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Some Results on Universal Optimality of Resolvable Designs with Unequal Block Sizes

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ABSTRACT
The optimality of the variance and efficiency balanced affine resolvable designs and resolvable designs with unequal block sizes has been checked and found it to be universal optimal. A method of constructing variance and efficiency balanced (α_1, α_2, …..,α_t) resolvable balanced incomplete block design with unequal block sizes is also proposed using 2^n-symmetrical factorial designs. Further, it is proposed that the designs constructed are universal optimal as well.

Introduction

Bose [1] introduced the concept of resolvability and affine resolvability. The concept of resolvability and affine resolvability was generalized to α - resolvability and affine α- resolvability by Shirkhande and Raghavarao [2]. The concept of α - resolvability was further generalized to (α_1, α_2, …..,α_t) resolvability by Kageyama [3] in 1976. A block design is said to be (α_1, α_2, …..,α_t) – resolvable if the blocks can be separated into t sets of m_i (≥ 2) blocks such that the set consisting of m_i blocks contains every treatment exactly α_i ($\geq$1) times, i.e. the set of m_i blocks form a α_i - replication set of each treatment (i = 1, 2, …t). Later different methods of construction of resolvable and affine resolvable designs have been given in the literature like, Bailey et al. [4], Banerjee et al. [5], Caliński et al.[6], Kageyama [7]-[10], Kageyama et al.[11], [12], Rai et al. [13], Rudra et al. [14], Mukerjee et al. [15], Agrawal et al. [16].

Let us consider v treatments arranged in b blocks, such that the j^{th} block contains k_j experimental units and i^{th} treatment appears r_i times in the entire design, i = 1, 2, ...,v ; j = 1, 2, …, b. For any block design there exist a incidence matrix $N=[n_{ij}]$ of order $v\times b$, where n_{ij} denotes the number of experimental units in the j^{th} block getting the i^{th} treatment. When n_{ij} = 1 or 0 ∀ i and j, the design is said to be binary. Otherwise it is said to be nonbinary. In this paper we consider binary block designs only. The following additional notations are used $\underline{k}=[k_1,k_2,....k_b]'$ is the column vector of block sizes, $\underline{r}=[r_1,r_2,.....r_v]'$ is the column vector of treatment replication, $K_{b\times b}=\text{diag }[k_1,k_2,....k_b]$, $R_{v\times v}=\text{diag }[r_1,r_2,.....r_v]$, $\sum r_i=\sum k_j=n$, the total number of experimental units, with this $N1_b=\underline{r}$ and $N'1_v=\underline{k}$, where 1_a is the $a\times 1$ vector of ones.

A balanced incomplete block design is an arrangement of v symbols (treatment) into b sets (blocks) such that (i) each block contains $k(<v)$ distinct treatments; (ii) each treatment appears in $r(>\lambda)$ different blocks and (iii) every pair of distinct treatments appears together in exactly λ blocks. Here, the parameters of balanced incomplete block design (v,b,r,k,λ) are related by the following relations $vr = bk$, $r\,(k\text{-}1)=\lambda\,(v\text{-}1)$ and $b\geq v$ (Fisher's inequality).

In many experimental situations, it is a severe restriction that all blocks in the experiment are of the same size. Variance Balanced (VB) designs forms a class of designs that are flexible extensions to balanced incomplete block designs. They provide the ability to design an experiment with equal precision among all pairwise comparisons, without being restricted to equal block size and equal replication of the treatments.

The importance of VB designs in the context of experimental material is well known as it yields optimal design apart from ensuring simplicity in the analysis. Many practical situations demand designs with varying block sizes (Pearce [17]), or resolvable VB designs with unequal replications (Kageyama [18], Mukerjee and Kageyama [19]).

Rao [20] gave the necessary and sufficient condition for a general block design to be variance balanced. The concept of efficiency balanced was introduced by Jones [21] and the nomenclature "Efficiency Balanced" is due to Puri et al. [22] and Williams [23].The importance of variance-balance and resolvability in the context of experimental planning is well known; the

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former yields optimal designs apart from ensuring simplicity in the analysis and the latter is helpful among other respects, in the recovery of interblock information. For the definition of variance balanced, efficiency balanced and factorial experiments along with their properties, refer, Dey [24] and Raghavarao [25].

In a given class of designs, one should attempt to choose a design which is good according to some well defined statistical criterion. This has led to the study of optimality of experimental designs. Optimal designs are experimental designs that are generated based on a particular optimality criterion and are generally optimal only for a specific statistical model. The optimality of a design depends on the statistical model and is assessed with respect to a statistical criterion, which is related to the variance matrix of the estimator.

Kiefer [26] introduced Balanced Block Designs (BBD) as a generalization of Balanced Incomplete Block (BIB) designs and proved the A-, D- and E-optimality of BBD's in D (v, b, k), where D (v, b, k) is the class of all connected block designs with v treatments, b blocks, and constant block size k.

