac{u_0^2 T}{4D_0}\right\}erfc\left\{\frac{X}{2\sqrt{D_0 T}}+\alpha\sqrt{T}\right\}\right]+exp\left\{-\frac{u_0}{2D_0}X+\frac{u_0^2 T}{4D_0}\right\}\left\{c_i+\frac{\gamma_0 X}{u_0}-\gamma_0 T\right\} \quad (30)$$

Now using equation (14), we may get,

$$C(X,T)=\left(c_i+\frac{\gamma_0 X}{u_0}\right)+\frac{(C_0-c_i)}{2}\left[erfc\left\{\frac{X}{2\sqrt{D_0 T}}-\alpha\sqrt{T}\right\}\right]+\frac{(C_0-c_i)}{2}\left[\exp\left(\frac{u_0 X}{D_0}\right)\right]\left[erfc\left\{\frac{X}{2\sqrt{D_0 T}}+\alpha\sqrt{T}\right\}\right] \quad (31)$$

where $X=\frac{x}{f(mt)}$ and $T=\int_0^t\frac{dt}{f(mt)}$, $\alpha^2=\left\{\frac{u_0^2}{4D_0}\right\}$.

**Numerical Example and Discussions**

Todd (1980) has clearly mentioned that groundwater velocities varying between 2m/year to 2m/day. In the present discussion hypothetical value of the seepage velocity has been considered. The numerical values are chosen to illustrate the application of the present problem.

The concentration values are evaluated from the analytical solution described by equation (31) for uniform input source in a finite domain $0 \leq x \leq 10$ (km) of semi-infinite extent. The other input parameters are considered as; $C_0 = 1.0$, $D_0 = 1.45$ (km$^2$/year), $u_0 = 1.14$ (km/year). In addition to these, $m = 0.01(\text{year}^{-1})$, $\gamma_0 = 0.004$ have been considered. The solutions are computed for three time $t$ (year) $= 0.4$, $0.7$ and $1.0$. The figure (1) is drawn for an increasing function $f(mt) = \exp(mt)$. In figure (1), the concentration distribution behavior for different time at particular position are different and increases with increasing time and behavior of the concentration at the particular position are increasing nature and is higher for higher time.

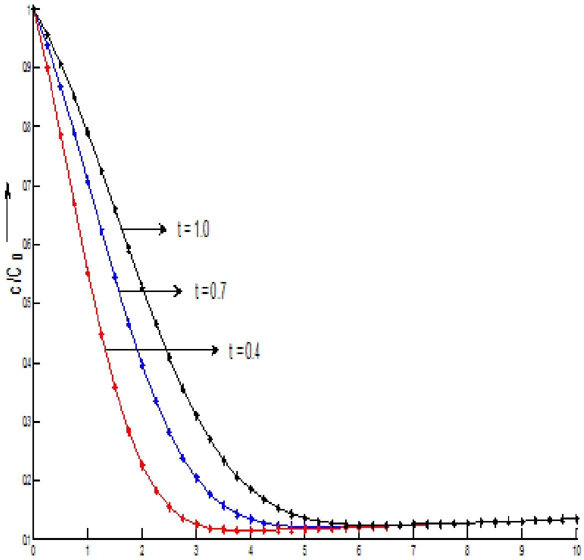


**Figure 1. Concentration distribution of function $f(mt) = \exp(mt)$ for solution (31)**

Comparison of solute transport for decreasing $f(mt) = exp(-mt)$ and increasing functions $f(mt) = \exp(mt)$, at time $t = 0.7$ (year) for uniform input concentration is shown in figure (2). Solute concentrations are higher for increasing function than decreasing function at that time.

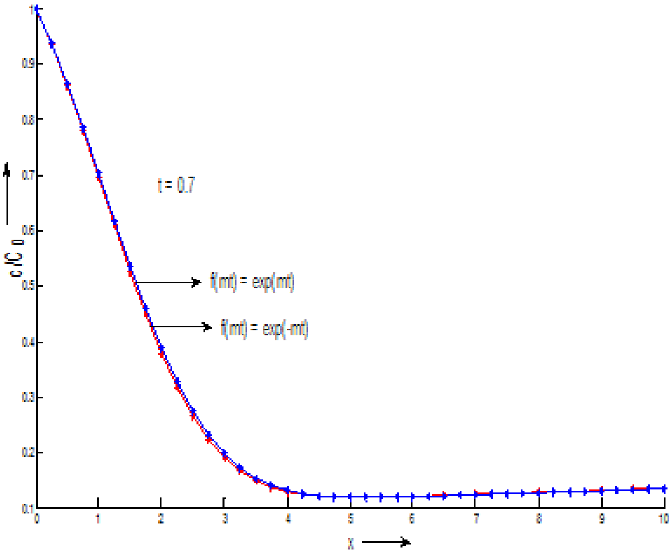


**Figure 2. Comparison between increasing and decreasing functions for solution (31).**

Problems of solute transport involving sequential zero order production reactions frequently occurs in soil and groundwater systems. The accuracy of the numerical solution is validated by direct comparisons with the analytical solution of advection-dispersion equation given by equation (31).

