Construction of Variance and Efficiency Balanced Designs using 2^n-factorial design

Awad Rashmi[*] and Banerjee Shakti

School of Statistics, Devi Ahilya University, Indore, India.

ABSTRACT

A method of constructing equireplicate Variance Balanced (VB) and Efficiency Balanced (EB) design with unequal block sizes is proposed using 2^n-symmetrical factorial design by deleting the control treatment and merging all the main effects with highest order interaction separately. Further optimality of the constructed design has been checked and found it to be universally optimal.

Introduction

The concept of blocking in statistically planned experiments was given by Prof. R. A. Fisher in the year 1935, in his book "Design of Experiments". Further, the research work on the construction of the statistical designs of incomplete type was started by Prof. R.C. Bose and others [1]. Later different methods of construction of balanced incomplete block designs have been given in the literature like, Shrikhande et al. [2], Calinski [3-4], Hanani [5], Agrawal et al. [6] etc.

Let us consider a block design in which v treatments arranged in b blocks and elements of the incidence matrix N are denoted by $[n_{ij}]$, for all $i = 1, 2,\ldots,v$; $j = 1, 2,\ldots,b$; such that the jth block contains k_j experimental units and the ith treatment appears r_i times in the entire design. A balanced block design is said to be (a) binary when $n_{ij} = 0$ or 1, $\forall\ i = 1, 2,\ldots, v$; $j = 1, 2,\ldots, b$. Otherwise, it is said to be nonbinary (see Raghavarao [7]); (b) ternary if $n_{ij} = 0,1$ or 2, $\forall\ i = 1, 2,\ldots,v$; $j = 1, 2,\ldots,b$ and it has parameters v, b, ρ_1 , ρ_2, r, k, Λ ; where ρ_1 , ρ_2 are the number of times 1, 2 occurs in the incidence matrix, respectively (see Billington et al. [8]); (c) generalized binary if $n_{ij} = 0$ or x, $\forall\ i = 1, 2,\ldots, v$; $j = 1, 2,\ldots, b$ and some positive integer x (> 1), and it has parameters v, b, r, k, Λ (see Angelis et al.[9]).

An equi-replicate, equi-block sized, incomplete design, which is also balanced in the sense given above is called balanced incomplete block design, which is an arrangement of v symbols (treatments) into b sets (blocks) each containing k $(< v)$ distinct symbols, such that any pair of distinct symbols occurs in exactly λ sets. Then it is easy to see that each treatment occurs in r $(> \lambda)$ sets. v, b, r, k, λ are called parameters of the BIBD and the parameters satisfies the relations $v\ r = bk$, $r\ (k-1) = \lambda\ (v\ -1)$ and $b \geq v$ (Fisher's Inequality). A BIB design is said to be symmetric if b= v and r=k.

The idea of merging the levels of factors was exploited by Addelman [10] to construct orthogonal resolution-III plans for symmetrical and asymmetrical factorial experiments. The effect of merging of treatments in block designs was studied by Pearce [11] and it was shown that merging of treatments in general leads to an increase in precision of the resulting design, if the merging has any effect.

We always need to setup a design in such a way that the variability in response due to uncontrolled variables (sometimes called experimental error) is not so great that it makes the effects of the controlled variables and usually the effect of the highest order interaction is very low but it contributes the effects of the treatments. We also want designs which are efficient, i.e., designs where we can answer the questions of interest with a minimal amount of data because of the expense associated with data collection. Especially in agricultural experiments, we preferred incomplete block designs when experimental units are very large. Thus in the construction method discussed here we are deleting the control treatment and merging all the main effects with less effective highest order interaction separately in order to increase the efficiency of the constructed design in our research.

In many experimental situations, it is a severe restriction that all blocks in the experiment are of the same size. Variance Balanced (VB) designs forms a class of designs that are flexible extensions to balanced incomplete block designs. They provide the ability to design an experiment with equal precision among all pairwise comparisons, without being restricted to equal block size and equal replication of the treatments.

The importance of VB designs in the context of experimental material is well known as it yields optimal design apart from ensuring simplicity in the analysis. Many practical situations demand designs with varying block sizes (Pearce [12]), or resolvable VB designs with unequal replications (Kageyama [13], Mukerjee and Kageyama [14])

Tele:
E-mail address: awad.rashmi@gmail.com

Rao [15] gave the necessary and sufficient condition for a block design to be variance balanced. Pearce [12] observed that it is sufficient to ensure the constancy of off diagonal elements of the matrix C (= R – NK^{-1}N') for a design to have variance balance. Hedayat and Federer [16] defined that a design is said to be VB if every normalized estimable linear function of treatment effect can be estimated with the same precision. Designs require equal number of replications on all the treatments and equal block sizes. These two conditions were relaxed with the introduction of new class of balanced designs called VB designs.