Let C_p denote the class of all acceptable designs with reference to P. C_p consists of only connected designs. For any design d $\in C_p$; let V_d denote the dispersion matrix, using d. Then

A- optimality A design $d^* \in C_p$ is said to be A-optimal in C_p if $tr\ (V_d^*) \leq tr\ (V_d)$ i.e. A-optimality criterion seeks to minimize the trace of the inverse of the information matrix. This criterion results in minimizing the average variance of the estimates of the regression coefficients.

D- optimality A design $d^* \in C_p$ is said to be D-optimal in C_p if $det\ (V_d^*) \leq det\ (V_d)$ i.e. D-optimality criterion seeks to minimize $|(X'X)^{-1}|$, or equivalently maximize the determinant of the information matrix X'X of the design.

E- optimality A design $d^* \in C_p$ is said to be E-optimal in C_p if $\max(\lambda_d^*) \leq \max(\lambda_d)$ i.e. E-optimality criterion seeks to maximizes the minimum eigen value of the information matrix.

In fact Subsequently, Kiefer [27] proved the stronger result regarding the optimality of balanced block designs, by introducing the concept of *universal optimality* of BBD's in D (v, b, k). The original concept of universal optimality in Kiefer [33] dealt with information matrices with zero row and column sums.

Let C_d be the C-matrix of a design d. An optimality criterion is a function $\emptyset: R_v \rightarrow (-\infty, \infty)$, where R_v is the set of $v \times v$ non-negative definite matrices with zero row and column sums. A design d^* is called $\emptyset$ – optimal if it minimizes $\emptyset\ (C_d^*)$ over the class of competing designs. A design is said to be universally optimal if $\emptyset$ satisfies,

(i) $\emptyset$ is convex.

(ii) $\emptyset$ is non-increasing in the scalar $b \geq 0$.

(iii) $\emptyset$ is invariant under any simultaneous permutation of rows and columns of C.

Kiefer [27] obtained a sufficient condition for universal optimality. He proved that the balanced block design (if it exists) is universally optimal in the class of all connected designs. If a design is universally optimal then it is A-, D- and E-optimal as well and a vice a versa.

Although a considerable amount of work is available on optimality of designs in D (v, b, k), not much appears to have been done on the optimality of designs with unequal block sizes, except by Lee and Jacroux [28-30], Dey and Das [31], Gupta and Singh [32], Gupta et al. [33].

Kageyama [34] gave the construction method of obtaining affine resolvable variance balanced deigns with unequal block sizes by using incidence matrices of known affine resolvable balanced incomplete block designs. Further, Agrawal et al. [35] proposed that these constructed designs are efficiency balanced as well. In this paper we proposed that those constructed affine resolvable variance and efficiency balanced design with unequal block sizes leading to the universal optimal designs. We have also proposed construction methods $(\alpha_1, \alpha_2, \ldots..\alpha_t)$-resolvable variance and efficiency balanced designs by using 2^n-symmetrical factorial designs. Further, we proposed that the designs constructed are universal optimal as well.

2. Method of Construction of Design Matrix-I

The construction method of obtaining affine resolvable variance balanced deigns with unequal block sizes by using incidence matrices of known affine resolvable balanced incomplete block designs were given by Kageyama [34]. The following result is from Kageyama [34]

Proposition 2.1

The existence of an affine resolvable balanced incomplete block design D with parameter $v = 2k$, $b = 4k$-2, $r = 2k$-1, k, $\lambda = k$-1 implies the existence of an affine resolvable efficiency balanced block design with unequal block sizes and parameter $v_* = 4k$, $b_* = 6k$, $r_* = 2k+1$, $k_{j*} = 2$ or v, $q_{1l} = 1(1,\ldots r_*)$, $q_{2l} = k(l=3,\ldots r_*)$, $q_{ll'} = k(l \neq l' = 3,\ldots.r_*)$ and $\eta_* = 2k$.

Further, Agrawal et al. [35] proposed that the designs constructed by Kageyama [34] are efficiency balanced as well. The following result is from Agrawal et al. [35]

Proposition 2.2

The existence of an affine resolvable balanced incomplete block design D with parameter $v = 2k$, $b = 4k$-2, $r = 2k$-1, k, $\lambda = k$-1 implies the existence of an affine resolvable efficiency balanced block design with unequal block sizes and parameter $v_* = 4k$, $b_* = 6k$, $r_* = 2k+1$, $k_{1*} = 2$, $k_{2*} = 2k$, $\lambda_* = k$, $q_{ll} = 0(l=1,\ldots,r_*)$, $q_{1l} = 1(1,\ldots r_*)$, $q_{2l} = k(l=3,\ldots r_*)$, $q_{ll'} = k(l \neq l' = 3,\ldots.r_*)$ and $\mu_* = 1/(2k+1)$.

In this section we are checking the universal optimality of these designs.

Theorem 2.3

The variance and efficiency balanced affine resolvable designs with unequal block sizes given in Kageyama [34] and Agrawal et al. [35] are universal optimal.