**Conclusion**

Numerical solution is validating by the analytical solution. For this, analytical solution is bench mark tool to check the accuracy of field, experimental and numerical data. Analytical solution is obtained for uniform input source. Advection-dispersion equation is considered one-dimensional and porous domain is semi-infinite domain. The solute dispersion parameter is considered temporally dependent with uniform flow velocity. Zero order production term is also considered. At the origin of the domain the source concentration is uniform.

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**References**
Al-Niami, A. N. S. and Rushton, K. R., (1977), Analysis of flow against dispersion in porous media, Journal of Hydrology, 33, 87-97.
Banks, R. B. and Ali, J., (1964), Dispersion and adsorption in porous media flow, Journal of Hydraulics Division, 90, 13-31.
Bastian, W. C. and Lapidus, L., (1956), Longitudinal diffusion in ion-exchange and chromatographic columns, Journal of Physical Chemistry, 60, 816-817.
Bear, J., (1972), Dynamics of Fluid in Porous Media, Elsvier Publ. Co. New York.
Crank, J., (1975), The Mathematics of Diffusion, Oxford Univ. Press, London, 2nd Ed.

Jaiswal, D. K., Kumar, A., Kumar, N. and Yadav, R. R., (2009), Analytical solutions for temporally and spatially dependent solute dispersion of pulse type input concentration in one-dimensional semi-infinite media, Journal of Hydro-environment Research, 2, 254-263.
Jaiswal, D. K., Kumar, A., Kumar, N. and Singh, M. K., 2011a, Solute Transport along Temporally and Spatially Dependent flows through Horizontal Semi-infinite Media: Dispersion being proportional to square of velocity,Journal of Hydrologic Engineering (ASCE),vol. 16, No. 3, pp. 228-238.
Jaiswal, D. K., 2012, Temporally dependent solute dispersion in one-dimensional porous media, ICMMSC 2012, CCIS 283, pp. 220–228, © Springer-Verlag Berlin Heidelberg.
Jaiswal, D. K., Yadav, R. R. and Gulrana, 2013, “Solute-Transport under Fluctuating Groundwater Flow in Homogeneous Finite Porous Domain”. Journal of Hydrogeology and Hydrologic Engineering, vol. 2(1), pp. 01-07.
Jaiswal, D. K. and Yadav, R. R., 2014, Analytical Solution of Advection-Dispersion Equation in Porous Medium with Temporally Dependent Coefficients, International Conference on Mathematical Sciences (Elsevier) India, pp. 291-294, ISBN – 978-93-5107-261-4.
Jaiswal, D. K. and Yadav, R. R., 2015, Solute transport through porous medium with effective adsorption coefficient, Research Desk, 4(1). 517-531.
Kumar, N., (1983), Unsteady flow against dispersion in finite porous media; Journal of Hydrology, 63, 345–358.
Kumar, A., Jaiswal, D. K., and Kumar, N., 2010, Analytical solutions to one-dimensional advection-diffusion equation with variable coefficients in semi-infinite media. Journal of Hydrology (Elsevier), vol. 380, pp. 330-337.
Lindstrom, F. T. and Boersma, L., (1989), Analytical solutions for convective–dispersive transport in confined aquifers with different initial and boundary conditions, Water Resources Research, 25(2), 241–256.
Smedt, F. D., (2006), Analytical solutions for transport of decaying solutes in rivers with transient storage, Journal of Hydrology, 330, 672– 680.
Todd, D. K., (1980), Groundwater Hydrology, John Wiley, N.Y., 2nd Ed.
van Genuchten, M. Th. and Alves, W. J., (1982), Analytical solutions of one dimensional convective-dispersive solute transport equations, United State Dept. of Agriculture, Technical Bulletin No. 1661.
Yadav, R. R., Jaiswal, D. K., Yadav, H.K., and Gulrana, 2010, One dimensional temporally dependent advection-dispersion equation in porous media: Analytical solution. Natural Resource Modeling (Wiley-Inter Science), vol. 23 No. 12, pp. 521-539.
Yadav, R. R., Jaiswal, D. K., Yadav, H. K., and Gulrana, 2012a, Three-dimensional temporally dependent dispersion through porous media: Analytical solution. Environmental Earth Sciences (Springer), vol. 65, pp. 849-859.
Yadav, R. R., Jaiswal, D. K. and Gulrana, 2012b, “Two-Dimensional Solute Transport for Periodic Flow in Isotropic Porous Media: An Analytical Solution”. Hydrological Processes (Wiley-Inter Science), vol. 26, pp. 3425-3433.
Yates, S. R., (1990), An analytical Solution for one-dimensional transport in heterogeneous porous media. Water Resources Research, 26(10), 2331-2338.
Yates, S. R., (1992), An analytical Solution for one-dimensional transport in heterogeneous porous media with an exponential dispersion function, Water Resources Research, 28(18), 2149-2154.