A block design is called variance balanced if and only if
1. it permits the estimation of all normalized treatment contrasts with the same variance.
2. if the information matrix for treatment effects C = R – NK^{-1}N' satisfies C = ψ [I$_v$ – (1/ v) 1$_v$1$_v$'].
where ψ is the unique nonzero eigen value of the matrix C with the multiplicity $(v – 1)$, I$_v$ is the $v \times v$ identity matrix.

Hedayat and Federer [16], Khatri [17], Agarwal and Kumar [18-19] have provided several methods for construction of VB designs. Kageyama [20] gave some methods for constructing block designs with unequal treatment replications and unequal block sizes. Das and Ghosh [21] have defined generalized efficiency balanced (GEB) designs which includes both, VB as well as efficiency balanced (EB) designs. Ghosh et al. [22] gave methods to construct binary and non-binary VB design.

The concept of efficiency balanced (EB) was introduced by Jones [23] and the nomenclature "efficiency balance" is due to Puri and Nigam [24], Williams [25]. Jones [23] showed that if s is a right eigenvector of the matrix M = $R^{-1}NK^{-1}$ N' corresponding to an eigenvalues μ $(\neq 1)$, then the loss of information on the 'intra-block component' of s'T is μ so that the efficiency-factor of the 'intra-block component' is $1 - \mu$. Calinski [3] and Puri and Nigam [24] established a sufficient condition for a design to be efficiency balanced is that its M matrix.

A block design is called efficiency balanced if
1. Every contrast of treatment effects is estimated through the design with the same efficiency factor.
2. $M = \mu I_v + (1 - \mu) J_v r' / n$; (See Caliski [3])

and since M s = μ s, where μ is the unique non zero eigen value of M with multiplicity (1-v). For the EB block design N, the information matrix C is given as C = (1- μ) (R – (1/n) r r'); (see Kageyama [26]).

Mukerjee and Saha [27] derived some optimality results on efficiency balanced designs. Gupta et al. [28] gave a method for constructing general efficiency balanced designs with equal and unequal block sizes. Gupta [29] gave a method for constructing efficiency balanced designs through BIB and GD designs. Ghosh et al. [30] introduced efficiency balanced and variance balanced ternary block designs. Ceranka and Graczyk [31] discussed some problems for a class of EB block design based on balanced incomplete block designs with repeated blocks. Sun and Tang [32] gave the optimal efficiency balanced designs and their constructions.

In a given class of designs, one should attempt to choose a design which is good according to some well defined statistical criterion. This has led to the study of optimality of experimental designs. Optimal designs are experimental designs that are generated based on a particular optimality criterion and are generally optimal only for a specific statistical model. The optimality of a design depends on the statistical model and is assessed with respect to a statistical criterion, which is related to the variance-matrix of the estimator.

Kiefer [33] introduced Balanced Block Designs (BBD) as a generalization of Balanced Incomplete Block (BIB) designs and proved the A-, D- and E-optimality of BBD's in D (*v, b, k),* where D (*v, b, k)* is the class of all connected block designs with *v* treatments, *b* blocks, and constant block size *k*.

Let C_P denote the class of all acceptable designs with reference to *P*. C_P consists of only connected designs. For any design $d \in C_P$; let V$_d$ denote the dispersion matrix, using *d*. Then

A- optimality A design $d^* \in C_P$ is said to be A-optimal in C_P if

$$tr (V_{d^*}) \leq tr (V_d)$$

i.e. A-optimality criterion seeks to minimize the trace of the inverse of the information matrix. This criterion results in minimizing the average variance of the estimates of the regression coefficients.

D- optimality A design $d^* \in C_P$ is said to be D-optimal in C_P if

$$\det (V_{d^*}) \leq \det (V_d)$$

i.e. D-optimality criterion seeks to minimize $|(X'X)^{-1}|$, or equivalently maximize the determinant of the information matrix X'X of the design.

E- optimality A design $d^* \in C_P$ is said to be E-optimal in C_P if

$$\max (\lambda_{d^*}) \leq \max (\lambda_d)$$

i.e. E-optimality criterion seeks to maximizes the minimum eigen value of the information matrix.