Proof

The C-matrix of affine resolvable designs with unequal block sizes given by Kageyama [34] and Agrawal et. al [35] is

$$C=\theta\begin{bmatrix} v-1 & -1 & -1 & \Lambda & -1 \\ -1 & v-1 & -1 & \Lambda & -1 \\ -1 & -1 & v-1 & \Lambda & -1 \\ M & M & M & M & M \\ -1 & -1 & -1 & \Lambda & v-1 \end{bmatrix}_{(v\times v)}$$

Let η_1, η_2, η_3,…., $\eta_{(v-1)}$ be non-zero eigen values of C matrix of the resultant design D . As we know that for variance balanced design there will be only one non-zero eigen value with multiplicities (v–1) of C matrix of design D. That is, $\eta_1 = \eta_2 = \eta_3 =\ldots= \eta_{(v-1)} = \eta = 2k$, as C– matrix is positive semi-definite. Therefore,

$$C=\eta\left[I_v-\frac{1}{v}1_v\,1_v^{'}\right]$$

Where, η is unique non-zero eigen value of C-matrix with multiplicity (v-1). The trace of the C-matrix of design D_* is tr (C) = tr (R - $NK^{-1}N^{'}$) = η (v - 1). Then

A-Optimality

The design is A-Optimal, since the inequality

$$\sum_{i=1}^{(v-1)}\frac{1}{\eta_i}\geq\frac{(v-1)^2}{tr(C\ \ )}$$

Holds true, which is the required condition of a variance balanced design to be A-optimal, with equal replication and unequal block sizes. Thus the variance and efficiency balanced design constructed is A-optimal.

D-Optimality

The design is D-Optimal, since the inequality

$$\prod_{i=1}^{(v-1)}\frac{1}{\eta_i}\leq\prod_{i=1}^{(v-1)}\left\{\sum_{i=1}^{v-1}\frac{1}{\eta_i}\Big/(v-1)\right\}$$

Holds true, which is the required condition of a variance balanced design to be D-optimal, with equal replication and unequal block sizes. Thus the design constructed is D-optimal.

E-Optimality

The design is E-Optimal, since the inequality

$$Min(\eta_i)\leq\frac{tr(C)}{(v-1)}$$

Holds true, which is the required condition of a variance balanced design to be E-optimal, with equal replication and unequal block sizes. Thus the design constructed is E-optimal.

Since, the variance and efficiency balanced affine resolvable balanced incomplete block design with unequal block sizes given by Kageyama [36] and Agrawal et. al [37] are A, D and E-Optimal. Hence the constructed design is the universally optimal.

Example 2.4

Consider a variance and efficiency balanced affine resolvable balanced incomplete block design with unequal block sizes and parameters v_* = 8, b_* =12, r_* =5, k_{1*} =2, k_{2*} =4, λ_*=2, η_*=4 and μ_* =1/5. The incidence matrix of the design is given as follows

$$N_*=\begin{bmatrix} 1&0&0&0&1&0&1&0&1&0&1&0\\ 0&1&0&0&1&0&0&1&1&0&0&1\\ 0&0&1&0&1&0&1&0&0&1&0&1\\ 0&0&0&1&1&0&0&1&0&1&1&0\\ 1&0&0&0&0&1&0&1&0&1&0&1\\ 0&1&0&0&0&1&1&0&0&1&1&0\\ 0&0&1&0&0&1&0&1&1&0&1&0\\ 0&0&0&1&0&1&1&0&1&0&0&1 \end{bmatrix}$$

C- matrix for the incidence matrix given above can be written as

$$C\ \ =1/2\begin{bmatrix} 7&-1&-1&-1&-1&-1&-1&-1\\ -1&7&-1&-1&-1&-1&-1&-1\\ -1&-1&7&-1&-1&-1&-1&-1\\ -1&-1&-1&7&-1&-1&-1&-1\\ -1&-1&-1&-1&7&-1&-1&-1\\ -1&-1&-1&-1&-1&7&-1&-1\\ -1&-1&-1&-1&-1&-1&7&-1\\ -1&-1&-1&-1&-1&-1&-1&7 \end{bmatrix}_{(8\times8)}$$

Further, it is simplified as

$C = 4[I_8 - \frac{1}{8}\,1_v1_v']$

The trace of C matrix is comes out to as 28 and non-zero eigen value of C matrix is η^*=4 with multiplicity 7.

I) Checking A- Optimality

Here, the inequality

$$\sum_{i=1}^{(8-1)} \frac{1}{\eta_{i*}} \geq \frac{(8-1)^2}{tr(C)} \Rightarrow \frac{7}{4} = \frac{7}{4}$$

Holds true and hence the design is A-optimal.

II) Checking D-Optimality

Here, the inequality

$$\prod_{i=1}^{(8-1)} \frac{1}{\eta_{i*}} \leq \prod_{i=1}^{(8-1)} \left\{ \sum_{i=1}^{8-1} \frac{1}{\eta_{i*}} \Big/ (8-1) \right\} \Rightarrow \left(\frac{1}{4}\right)^7 = \left(\frac{1}{4}\right)^7$$

Holds true and hence the design is D-optimal.

III) Checking E-optimality

Here, the inequality

$$\min(\eta_{i*}) \leq \frac{tr(C)}{(8-1)} \Rightarrow 4 = 4$$

Holds true and hence the design is E-optimal. Since, the constructed affine resolvable design with unequal block sizes is A, D and E-Optimal. Hence the constructed design is the universally optimal.

3. Method of Construction of Design Matrix-II

Let as consider a 2^n-factorial design ($n \geq 4$). There are 2^n-treatment combinations and n-main effects are there in the design. Now delete the control treatment (i.e. a treatment combination whose level of all factor is zero), highest order treatment combination (which is of less importance in estimation point of view) and n- main effects (i.e. a treatment combination, where level of one factor is one while level of other factor are zero), which is of less importance as the block size is one. Thus we get 2^n-(n+2) treatment combinations. Now consider these 2^n-(n+2) treatment combinations as blocks for the required design with unequal block sizes.

Theorem 3.1

The existence of 2^n-symmetrical factorial experiment implies the existence of equireplicated variance and efficiency balanced ($\alpha_1, \alpha_2, \ldots\alpha_t$)-resolvable designs with unequal block sizes, having the parameters

$v^* = n$, $b^* = 2^n-(n+2)$, $r^* = 2^{n-1}-2$, $k^* = [2,2,\ldots,2;3,3,\ldots,3;4,4,\ldots,4;\ldots..n-1,n-1,\ldots,n-1]$ and $\lambda^* = 2^{n-2}-1$

Proof

Consider 2^n-symmetrical factorial experiment. This has 2^n treatment combinations in all. Considering "n" factors as rows and 2^n- treatment combinations as columns. Now deleting the control treatment, highest order treatment combinations and n-main effect treatment combinations, we get the 2^n-(n+2) treatment combinations (which are treated as blocks); then incidence matrix N^* of the resultant design D^* with unequal block sizes is given as

$$N^* = \begin{bmatrix} \overbrace{\begin{matrix} 1 & 1 & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & \cdots & 1 \end{matrix}}^{2-\text{factor interaction}} & \vdots & \overbrace{\begin{matrix} 1 & 1 & \cdots & \cdots & 0 \\ 1 & 1 & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & \cdots & 1 \\ 0 & 0 & \cdots & \cdots & 1 \end{matrix}}^{3-\text{factor interaction}} & \vdots & \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} & \vdots & \overbrace{\begin{matrix} 1 & 1 & \cdots & \cdots & 0 \\ 1 & 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & \cdots & \cdots & 1 \\ 0 & 1 & \cdots & \cdots & 1 \end{matrix}}^{(n-1)-\text{factor interaction}} \end{bmatrix} \quad (1)$$

Since in N^*; there are "n" rows. Considering these n-rows as treatments, obviously we have $v^*=n$.

In incidence matrix N^*, among ${}^nC_2$ columns, element one occurs twice while zero occurs (n-2) times, among ${}^nC_3$ columns, element one occurs thrice while zero occurs (n-3) times in each column and so on. Thus we get

$b^* = {}^nC_2 + {}^nC_3 + \ldots\ldots\ldots\ldots\ldots + {}^nC_{n-1}$

Since ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots.. + {}^nC_n = 2^n$ and therefore

$b^* = 2^n - ({}^nC_0 + {}^nC_1 + {}^nC_n)$

$\Longrightarrow b^* = 2^n - (n+2)$

Also in incidence matrix N^*; ${}^nC_2$ blocks have block size 2, ${}^nC_3$ blocks have block size 3 and so on. Thus

$$k^* = \left[\underbrace{2,2,\cdots,2}_{{}^nC_2\ times}; \underbrace{3,3,\cdots,3}_{{}^nC_3\ times}; \underbrace{4,4,\cdots,4}_{{}^nC_4\ times}; \cdots; \underbrace{n-1,n-1,\cdots,n-1}_{{}^nC_{n-1}\ times} \right]$$

Since we have considered rows as treatments and columns as blocks. In N^* there are $v^* = n$ treatments and $b^* = 2^n - (n+2)$ blocks and in each row one occurs $2^{n-1} - 2$ times. Thus row sum is

$r^* = {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots\ldots\ldots + {}^{n-1}C_{n-2}$

Since ${}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots\ldots\ldots\ldots + {}^{n-1}C_{n-1} = 2^{n-1}$ and therefore

$r^* = 2^{n-1} - ({}^{n-1}C_0 + {}^{n-1}C_{n-1})$

$\Longrightarrow r^* = 2^{n-1} - 2$

In the present construction method, the frequency of (θ,Ø) pair can be calculated as

$\lambda^* = {}^{n-2}C_1 + {}^{n-2}C_2 + \ldots\ldots\ldots + {}^{n-2}C_{n-2}$

Since ${}^{n-2}C_0 + {}^{n-2}C_1 + {}^{n-2}C_2 + \ldots\ldots\ldots\ldots + {}^{n-2}C_{n-2} = 2^{n-2}$ and therefore

$\lambda^* = 2^{n-2} - {}^{n-2}C_0$

$\Longrightarrow \lambda^* = 2^{n-2} - 1$

Here in the present construction method on pooling all i-factor interaction (i=2,3,….,n-1) treatment combinations, the resultant design is (α_1, α_2, …..α_t)-resolvable design with unequal block sizes in each resolution set.