In fact Subsequently, Kiefer [34] proved the stronger result regarding the optimality of balanced block designs, by introducing the concept of *universal optimality* of BBD's in D (*v, b, k).* The original concept of universal optimality in Kiefer [33] dealt with information matrices with zero row and column sums.

Let C_d be the C-matrix of a design *d*. an optimality criterion is a function $\phi : R_v \rightarrow (-\infty, \infty)$, where R$_v$ is the set of $v \times v$ non-negative definite matrices with zero row and column sums. A design d^* is called ϕ – optimal if it minimizes $\phi(C_{d^*})$ over the class of competing designs. A design is said to be universally optimal if ϕ satisfies,

(i) ϕ is convex.
(ii) $\phi (bC)$ is non-increasing in the scalar b ≥ 0.
(iii) ϕ is invariant under any simultaneous permutation of rows and columns of C.

Kiefer [34] obtained a sufficient condition for universal optimality. He proved that the balanced block design (if it exists) is universally optimal in the class of all connected designs. If a design is universally optimal then it is A-, D- and E-optimal as well and a vice a versa.

Although a considerable amount of work is available on optimality of designs in D (v, b, k), not much appears to have been done on the optimality of designs with unequal block sizes, except by Lee and Jacroux [35-37], Dey and Das [38], Gupta and Singh [39], Gupta et al. [40].

Factorial experiments are experiments that investigate effects of two or more factors or input parameters on the output response of a process. It involves simultaneously more than one factor each at two or more levels. Several factors affect simultaneously the characteristic under study in factorial experiments and the experimenter is interested in the main effects and the interaction effects among different factors. Experiments in which the number of levels of all the factors are same, are called symmetrical factorial experiments.

Rajarathinam et al. [41-42] gave the construction of unequal block sizes and equireplicated binary variance balanced and efficiency balanced designs from symmetrical factorial design. Since they delete the control treatment and merged all the main effects and considered them as one block in the first method (See Rajarathinam et al. [41]) and in the second method delete the control treatment as well as all the main effects in 2^n-factorial design (See Rajarathinam et al. [42]). Thus in previously constructed designs, it is not possible to determine the main effects separately and hence cannot be checked its significance values. But in the construction method discussed here we are deleting the control treatment and merging the highest order treatment combination (which is of less important in the practical point of view) with each of main effect separately; because of this there is a scope to determine the main effects and also the efficiency of the design increases in our result.

Rajarathinam et al. (41-42) gave the following theorems

Lemma 1.1

Let us consider a 2^n Symmetrical factorial experiment.

(i) from these 2^n treatment combinations let delete the control treatments (i.e. a treatment combination whose level of all factor is zero).

(ii) merge all those treatment combinations, which represent n main effects (i.e. treatment combinations, where level of one factor is one while level of other factors are zero) and further consider these n merged treatment combinations as one treatment combinations. This way, we have (2^n–n) treatment combinations.

(iii) finally consider (2^n – n) treatment combinations as the blocks for the required design.

Construction of an variance balanced design is given in the theorem that follows.

Theorem 1.2

If there exists a 2^n symmetrical factorial experiment then there always exists a unequal block sizes, equireplicated, binary variance balanced design, by merging all the treatment combinations belonging to main effects and then considering these merged n treatment combinations as one, with the following parameters

$v = n$, $b = 2^n - n$, $r = 2^n - 1$, $k = [2,2,….,2; 3,3,….,3; 4,4,….,4; ……; n, n]$

Lemma 1.3

Consider a 2^n symmetrical factorial experiment. From the 2^n treatment combinations, delete the control treatments (i.e. treatment combinations where the levels of all factors is zero) as well as all main effects (i.e. treatment combinations where the level of one factor is one while the levels of all other factors are zero). Hence we have $2^n − n − 1$ treatment combinations as the blocks in the required design.

Construction of an efficiency balanced design is given in the theorem that follows.

Theorem 1.4

If there exists a 2^n symmetrical factorial experiment then there always exists a unequal block sizes, equireplicated, binary efficiency balanced design, by deleting the control treatment and main effects with the following parameters

$v = n$, $b = 2^n – n-1$, $r = 2^{n-1}-1$, $k = [2, 2,…., 2; 3,3,….,3;……..; n-1, n-1,…., n-1; n]$

2 Method of Construction

Let as consider a 2^n-factorial design. There are 2^n-treatment combinations and n-main effects are there in the design. Now delete the control treatment and merge the highest order treatment combination (which is of less importance in estimation point of view) with each main effects separately. Thus we get 2^n-1 treatment combinations. Now consider these (2^n-1) treatment combinations as blocks for the required design with unequal (varying) block sizes.