Now calculation of variance and efficiency can be done as follows

A block design is said to be variance balanced iff

$$C = (I_v - \frac{1}{v} 1_v 1_v') \tag{2}$$

Where η^* is the unique non zero eigen value of C-matrix with multiplicity (v-1). The C-matrix for the design having incidence matrix given in (1) can be written as

$$C = \theta \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{bmatrix}_{(n \times n)} \tag{3}$$

Where,

$$\theta = \frac{{}^{n-2}C_1}{n-1} + \frac{{}^{n-2}C_2}{n-2} + \frac{{}^{n-2}C_3}{n-3} + \cdots\cdots + \frac{{}^{n-2}C_m}{n-m} \quad \text{;where m = 1,2,….., (n-2)} \tag{4}$$

Comparing (2) and (3)

$$\eta^* = n\left[\frac{{}^{n-2}C_1}{n-1} + \frac{{}^{n-2}C_2}{n-2} + \frac{{}^{n-2}C_3}{n-3} + \cdots\cdots + \frac{{}^{n-2}C_m}{n-m}\right] \text{; where m = 1,2,…..,(n-2)} \tag{5}$$

Hence the incidence matrix defined in (1) of design D* gives equireplicated variance balanced design with unequal block sizes.

We know that M-matrix is defined as

$$M = I - R^{-1}C \tag{6}$$

After simplification we get

$$M = \begin{bmatrix} 1 - \frac{\theta(n-1)}{r^*} & & & \\ & \ddots & \frac{\theta}{r^*} & \\ & \ddots & & \\ & \frac{\theta}{r^*} & \ddots & \\ & & & 1 - \frac{\theta(n-1)}{r^*} \end{bmatrix} \tag{7}$$

Since $MJ = J$, where J is the unit vector of order (v×1). Also M matrix is given as

$$M = \mu^* I_v + \left[(1-\mu^*)/\sum_{i=1}^{v} r_i^*\right] J_v (r^*) J_v \tag{8}$$

Where μ* is the loss of information, I is the identify matrix of order (v×v), J_v is the unit vector of order (v×1) and $\sum_i r_i^*$ is the total number of experimental units. On simplification we get

$$M = \begin{bmatrix} 1 - \frac{r^*}{\sum_{i=1}^{v} r_i^*} & & & & \\ & \ddots & & -\frac{r^*}{\sum_{i=1}^{v} r_i^*} & \\ & & \ddots & & \\ & -\frac{r^*}{\sum_{i=1}^{v} r_i^*} & & \ddots & \\ & & & & 1 - \frac{r^*}{\sum_{i=1}^{v} r_i^*} \end{bmatrix} + \begin{bmatrix} \frac{r^*}{\sum_{i=1}^{v} r_i^*} & \cdots & \cdots & \cdots & \frac{r^*}{\sum_{i=1}^{v} r_i^*} \\ \vdots & \frac{r^*}{\sum_{i=1}^{v} r_i^*} & & \frac{r^*}{\sum_{i=1}^{v} r_i^*} & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & \frac{r^*}{\sum_{i=1}^{v} r_i^*} & & \frac{r^*}{\sum_{i=1}^{v} r_i^*} & \vdots \\ \frac{r^*}{\sum_{i=1}^{v} r_i^*} & \cdots & \cdots & \cdots & \frac{r^*}{\sum_{i=1}^{v} r_i^*} \end{bmatrix} \tag{9}$$

Comparing (7) and (9) we get,

$$\mu^* = \left[\sum_{i=1}^{v} r_i^* - \frac{\theta(n-1)\sum_{i=1}^{v} r_i^*}{r^*} - r^* \right] \times \frac{1}{\sum_{i=1}^{v} r_i^* - r^*} = 1 - \frac{\theta \sum_{i=1}^{v} r_i^*}{\left(r^*\right)^2} \tag{10}$$

Where θ is defined in (4). Thus the design is efficiency balanced with unequal block sizes.

Theorem 3.2

The variance and efficiency balanced $(\alpha_1,\alpha_2,\ldots,\alpha_t)$-resolvable balanced incomplete block designs with unequal block sizes given in Theorem 3.1 are universal optimal.

Proof

The C-matrix of the design is defined as below

$$C = R - NK^{-1}N'$$

Let η_1, η_2, η_3,…., $\eta_{(v-1)}$ be non-zero eigen values of C matrix of the resultant design D given in Theorem 3.1. In the construction method given above, the C-matrix is

$$C = \theta \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{bmatrix}_{(n\times n)}$$

Since the design constructed is variance balanced. Therefore, its C-matrix is given as

$$C = \eta \left[I_v - \frac{1}{v} 1_v 1_v' \right]$$

Also we know that for the resultant variance balanced design, there will be only one non-zero eigen value with multiplicities (v–1) of C-matrix. That is, $\eta_1 = \eta_2 = \eta_3 = \ldots = \eta_{(v-1)} = \eta$, as C– matrix is positive semi-definite. The trace of the C-matrix of design D is tr (C) = tr (R - $NK^{-1}N'$) = $\eta\,(v$-1). Then

A-Optimality

The design is A-Optimal, since the inequality

$$\sum_{i=1}^{(v-1)} \frac{1}{\eta_i} \geq \frac{(v-1)^2}{tr(C)}$$

Holds true, which is the required condition of a variance balanced design to be A-optimal, with equal replication and unequal block sizes. Thus the design constructed here is A-optimal.

D-Optimality

The design is D-Optimal, since the inequality

$$\prod_{i=1}^{(v-1)} \frac{1}{\eta_i} \leq \prod_{i=1}^{(v-1)} \left\{ \sum_{i=1}^{v-1} \frac{1}{\eta_i} \Big/ (v-1) \right\}$$

Holds true, which is the required condition of a variance balanced design to be D-optimal, with equal replication and unequal block sizes. Thus the design constructed here is D-optimal.

E-Optimality

The design is E-Optimal, since the inequality

$$\min(\eta_i) \leq \frac{tr(C)}{(v-1)}$$

Holds true, which is the required condition of a variance balanced design to be E-optimal, with equal replication and unequal block sizes. Thus the design constructed here is E-optimal.

Since, the constructed variance and efficiency balanced $(\alpha_1,\alpha_2,\ldots,\alpha_t)$-resolvable balanced incomplete block design with unequal block sizes is A, D and E-Optimal and hence the constructed design is the universally optimal.

Example 3.3

Let n = 5, then in 2^5-factorial design after deleting the control treatment, highest order treatment combinations and n-main effect treatment combinations; theorem 3.1 yields a (4,6,4)-resolvable variance and efficiency balanced design with unequal block sizes and parameters

$$v^* = 5, b^* = 25, r^* = 14, k^* = \left[\underbrace{2,2,2,2,2,2,2,2,2,2}_{^5C_2}, \underbrace{3,3,3,3,3,3,3,3,3,3}_{^5C_3}, \underbrace{4,4,4,4,4}_{^5C_4} \right], \lambda^* = 7$$

The incidence matrix N^* of the resultant design is given below

$$N^* = \left[\begin{array}{cccccccccc:cccccccccc:ccccc} 1&1&0&1&0&0&1&0&0&0&1&1&1&0&1&1&0&1&0&0&1&1&1&1&0\\ 1&0&1&0&1&0&0&1&0&0&1&1&0&1&1&0&1&0&1&0&1&1&1&0&1\\ 0&1&1&0&0&1&0&0&1&0&1&0&1&1&0&1&1&0&0&1&1&1&0&1&1\\ 0&0&0&1&1&1&0&0&0&1&0&1&1&1&0&0&0&1&1&1&1&0&1&1&1\\ 0&0&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&1&1&0&1&1&1&1 \end{array}\right]$$

Obviously the design satisfies the conditions of ($\alpha_1,\alpha_2,\ldots..,\alpha_t$)-resolvable balanced incomplete block design. Hence the design is (4,6,4)-resolvable balanced incomplete block design.

The C-matrix for the incidence matrix given above can be written as

$$C = \frac{9}{4}\begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}_{(5\times 5)} \tag{11}$$

Also we know that

$$C = \eta^*\left[I_5 - \frac{1}{5}J_5 J_5{}'\right] \tag{12}$$

Where η^* is the unique non – zero eigen value of the C-matrix with multiplicity 4.

Comparing (11) and (12)

$$\eta^* = \frac{45}{4}$$

Thus the design is variance balanced.

Now the M matrix of the above design is given as

$$M = \begin{bmatrix} 5/14 & 9/56 & 9/56 & 9/56 & 9/56 \\ 9/56 & 5/14 & 9/56 & 9/56 & 9/56 \\ 9/56 & 9/56 & 5/14 & 9/56 & 9/56 \\ 9/56 & 9/56 & 9/56 & 5/14 & 9/56 \\ 9/56 & 9/56 & 9/56 & 9/56 & 5/14 \end{bmatrix} \tag{13}$$

Obviously this matrix satisfies the condition of efficiency balanced design i.e. MJ = J; where J is the $v\times 1$ vector of ones. The efficiency factor is calculated using the formula

$$M = \mu^* I_v + \left[(1-\mu^*)/\sum_i r_i^*\right] J_v\left(r^*\right)J_v{}' \tag{14}$$

On simplification we get

$$M = \mu^*\begin{bmatrix} 4/5 & -1/5 & -1/5 & -1/5 & -1/5 \\ -1/5 & 4/5 & -1/5 & -1/5 & -1/5 \\ -1/5 & -1/5 & 4/5 & -1/5 & -1/5 \\ -1/5 & -1/5 & -1/5 & 4/5 & -1/5 \\ -1/5 & -1/5 & -1/5 & -1/5 & 4/5 \end{bmatrix} + \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix} \tag{15}$$

Equating (13) and (15), we get

$$\mu^* = \frac{11}{56}$$

Thus the design if Efficiency balanced.