For example, let n=3, Then in 2^3-factorial experiment there are 2^3 =8 treatment combinations in all, which are as follows

0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Delete a treatment combination whose levels of all factors are zero i.e. delete the control treatment. Next merge the highest order treatment combination with each main effect i.e. the treatment combination where the level of only one factor is one while the levels of the other factors are zero. Here there are three main effects in 2^3-factorial experiment. The remaining treatment combinations remain as it is. Thus we get

2	1	1
1	2	1

$$
\begin{array}{ccc}
1 & 1 & 2 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{array}
$$

Finally we get $2^3-1 = 7$ treatment combinations. Now transposing all the treatment combinations and treated them as blocks; we get the incidence matrix of variance balanced and efficiency balanced design, with block sizes 2, 3 and 4 respectively.

Theorem 2.1

The existence of 2^n- symmetrical factorial experiment implies the existence of equireplicate variance balanced and efficiency balanced design with unequal block sizes, having parameters

$v^* = n, \quad b^* = 2^n - 1, \quad r^* = 2^{n-1} + n, \quad k^* = [2,2,....,2; 3,3,....,3; 4,4,....,4; ....; n-1, n-1,...., n-1; n; n+1, n+1,...., n+1]$

Proof

In 2^n-symmetric factorial experiment, there are 2^n- treatment combinations in all. Considering "n" factors as rows and 2^n- treatment combinations as columns. Now deleting the control treatment and merging the highest order treatment combination with each main effect separately, we get the $2^n - 1$ treatment combinations (which are treated as blocks); then incidence matrix N^* of design D^* with unequal block sizes is given as

$$\tag{1}$$

Since in N^*; there are "n" rows. Considering these n-rows as treatments, we have $v^* = n$.

In incidence matrix N^*, among $^nc_2$ columns, in each column, element one occurs twice while zero occurs (n-2) times, among $^nc_3$ columns, element one occurs thrice while zero occurs (n-3) times in each column and so on. Also due to merging of blocks, $^nc_1$ columns are obtained in which one occurs (n-1) times and two occurs once.

Thus we get

$$b^* = {}^nC_2 + {}^nC_3 + \cdots\cdots\cdots + {}^nC_n + {}^nC_1$$

$$\Rightarrow b^* = {}^nC_2 + {}^nC_3 + \cdots\cdots\cdots + {}^nC_n + {}^nC_1 + {}^nC_0 - {}^nC_0$$

$$\Rightarrow b^* = 2^n - 1$$

Also in N^*; $^nc_2$ block have block size 2, $^nc_3$ blocks have block size 3, and so on and due to merging of blocks $^nc_1$ blocks have block size (n+1). Thus

$$k^* = \left[\underbrace{2,2,\ldots\ldots\ldots,2}_{{}^nC_2 \text{ times}} ; \underbrace{3,3,\ldots\ldots\ldots,3}_{{}^nC_3 \text{ times}} ; \ldots\ldots\ldots\ldots \underbrace{n-1,n-1,\ldots\ldots\ldots,n-1}_{{}^nC_{n-1} \text{ times}} ; \underbrace{n}_{{}^nC_n \text{ times}} ; \underbrace{n+1,n+1,\ldots\ldots\ldots,n+1}_{{}^nC_1 \text{ times}} \right]$$

Since we have considered rows as treatments and columns as blocks. In N^* there are $v^* = n$ treatments and $b^* = 2^n - 1$ blocks. In each row due to pooling one occurs $(2^{n-1}-1+n)$ times and 2 occurs once. Thus row sum becomes

$$r^* = {}^{n-1}C_1 + {}^{n-1}C_2 + \cdots\cdots\cdots + {}^{n-1}C_{n-1} + (n+1)$$

$$\Rightarrow r^* = \left({}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \cdots\cdots\cdots + {}^{n-1}C_{n-1} \right) - {}^{n-1}C_0 + (n+1)$$

$$\Rightarrow r^* = 2^{n-1} - 1 + (n+1)$$

$$\Rightarrow r^* = 2^{n-1} + n$$

Now calculation of variance and efficiency can be done as follows

Since a block design is variance balanced iff

$$C = \psi^* \left[I_v - \frac{1}{v} J_v J_v' \right]$$

$$\tag{2}$$

Where ψ^* is the unique non-zero eigen value of the C -matrix with multiplicity $(v-1)$.