The trace of C-matrix given in (11) is comes out to as 45 and non-zero eigen value of C-matrix is 45/4 with multiplicity 4.

I) Checking A- Optimality

Here, the inequality

$$\sum_{i=1}^{(5-1)}\frac{1}{\eta_i^*} \geq \frac{(5-1)^2}{tr(C)} \Rightarrow \frac{16}{45} = \frac{16}{45}$$

Holds true and hence the design is A-optimal.

II) Checking D-Optimality

Here, the inequality

$$\prod_{i=1}^{(5-1)}\frac{1}{\eta_i^*} \leq \prod_{i=1}^{(5-1)}\left\{\sum_{i=1}^{5-1}\frac{1}{\eta_i^*}\Big/(5-1)\right\} \Rightarrow \left(\frac{4}{45}\right)^4 = \left(\frac{4}{45}\right)^4$$

Holds true and hence the design is D-optimal.

III) Checking E-optimality

Here, the inequality

$$\min(\eta_i^*) \le \frac{tr(C)}{(5-1)} \Rightarrow \frac{45}{4} = \frac{45}{4}$$

Holds true, hence the design is E-optimal. Since, the constructed variance and efficiency balanced ($\alpha_1,\alpha_2,\ldots,\alpha_t$)-resolvable balanced incomplete block design with unequal block sizes is A, D and E-Optimal. Hence the constructed design is the universally optimal.

4. Method of Construction of Design Matrix-III

The construction method of obtaining pairwise balanced design which is efficiency and variance balanced by using incidence matrices of known balanced incomplete block designs were given by Kageyama et al. [36]. The following result is from Kageyama et al. [36]

Proposition 4.1

A pairwise balanced design *D* is efficiency balanced, if its equiblock component D_i, $i=1,2,\ldots m$, is equireplicated and efficiency balanced.

It was already proposed that, if N is the incidence matrix of a variance as well as efficiency balanced incomplete block design D and N^c is the incidence matrix of complement of the design D, which is also variance as well as efficiency balanced. Then the following incidence matrix N^{**} yields an variance and efficiency balanced resolvable design with unequal block sizes

$$N^{**} = [N \cup N^c] \tag{16}$$

In this section we obtain the universal optimality of these designs with unequal block sizes

Theorem 4.2

The variance and efficiency balanced resolvable design with unequal block sizes, whose incidence matrix is given by (16) are universal optimal.

Proof

The C-matrix of the resultant resolvable design with unequal block sizes is given as

$C = RI_v - NK^{-1}N'$

The constructed design D will be A-optimal if it maximizes tr (C).That is, tr (C) = tr (R - $NK^{-1}N'$) = η (v-1).

For a design, it can be shown that the sum of the variances of the estimates of all elementary treatment contrast is proportional to the sum of the reciprocals of the non-zero eigen values of C. Let $\eta_1, \eta_2, \eta_3,\ldots, \eta_{(v-1)}$ be non-zero eigen values of C matrix of the resultant design D . As we know that for variance balanced design there will be only one non-zero eigen value with multiplicities (v–1) of C matrix. That is, $\eta_1 = \eta_2 = \eta_3 = \ldots = \eta_{(v-1)} = \eta$, as C– matrix is positive semi-definite. Then

A-Optimality

The design is A-Optimal, since the inequality

$$\sum_{i=1}^{(v-1)} \frac{1}{\eta_i} \ge \frac{(v-1)^2}{tr(C)}$$

Holds true, which is the required condition of a variance balanced design to be A-optimal, with equal replication and unequal block sizes. Thus the design constructed is A-optimal.

D-Optimality

The design is D-Optimal, since the inequality

$$\prod_{i=1}^{(v-1)} \frac{1}{\eta_i} \le \prod_{i=1}^{(v-1)} \left\{ \sum_{i=1}^{v-1} \frac{1}{\eta_i} \Big/ (v-1) \right\}$$

Holds true, which is the required condition of a variance balanced design to be D-optimal, with equal replication and unequal block sizes. Thus the design constructed here is D-optimal.

E-Optimality

The design is E-Optimal, since the inequality

$$\text{Min}\,(\eta_i) \le \frac{tr(C)}{(v-1)}$$

Holds true, which is the required condition of a variance balanced design to be E-optimal, with equal replication and unequal block sizes. Thus the design constructed here is E-optimal.

Since, the constructed variance and efficiency balanced resolvable balanced incomplete block design with unequal block sizes are A, D and E-Optimal and hence the constructed design is the universally optimal.