The C- matrix for the design having incidence matrix given in (1) can be written as

$$
C = \theta \begin{bmatrix} n-1 & & & & \\ & \ddots & & -1 & \\ & & \ddots & & \\ & -1 & & \ddots & \\ & & & & n-1 \end{bmatrix} \tag{3}
$$

Where,

$$
\theta = \frac{n+2}{n+1} + \left\{ \frac{{}^{n-2}C_0}{n} + \frac{{}^{n-2}C_1}{n-1} + \frac{{}^{n-2}C_2}{n-2} + \cdots\cdots\cdots + \frac{{}^{n-2}C_m}{n-m} \right\}; \quad \text{where } m = 1, 2, \ldots\ldots \tag{4}
$$

Comparing (2) and (3)

$$
\psi^* = n \left[\frac{n+2}{n+1} + \left\{ \frac{{}^{n-2}C_0}{n} + \frac{{}^{n-2}C_1}{n-1} + \frac{{}^{n-2}C_2}{n-2} + \cdots\cdots\cdots + \frac{{}^{n-2}C_m}{n-m} \right\} \right]; \text{ where } m = 1, 2, \ldots\ldots \tag{5}
$$

Hence the incidence matrix defined in eq. (1) of design D^* gives equireplicated variance balanced design with unequal block sizes.

We know that M-matrix is defined as

$$
M = I - R^{-1} C \tag{6}
$$

After simplification we get,

$$
M = \begin{bmatrix} 1 - \dfrac{\theta(n-1)}{r^*} & & & & \\ & \ddots & & \dfrac{\theta}{r^*} & \\ & & \ddots & & \\ & \dfrac{\theta}{r^*} & & \ddots & \\ & & & & \dfrac{\theta(n-1)}{r^*} \end{bmatrix} \tag{7}
$$

Since MJ = J, where J is the unit vector of order (vx1).

Also M-matrix is given as

$$
M = \mu^* I_v + \left[(1-\mu^*) / \sum_i r_i^* \right] J_v \left(r^* \right) J_v' \tag{8}
$$

Where μ is the loss of information, I is the identify matrix of order ($v \times v$), J_v is unit vector of order ($v \times 1$) and $\sum_i r_i^*$ is the total number of observations.

On simplification we get

$$
M = \begin{bmatrix} 1 - \dfrac{r^*}{\sum_i r_i^*} & & & & \\ & \ddots & & -\dfrac{r^*}{\sum_i r_i^*} & \\ & & \ddots & & \\ & -\dfrac{r^*}{\sum_i r_i^*} & & \ddots & \\ & & & & 1-\dfrac{r^*}{\sum_i r_i^*} \end{bmatrix} + \begin{bmatrix} \dfrac{r^*}{\sum_i r_i^*} & \cdots & \cdots & \cdots & \dfrac{r^*}{\sum_i r_i^*} \\ \vdots & \dfrac{r^*}{\sum_i r_i^*} & \dfrac{r^*}{\sum_i r_i^*} & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & \dfrac{r^*}{\sum_i r_i^*} & \dfrac{r^*}{\sum_i r_i^*} & \vdots \\ \dfrac{r^*}{\sum_i r_i^*} & \cdots & \cdots & \cdots & \dfrac{r^*}{\sum_i r_i^*} \end{bmatrix} \tag{9}
$$

Comparing (7) and (9) we get,

$$
\mu^* = \left[\sum_i r_i^* - \frac{\theta(n-1)\sum_i r_i^*}{r^*} - r^* \right] \times \frac{1}{\sum_i r_i^* - r^*} = 1 - \frac{\theta \sum_i r_i^*}{(r^*)^2} \tag{10}
$$

where θ is defined in (4).

Thus the design is efficiency balanced with unequal block sizes.

Example 2.2 Let n=4, then in 2^4- factorial design; theorem 2.1 yields an incidence matrix N^* as given below

$$
N^* = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}
$$

Here,

$$v^* = 4, b^* = 15, r^* = 12, k^* = \left[\underbrace{2,2,2,2,2,2}_{^4c_2 \, times} \; ; \; \underbrace{3,3,3,3}_{^4c_3 \, times} \; ; \; \underset{^4c_4 \, times}{4} \; ; \; \underbrace{5,5,5,5}_{^4c_1 \, times} \right]$$

The C- matrix for the design having incidence matrix given above can be written as

$$C = \frac{157}{60} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$
(11)

Also we know tha

$$C = \psi^* \left[I_4 - \frac{1}{4} J_4 J_4' \right]$$
(12)

where ψ^* is the unique non-zero eight value of the C-matrix with multiplicity 3.

Comparing (11) and (12)

$$\psi^* = \frac{157}{15}$$
(13)

Thus the design is Variance balanced.

Now the M-matrix of the above design is given as

$$M = \begin{bmatrix} 83/240 & & & \\ & \ddots & \frac{157}{720} & \\ & \frac{157}{720} & \ddots & \\ & & & 83/240 \end{bmatrix}$$
(14)

Obviously this matrix satisfies the condition of efficiency balanced design i.e. MJ=J; where J is the v×1 vector of ones.

The efficiency factor is calculated using the formula

$$M = \mu^* I_v + \left[(1 - \mu^*) / \sum_i r_i^* \right] J_v \left(r^* \right) J_v'$$
(15)

On simplification we get

$$M = \mu^* \begin{bmatrix} 3/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 3/4 \end{bmatrix} + \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$
(16)

equating (14) and (16), we get

$$\mu^* = \frac{23}{180}$$
(17)

Thus the design is Efficiency balanced.

3 Optimality of the design

Let $\theta_1, \theta_2, \theta_3, \ldots, \theta_{(v-1)}$ be non-zero eigen values of C_d matrix of design d. As we know that for variance balanced design there will be only one non-zero eigen value with multiplicities $(v–1)$ of C_d matrix of design d. That is, $\theta_1 = \theta_2 = \theta_3 = \ldots = \theta_{(v-1)} = \theta$ as C– matrix is positive semi-definite. Then,

A-Optimality: A design is A-Optimal if

$$\sum_{i=1}^{v-1} \frac{1}{\theta_i} \geq \frac{(v-1)^2}{tr(C_d)}$$

D-Optimality: A design is D-Optimal if

$$\prod_{i=1}^{v-1} \frac{1}{\theta_i} \leq \prod_{i=1}^{v-1} \left\{ \sum_{i=1}^{v-1} \frac{1}{\theta_i} \Big/ (v-1) \right\}$$

E-Optimality: A design is E-Optimal if

$$\text{Min } (\theta_i) \leq \frac{tr(C_d)}{(v-1)}$$

Example 3.1 Consider a variance balanced and efficiency balanced design obtained in the example 2.2 with parameters $v^* = 4$, $b^* = 15$, $r^* = 12$, $k^* = [2,\ldots, 2; 3,\ldots, 3; 4; 5, \ldots, 5]$. The trace of C-matrix is comes out to be 157/5 and non-zero eight value of C-matrix is $\psi^* = \frac{157}{15}$ with multiplicity 3.

I) Checking A- Optimality

Here, the inequality

$$\sum_{i=1}^{4-1} \frac{1}{\psi_i^*} \geq \frac{(4-1)^2}{tr(C_d)} \quad \Rightarrow \quad \frac{45}{157} = \frac{45}{157}$$

holds true, which is the required condition of a variance balanced design to be A-optimal, with equal replication and unequal block sizes. Thus the variance balanced and efficiency balanced design constructed here is A-optimal.

II) Checking D-Optimality

Here, the inequality

$$\prod_{i=1}^{4-1} \frac{1}{\psi_i^*} \leq \prod_{i=1}^{4-1} \left\{ \sum_{i=1}^{4-1} \frac{1}{\psi_i^*} \middle/ (4-1) \right\} \quad \Rightarrow \quad \left(\frac{15}{157}\right)^3 = \left(\frac{15}{157}\right)^3$$

holds true, which is the required condition of a variance balanced design to be D-optimal, with equal replication and unequal block sizes. Thus the variance balanced and efficiency balanced design constructed here is D-optimal.

III) Checking E-optimality

Here, the inequality

$$\text{Min } (\psi_i^*) \leq \frac{tr(C_d)}{(4-1)} \quad \Rightarrow \quad \frac{157}{15} = \frac{157}{15}$$

holds true, which is the required condition of a variance balanced design to be E-optimal, with equal replication and unequal block sizes. Thus the variance balanced and efficiency balanced design constructed here is E-optimal.

　　　　Since, the constructed VB is A-Optimal, D-Optimal as well as E-Optimal and hence the constructed variance balanced and efficiency balanced design is the universally optimal.

Conclusion

In this research we have significantly shown that the constructed variance balanced and efficiency balanced designs are universally optimal and more efficient in terms of variance and efficiency both as compared to the previously constructed designs by Rajarathinam et al. [41-42].

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