Example 4.3

Let us consider a variance and efficiency balanced symmetric balanced incomplete block design D with parameters v = b = 7, r = k = 4, λ = 2 and N^c is complement of the design D with parameters $v^* = b^* = 7$, $r^* = k^* = 3$, λ = 1. Then the incidence matrix given in (16) yields resolvable design with unequal block sizes and parameters $v^{**} = 7$, $b^{**} = 14$, $r^{**} = 7$, $k_1^{**} = 4$, $k_2^{**} = 3$ and λ^{**} = 3. The incidence matrix of the resultant design is given as follows

$$N^{**}=\left[\begin{array}{cc:cc:cc:cc:cc:cc:cc} 1&0&1&0&1&0&1&0&0&1&0&1&0&1\\ 1&0&1&0&0&1&0&1&1&0&1&0&0&1\\ 1&0&0&1&1&0&0&1&1&0&0&1&1&0\\ 1&0&0&1&0&1&1&0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0&1&0&1&1&0&1&0\\ 0&1&1&0&0&1&1&0&1&0&0&1&1&0\\ 0&1&0&1&1&0&1&0&1&0&1&0&0&1 \end{array}\right]$$

The C-matrix for the incidence matrix given above can be written as

$$C=5\begin{bmatrix} 1 & -1/6 & -1/6 & -1/6 & -1/6 & -1/6 & -1/6\\ -1/6 & 1 & -1/6 & -1/6 & -1/6 & -1/6 & -1/6\\ -1/6 & -1/6 & 1 & -1/6 & -1/6 & -1/6 & -1/6\\ -1/6 & -1/6 & -1/6 & 1 & -1/6 & -1/6 & -1/6\\ -1/6 & -1/6 & -1/6 & -1/6 & 1 & -1/6 & -1/6\\ -1/6 & -1/6 & -1/6 & -1/6 & -1/6 & 1 & -1/6\\ -1/6 & -1/6 & -1/6 & -1/6 & -1/6 & -1/6 & 1 \end{bmatrix}_{(7\times 7)}$$

Further, it is simplified as

$$C=\frac{35}{6}\left[I_7-\frac{1}{7}J_7J_7^{'}\right]$$

The trace of C matrix is comes out to as 35 and non-zero eigen value of C matrix is η^{**}=35/6 with multiplicity 6.

I) Checking A- Optimality

Here, the inequality

$$\sum_{i=1}^{(7-1)}\frac{1}{\eta_i^{**}}\geq\frac{(7-1)^2}{tr(C)}\Rightarrow\frac{36}{35}=\frac{36}{35}$$

Holds true, thus the design is A-optimal.

II) Checking D-Optimality

Here, the inequality

$$\prod_{i=1}^{(7-1)}\frac{1}{\eta_i^{**}}\leq\prod_{i=1}^{(7-1)}\left\{\sum_{i=1}^{7-1}\frac{1}{\eta_i^{**}}\Big/(7-1)\right\}\Rightarrow\left(\frac{6}{35}\right)^6=\left(\frac{6}{35}\right)^6$$

Holds true, thus the design is D-optimal.

III) Checking E-optimality

Here, the inequality

$$\text{Min}\,(\eta_i^{**})\leq\frac{tr(C)}{(7-1)}\Rightarrow\frac{35}{6}=\frac{35}{6}$$

Holds true, thus the design is E-optimal. Since, the constructed resolvable design with unequal block sizes is A, D and E-Optimal. Hence the constructed design is the universally optimal.

5. Results and Discussion

The following tables provides a list of universal optimal (α_1, α_2, …..α_t)-resolvable and affine resolvable designs with unequal block sizes for $r \leq 30$

Table 1

S. No.	v^*	b^*	r^*	k_1^*	k_2^*	k_3^*	k_4^*	λ^*	**Type of resolvability**	**Reference**
1	4	10	6	2	3	-	-	3	3-resolvable	Theorem 3.1
2	5	25	14	2	3	4	-	7	(4,6,4)-resolvable	Theorem 3.1
3	6	56	30	2	3	4	5	17	(5,10,10,5)-resolvable	Theorem 3.1
4	7	14	7	4	3	-	-	3	1-resolvable	Theorem 4.2
5	8	12	5	2	4	-	-	2	Affine-1-resolvable	Theorem 2.3
6	11	22	11	6	5	-	-	5	1-resolvable	Theorem 4.2
7	15	30	15	8	7	-	-	7	1-resolvable	Theorem 4.2
8	16	24	9	2	8	-	-	4	Affine-1-resolvable	Theorem 2.3
9	19	38	19	10	9	-	-	9	1-resolvable	Theorem 4.2
10	23	46	23	12	11	-	-	11	1-resolvable	Theorem 4.2
11	24	36	13	2	12	-	-	6	Affine-1-resolvable	Theorem 2.3
12	27	54	27	14	13	-	-	13	1-resolvable	Theorem 4.2
13	32	48	17	2	16	-	-	8	Affine-1-resolvable	Theorem 2.3
14	40	60	21	2	20	-	-	10	Affine-1-resolvable	Theorem 2.3
15	48	72	25	2	24	-	-	12	Affine-1-resolvable	Theorem 2.3
16	56	84	29	2	28	-	-	14	Affine-1-resolvable	Theorem 2.3

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